# ON THE RIEMANN DERIVATIVES FOR INTEGRABLE FUNCTIONS

P. L. BUTZER AND W. KOZAKIEWICZ

**1. Introduction.** The central difference of order $s$ of the function $f(x)$, $\Delta_{2h}^{s} f(x)$, corresponding to a number $h > 0$, is defined inductively by the relations

$$\Delta_{2h}^{1} f(x) = f(x+h) - f(x-h), \quad \Delta_{2h}^{s+1} f(x) = \Delta_{2h}^{1}[\Delta_{2h}^{s} f(x)].$$

If the limit of the difference quotient

$$\lim_{h \to 0} (2h)^{-s} \Delta_{2h}^{s} f(x)$$

exists at the point $x$, it is called the $s$th Riemann derivative or the generalized $s$th derivative of $f(x)$ at the point $x$.

This paper deals with the following problem: What are the necessary and sufficient conditions in order that a given integrable function $f(x)$ be p.p. (almost everywhere) equal to an indefinite repeated integral of another function $g(x)$? The main result (Theorem 2) gives this condition in terms of the weak convergence of the difference quotient of $f(x)$ to $g(x)$.

In particular, in §3 we prove by an elementary but apparently powerful method, a theorem which contains the well-known proposition of Brouwer (**3** or **1** or **6**, p. 70) which states:

A. *If $f(x)$ is continuous for $a < x < b$ and*

$$\Delta_{2h}^{s} f(x) = 0, \qquad a < x - sh < x + sh < b,$$

*then $f(x)$ is a polynomial of degree at most $(s-1)$ in $(a, b)$.*

In §4 we come to our main result mentioned above which in §5 we use to establish a certain type of extension of the following theorem of de la Vallée-Poussin (**12**, p. 274):

B. *If $f(x)$ is continuous in $[a, b]$ and has at every point of this interval a finite second Riemann derivative $g(x)$, with $g(x) \in L(a, b)$, then*

$$f(x) = \int_a^x dt_1 \int_a^{t_1} g(t_2)\, dt_2 + c_0 + c_1 x, \qquad a \leqslant x \leqslant b,$$

*where $c_0$ and $c_1$ are constants.*

This last theorem is fundamental in the uniqueness theory of trigonometrical series.

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Finally in §6 we state results due to Anghelutza, Marchaud, Popoviciu, and Reid which follow from our main theorem. We also consider in this section an application to generalized convex functions.

**2. Preliminary results.** Consider the space $L(a, b)$, that is, the space of functions which are Lebesgue integrable over $(a, b)$. The distance between two elements $f, h \in L(a, b)$ is defined as

$$||f - h|| = \int_a^b |f(x) - h(x)|\, dx.$$

If $||f_n - f|| \to 0$ as $n \to \infty$, then $f_n(x)$ is said to be *convergent in the mean* to $f(x)$. If

(i) $$||f_n|| \leqslant M, \text{ all } n$$

(ii) $$\int_a^x f_n(t)\, dt \to \int_a^x f(t)\, dt$$

for every $x \in [a, b]$, then $f_n(x)$ is said to be *weakly convergent* to $f(x)$ (with index 1).

It is known that convergence in the mean implies the weak convergence of $f_n(x)$ to $f(x)$ in the space $L(a, b)$.

We also define the space $L\{a, b\}$ as the class of functions integrable over every closed subinterval contained in the open interval $(a, b)$. Let $f(x) \in L\{a,b\}$. Define the operator

$$A_h^1 f(x) = \frac{1}{2h}\int_{-h}^{h} f(x + t)\, dt = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\, dt, \qquad a < x - h < x + h < b, \tag{1}$$

and in general,

$$A_h^{s+1} f(x) = A_h^1 [A_h^s f(x)].$$

These integral operators, or repeated average values, or integral means as they are sometimes called have been employed previously **(8** or **4)** and several of their properties necessary for our work will be recalled.

LEMMA 1. *If $f(x) \in L\{a, b\}$, the operator $A_h^s f(x)$ is continuous and has derivatives $[A_h^s f(x)]^{(i)}$ $(i = 1, 2, \ldots, s - 1)$, which are absolutely continuous and moreover,*

$$[A_h^s f(x)]^{(s)} = (2h)^{-s}\Delta_{2h}^s f(x) \qquad p.p. \text{ in } (a - sh, b + sh). \tag{2}$$

LEMMA 2. *If $f(x) \in L\{a, b\}$, then*

$$\lim_{h\to 0} A_h^s f(x) = f(x) \qquad (s = 1, 2, \ldots)$$

*p.p. on $(a, b)$.*

LEMMA 3. *If $f(x) \in L\{a, b\}$,*

$$\lim_{h\to 0}\int_\alpha^\beta |A_h^s f(x) - f(x)|\, dx = 0 \qquad (s = 1, 2, \ldots)$$

*for every closed subinterval $[\alpha, \beta]$ contained in $(a, b)$.*

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LEMMA 4.

$$A_{h_2}^{s_2}\, \Delta_{2h_1}^{s_1} f(x) = \Delta_{2h_1}^{s_1}\, A_{h_2}^{s_2} f(x),$$

$$a < x - s_1 h_1 - s_2 h_2 < x + s_1 h_1 + s_2 h_2 < b.$$

This relation follows readily from the linearity of the operator defined in (1) and since

$$\Delta_{2h}^{s} f(x) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} f[x + (s - 2j)\, h].$$

The following lemma will also be of use and for the proof, one may see (**10**, p. 73).

LEMMA 5. *If the sth derivative $f^{(s)}(x)$ exists at the point $x$, then*

$$\lim_{h \to 0} (2h)^{-s}\, \Delta_{2h}^{s} f(x) = f^{(s)}(x).$$

For the sake of brevity we put

$$\mathfrak{J}_c^s\, g(x) = \int_c^x dt_1 \int_c^{t_1} dt_2 \ldots \int_c^{t_{s-1}} g(t_s)\, dt_s;$$

$P_s(x)$ always denotes a polynomial in $x$ of degree not exceeding $s$.

**3. An integral-difference equation.** We shall now study an equation which connects the integral operators and the differences and in particular contains proposition $A$.

THEOREM 1. *Let $f(x)$ and $g(x)$ both $\in L\{a, b\}$. If, for every fixed $h$, $0 < h < (b - a)/2s$,*

$$\Delta_{2h}^{s} f(x) = (2h)^s A_h^s\, g(x) \tag{3}$$

*for almost every $x$ satisfying the inequality $a < x - sh < x + sh < b$, then there exists a $P_{s-1}$ such that*

$$f(x) = \mathfrak{J}_c^s\, g(x) + P_{s-1}(x) \tag{4}$$

*almost everywhere in $(a, b)$, with $a < c < b$.*

*Conversely, if the equality* (4) *is satisfied almost everywhere, then the relation* (3) *holds almost everywhere.*

*Proof.* To begin with, consider the equation

$$\Delta_{2h}^{s} f(x) = 0 \tag{5}$$

holding for every fixed $0 < h < (b - a)/2s$ and for almost every $x \in (a + sh, b - sh)$.

To solve (5), consider first the case $f(x) \in C^{(s)}\ (a, b)$. Then, by Lemma 5

$$\lim_{h \to 0} (2h)^{-s} \Delta_{2h}^{s} f(x) = f^{(s)}(x), \qquad a < x < b,$$

which implies that $f^{(s)}(x) \equiv 0$ $(a < x < b)$, and consequently $f(x) = P_{s-1}(x)$ $(a < x < b)$.

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Secondly, consider $f(x) \in L\{a, b\}$. Let $k$ be fixed, such that $0 < k < (b-a)/(2s+2)$. Applying the operator

$$A_k^{s+1}$$

to equation (5), by Lemma 4, we obtain

$$\Delta_{2h}^s A_k^{s+1} f(x) = 0$$

for every $h$ and $x$ such that

$$a + (s+1)k < x - sh < x + sh < b - (s+1)k.$$

Since

$$A_k^{s+1} f(x) \in C^{(s)},$$

we deduce from the first case that

$$(6) \qquad A_k^{s+1} f(x) = P_{s-1}(x;\, k), \qquad x \in (a + (s+1)k,\ b - (s+1)k),$$

where the polynomial $P_{s-1}(x; k)$ depends on $k$.

Let $[\alpha, \beta]$ be a closed subinterval of $(a, b)$. It is obvious that (6) implies that

$$A_{1/n}^{s+1} f(x) = P_{s-1}(x;\, 1/n)$$

for $\alpha \leqslant x \leqslant \beta$ and $n \geqslant N$, $N = N(\alpha, \beta)$. By Lemma 2,

$$A_{1/n}^{s+1} f(x)$$

approaches $f(x)$ p.p. in $[\alpha, \beta]$ as $n \to \infty$ and therefore $P_{s-1}(x; 1/n)$ must converge to a limit $P_{s-1}(x)$ p.p. in $[\alpha, \beta]$. This latter limit must be a polynomial of degree at most $s - 1$ for if a sequence of polynomials, the degree of each being at most $l$, converges for $l + 1$ different values of $x$, it converges for every value of $x$ and its limit is a polynomial of degree at most $l$. Consequently,

$$(7) \qquad f(x) = P_{s-1}(x) \qquad \text{p.p. in } [\alpha, \beta].$$

Since the relation (7) holds for every closed subinterval of $(a, b)$, it holds for $(a, b)$.

Let us now return to equation (3) with the conditions specified in the theorem. It can easily be established (by induction on $s$) that

$$(8) \qquad \Delta_{2h}^s \mathfrak{J}_c^s g(x) = (2h)^s A_h^s g(x),$$

and therefore the function

$$F(x) = f(x) - \mathfrak{J}_c^s g(x)$$

satisfies the equation (5) for $x \in (a + sh, b - sh)$. The theorem now follows readily. The converse is obvious.

In the particular case $f(x)$ is continuous in $(a, b)$ and $g(x)$ is zero we obtain proposition A. The known proofs of the former proposition (for references see §1) that the authors have seen appear to depend rather too heavily on intrinsic properties of the differences and thus perhaps cannot be applied to the type of problems we consider in this paper.

**4. The fundamental theorem.**

Put

(9) $$\sigma_n(x) = \tau_n(x) - g(x)$$

where

$$\tau_n(x) = (2h_n)^{-s}\Delta_{2h_n}^s f(x).$$

THEOREM 2. *Let $f(x)$ and $g(x) \in L\{a, b\}$. There exists a polynomial $P_{s-1}(x)$ such that*

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x)$$

*p.p. in $(a, b)$ with $a < c < b$, if and only if there exists a sequence $\{h_n\}$ of positive numbers converging to zero such that the sequence of functions $\{\tau_n(x)\}$ converges weakly to $g(x)$ in every closed subinterval $[\alpha, \beta]$ in $(a, b)$, in other words if the conditions*

$$\text{(i)} \quad \int_\alpha^\beta |\tau_n(x)|\, dx \leqslant M, \text{ all } n$$

$$\text{(ii)} \quad \int_\alpha^\beta \tau_n(x)\, dx \to \int_\alpha^\beta g(x)\, dx$$

*are satisfied in every $[\alpha, \beta]$ of $(a, b)$.*

*Proof.* We shall at first prove the sufficiency of our hypothesis.

Let $[\alpha, \beta]$ be an arbitrary subinterval in $(a, b)$ and let $h$ be a fixed number with $0 < h < (\beta - \alpha)/2s$. Applying the operator $A_h^s$ to both sides of the relation (9), for $x \in [\alpha + sh, \beta - sh]$ and sufficiently small $h_n$ we find that

(10) $$A_h^s \sigma_n(x) = (2h_n)^{-s}\Delta_{2h_n}^s A_h^s f(x) - A_h^s g(x)$$

where we have used Lemma 4 to invert the integral and difference operators of the first term on the right-hand side of this equality.

Now

$$A_h^1 \sigma_n(x) = \frac{1}{2h}\int_{x-h}^{x+h} \sigma_n(t)\, dt = \frac{1}{2h}\int_{x-h}^{x+h} [\tau_n(t) - g(t)]\, dt$$

and hence by (ii), $A_h^1 \sigma_n(x)$ converges to zero for $x \in [\alpha + h, \beta - h]$. But by (i),

$$|A_h^1 \sigma_n(x)| \leqslant \frac{1}{2h}\int_\alpha^\beta |\sigma_n(t)|\, dt \leqslant \frac{1}{2h}\left[M + \int_\alpha^\beta |g(t)|\, dt\right],$$

and so as $g(x) \in L\{a, b\}$,

$$A_h^1 \sigma_n(x)$$

converges dominatedly to zero for $x \in [\alpha + h, \beta - h]$. By Lebesgue's theorem on dominated convergence,

$$A_h^2 \sigma_n(x) = \frac{1}{2h}\int_{x-h}^{x+h} A_h^1 \sigma_n(t)\, dt$$

converges to zero for $x \in [\alpha + 2h, \beta - 2h]$.

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Repeating this argument $s - 2$ more times we finally find that

$$\lim_{n\to\infty} A_h^s \sigma_n(x) = 0 \tag{11}$$

for $x \in [\alpha + sh, \beta - sh]$.

On the other hand, by Lemmas 1 and 5, and the relation (2), we deduce

$$\begin{aligned} \lim_{n\to\infty} (2h_n)^{-s}\Delta_{2h_n}^s A_h^s f(x) &= [A_h^s f(x)]^{(s)} \\ &= (2h)^{-s}\Delta_{2h}^s f(x) \end{aligned} \tag{12}$$

for almost every $x$ in $[\alpha + sh, \beta - sh]$.

The relations (10), (11) and (12) show that

$$(2h)^{-s}\Delta_{2h}^s f(x) = A_h^s g(x)$$

for almost every $x \in [\alpha + sh, \beta - sh]$ and since $[\alpha, \beta]$ was an arbitrary closed subinterval of $(a, b)$, the last condition holds for every fixed $0 < h < (b - a)/2s$ and for almost every $x$ such that $a < x - sh < x + sh < b$. Applying Theorem 1, we deduce for almost every $x$ in $(a, b)$,

$$f(x) = \mathfrak{I}_c^s g(x) + P_{s-1}(x)$$

where $c$ is fixed with $a < c < b$.

To establish the converse, we note that

$$\int_\alpha^\beta |\tau_n(x) - g(x)|\, dx \leqslant \int_\alpha^\beta |\tau_n(x) - A_{h_n}^s g(x)|\, dx + \int_\alpha^\beta |A_{h_n}^s g(x) - g(x)|\, dx,$$

where the first term on the right-hand side is zero by (8) and the second approaches zero by Lemma 3. Hence $\tau_n(x)$ converges in the mean and therefore weakly converges to $g(x)$ in $[\alpha, \beta]$. The theorem is now complete.

It is obvious from the proof that in case $s = 1$ the hypothesis (i) of the above theorem is not necessary.

THEOREM 3. *Let $f(x)$ and $g(x) \in L\{a, b\}$. The existence of a sequence of positive numbers $\{h_n\}$ converging to zero such that the sequence of functions $\{\tau_n(x)\}$ converges in the mean to $g(x)$ in every $[\alpha, \beta]$ of $(a, b)$, is a necessary and sufficient condition in order that there exists a $P_{s-1}(x)$ with*

$$f(x) = \mathfrak{I}_c^s g(x) + P_{s-1}(x)$$

*for almost every $x$ in $(a, b)$ where $a < c < b$.*

Since mean convergence implies weak convergence, this theorem follows from the preceding.

**5. Riemann derivatives.** We now wish to express the previous theorem more directly in terms of the Riemann derivatives in a form, which, though weaker, can easily be recognized.

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THEOREM 4. *Let* $f(x) \in L\{a, b\}$. *If*

(i) *there exists a sequence of positive numbers* $\{h_n\}$ *converging to zero such that*

$$\lim_{n\to\infty} \tau_n(x) = g(x) \qquad p.p. \text{ in } (a, b),$$

(ii) *there exists a function* $\tau(x) \in L\{a, b\}$ *such that*

$$\sup_{n>0} |\tau_n(x)| \leqslant \tau(x), \qquad a < x - sh_n < x + sh_n < b,$$

*then there exists a* $P_{s-1}(x)$ *such that*

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x)$$

*p.p. in* $(a, b)$, *where* $a < c < b$.

*Proof.* By Lebesgue's theorem on dominated convergence, we obtain $g(x) \in L\{a, b\}$ and

$$\int_\alpha^\beta |\tau_n(x) - g(x)|\, dx \to 0, \qquad n \to \infty,$$

for every $[\alpha, \beta]$ in $(a, b)$. The theorem now follows from the above.

In the particular case of Theorem 4, when $f(x)$ is continuous, conditions (i) and (ii) remaining unaltered, it follows immediately that for every $x$ in $(a, b)$,

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x).$$

Theorems 2, 3 or 4 may be considered as certain types of extensions of proposition B of §1 on the second Riemann derivatives to those of higher order, but we must note that the convergence conditions are somewhat different. The direct generalization (in case of an open interval) would be the following:

C. *If* $f(x)$ *is continuous in* $(a, b)$ *and* $f^{(s-2)}(x)$ *exists everywhere in* $(a, b)$, $f(x)$ *has a finite sth Riemann derivative* $g(x)$, *with* $g(x) \in L\{a, b\}$, *then for* $a < x < b$

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x).$$

For $s = 3, 4$, this result is known (**9** or **11**). It is conjectured that the result would hold for $s \geqslant 5$. That one must assume the existence of $f^{(s-2)}(x)$ for $s \geqslant 3$ even in the case $g(x) = 0$ can be seen from the following counter-example:

$$f(x) = |x| x^{s-3}.$$

In fact, the first $s - 3$ ordinary derivatives of this function exist, but the $(s - 2)$nd ordinary derivative does not exist at $x = 0$, while the Riemann $s$th derivative is everywhere zero.

The importance of proposition B lies in the fact that it is used in proving the result that if a trigonometrical series converges, except in an enumerable set, to a finite and integrable function $g(x)$, then it is the Fourier series of $g(x)$ (**12**, p. 274).

**6. Related theorems.** We shall now state several corollaries to our theorems.

COROLLARY 1. *If $f(x) \in L\{a, b\}$ and $\tau_n(x)$ converges boundedly to $g(x)$ in every closed $[\alpha, \beta]$ of $(a, b)$, then for almost every $x$ in $(a, b)$,*

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x).$$

COROLLARY 2. *If $f(x)$ is continuous in $(a, b)$ and if*

$$\lim_{h \to 0} (2h)^{-s} \Delta_{2h}^s f(x) = g(x)$$

*uniformly in every $[\alpha, \beta]$ of $(a, b)$, then for every $x$ in $(a, b)$*

$$f(x) = \mathfrak{J}_c^s g(x) + P_{s-1}(x).$$

This corollary was previously established by Marchaud (**5**) and in the case $g(x) = 0$ by Anghelutza (**2**).

COROLLARY 3. *If $f(x) \in L\{a, b\}$ and*

$$\lim_{h \to 0} (2h)^{-s} \int_\alpha^\beta |\Delta_{2h}^s f(x)| \, dx = 0$$

*for every $[\alpha, \beta]$ in $(a, b)$, then*

$$f(x) = P_{s-1}(x)$$

*p.p. in $(a, b)$.*

This result is due to Reid (**8**), who used it to obtain integral criteria for a function to be p.p. equal to a solution of a linear differential equation.

Our final result concerns the class of functions defined in $(a, b)$, every one of whose members can be represented in every $[\alpha, \beta]$ of $(a, b)$ as the difference of two non-concave functions of order $l$. This class, which will be denoted by $DC^l\{a, b\}$, is connected with the class of functions of $l$th generalized bounded variation (**7**, p. 24).

At first we recall the definition of non-concave functions in general. A function $f(x)$ is said to be *non-concave of order $l$* in $(a, b)$, if it is continuous in $(a, b)$ and if for $a < x - lh < x + lh < b$,

(a) $$\Delta_{2h}^{l+1} f(x) \geqslant 0.$$

If $f(x)$ is non-concave of order $l$, it is known (**7**, pp. 48, 25) that

(b) $$\Delta_{2h}^l f(x) = O(h^l)$$

uniformly for $x$ in every $[\alpha, \beta]$ of $(a, b)$.

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THEOREM 5. *Let $f(x) \in L\{a, b\}$ and $s \geqslant 2$. The necessary and sufficient conditions that there exists a $P_{s-1}(x)$ with*

$$f(x) = P_{s-1}(x) \qquad p.p. \text{ in } (a, b),$$

*are*

(i) *$f(x)$ is p.p. in $(a, b)$ equal to a function $\phi(x) \in DC^{s-1}\{a, b\}$,*

(ii) $$\int_\alpha^\beta \Delta_{2h}^s f(x) = o(h^s), \qquad h \to 0,$$

*for every $[\alpha, \beta]$ of $(a, b)$.*

*Proof.* The necessity is obvious.

To prove the sufficiency, according to Theorem 2 (the case $g(x) = 0$) we only need to show that the condition (i) implies that

$$\int_\alpha^\beta |\Delta_{2h}^s f(x)|\, dx = O(h^s)$$

for every $[\alpha, \beta]$ of $(a, b)$.

It follows from the hypothesis (i) that $f(x) = \phi(x)$ p.p. in $(a, b)$ where $\phi(x)$ can be represented in $[\alpha, \beta]$ in the form

$$\phi(x) = \phi_1(x) - \phi_2(x)$$

where $\phi_1(x)$ and $\phi_2(x)$ satisfy (a) and (b) for $l = s - 1$.

We have

$$\begin{aligned}\int_\alpha^\beta |\Delta_{2h}^s f(x)|\, dx &= \int_\alpha^\beta |\Delta_{2h}^s\, \phi(x)|\, dx \leqslant \int_\alpha^\beta \Delta_{2h}^s\, \phi_1(x)\, dx + \int_\alpha^\beta \Delta_{2h}^s\, \phi_2(x) \\ &= \int_\alpha^\beta [\Delta_{2h}^{s-1}\, \phi_1(x+h) - \Delta_{2h}^{s-1}\, \phi_1(x-h)]\, dx \\ &\qquad + \int_\alpha^\beta [\Delta_{2h}^{s-1}\, \phi_2(x+h) - \Delta_{2h}^{s-1}\, \phi_2(x-h)]\, dx \\ &= \int_{\beta-h}^{\beta+h} \Delta_{2h}^{s-1}\, \phi_1(t)\, dt - \int_{\alpha-h}^{\alpha+h} \Delta_{2h}^{s-1}\, \phi_1(t)\, dt \\ &\qquad + \int_{\beta-h}^{\beta+h} \Delta_{2h}^{s-1}\, \phi_2(t)\, dt - \int_{\alpha-h}^{\alpha+h} \Delta_{2h}^{s-1}\, \phi_2(t)\, dt \\ &= O(h^s).\end{aligned}$$

The theorem is now established.

The authors believe no previous attention has been given to results of this type, showing a relation between the class $DC^s\{a, b\}$ and polynomials of degree $s$.

Finally we would like to add that results corresponding to every one of the above theorems may be established for the forward and also the backward differences.

## References

**1.** Th. Anghelutza, *Sur une équation fonctionelle caractérisant les polynomes*, Mathematica (Cluj), *6* (1932), 1–7.

**2.** ———, *Sur une propriété des polynomes*, Bull. Sci. Math. (2), *63* (1939), 239–46.

**3.** L. E. J. Brouwer, *Over differentiequotienten en differentialquotienten*, Verh. Nederl, Akad. Wetensch. Afd. Natuurk. (Amsterdam), *17* (1908), 38–45.

**4.** S. Mandelbrojt, *Analytic functions and classes of infinitely differentiable functions*, Rice Institute Pamphlet, no. 1 *29* (1942).

**5.** A. Marchaud, *Sur les dérivées et sur les différences des fonctions de variable réelles*, J. de Math. (9), *6* (1927), 337–425.

**6.** T. Popoviciu, *Sur les solutions bornées et les solutions mesurables de certaines équations fonctionelles*, Mathematica (Cluj), *14* (1938), 47–106.

**7.** ———, *Les fonctions convexes* (Paris, 1945).

**8.** W. T. Reid, *Integral criteria for solutions of linear differential equations*, Duke Math. J., *12* (1945), 685–694.

**9.** S. Saks, *On the generalized derivatives*, J. Lond. Math. Soc., *7* (1932), 247–251.

**10.** Ch. de la Vallée-Poussin, *Cours d'analyse infinitesimale* (New York, 1946).

**11.** S. Verblunsky, *The generalized fourth derivative*, J. Lond. Math. Soc., *6* (1931), 82–84.

**12.** A. Zygmund, *Trigonometrical series* (Warszawa, 1935).

*McGill University*