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A DUALITY RELATION BETWEEN THE WORKLOAD AND ATTAINED WAITING TIME IN FCFS G/G/s QUEUES

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Abstract

Sengupta (1989) showed that, for the first-come–first-served (FCFS) G/G/1 queue, the workload and attained waiting time of a customer in service have the same stationary distribution. Sakasegawa and Wolff (1990) derived a sample path version of this result, showing that the empirical distribution of the workload values over a busy period of a given sample path is identical to that of the attained waiting time values over the same period. For a given sample path of an FCFS G/G/s queue, we construct a dual sample path of a dual queue which is FCFS G/G/s in reverse time. It is shown that the workload process on the original sample path is identical to the total attained waiting time process on the dual sample path. As an application of this duality relation, we show that, for a time-stationary FCFS M/M/s/k queue, the workload process is equal in distribution to the time-reversed total attained waiting time process.

Keywords: Duality; workload process; sample path; reverse time

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Sengupta [3] showed that, for the first-come–first-served (FCFS) G/G/1 queue, the workload and attained waiting time of a customer in service have the same stationary distribution. Sakasegawa and Wolff [2] derived a sample path version of this result, showing that the empirical distribution of the workload values over a busy period of a given sample path is identical to that of the attained waiting time values over the same period. See also [1] for a different proof and [4] for related results. For a general multiserver queueing system, Yamazaki and Miyazawa [5] established that the workload and total attained waiting time of customers in service are identical in average, but their distributions are unequal in general. We complement this result by deriving a duality relation between the workload process and the total attained waiting time process (in reverse time) in FCFS G/G/s queues. As an application of this duality relation, we show that, for a time-stationary FCFS M/M/s/k queue, the workload process is equal in distribution to the time-reversed total attained waiting time process.

Specifically, consider a sample path ω of an FCFS G/G/s queue. We will construct a corresponding sample path ω^* of a dual queue which is FCFS G/G/s in reverse time, and show that the workload process on ω is identical to the total attained waiting time process on ω^*. Let $[\underline{t}, \bar{t}]$ be a busy period of ω during which n customers $C_1, \dots, C_n$ enter the system with corresponding arrival and departure times, $A_i < D_i$, $i = 1, \dots, n$, where $\underline{t} = A_1 < A_2 < \cdots < A_n < \bar{t}$ (implying no batch arrivals). We also assume that the departure times $D_1, \dots, D_n$ are all distinct. Let π be the permutation of $(1\,2 \cdots n)$ such that

$$\underline{t} < D_{\pi(1)} < D_{\pi(2)} < \cdots < D_{\pi(n)} = \bar{t}. \tag{1}$$

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To be consistent with the assumption of an FCFS service discipline, π needs to satisfy the requirement that

$$\pi(i) < s + i \quad \text{for all } 1 \le i \le n. \tag{2}$$

In words, the ith departing customer must be one of $C_1, \dots, C_{s+i-1}$. Note that (2) is equivalent to

$$j > \pi(i) \quad \text{for all } 1 \le i, j \le n \text{ with } i \le j - s. \tag{3}$$

The service time of each customer can be determined as follows. Each of $C_1, \dots, C_s$ begins service upon arrival, so that C_i's service time $S_i = D_i - A_i$, $i = 1, \dots, s$. Customer C_{s+1} begins service either upon arrival if at least one of $C_1, \dots, C_s$ has already completed service, or at the first departure time $D_{\pi(1)}$ otherwise. More generally, C_{s+i} begins service at $\max\{A_{s+i}, D_{\pi(i)}\}$, so that

$$S_{s+i} = D_{s+i} - \max\{A_{s+i}, D_{\pi(i)}\}, \qquad i = 1, \dots, n - s. \tag{4}$$

The workload $V(t)$ at time $t \in [\underline{t}, \bar{t}]$ is the total remaining service time of customers in the system at time t (which is also known as the virtual waiting time for the case $s = 1$). The arrival and departure times, A_i, $i = 2, \dots, n$ and $D_{\pi(i)}$, $i = 1, \dots, n-1$, divide the interval $[\underline{t}, \bar{t}] = [A_1, D_{\pi(n)}]$ into $2n - 1$ subintervals. In each (open) subinterval, $V(t)$ is linear with a slope belonging to $\{-1, \dots, -s\}$. (More precisely, the absolute value of the slope equals the smaller of s and the number of customers in the system during this subperiod.) Let $L(t) = |\{i : A_i < t < D_i\}|$, the number of customers in the system at $t \notin \{A_j, D_j, j = 1, \dots, n\}$, which is constant in each of the $2n - 1$ (open) subintervals. The following lemma can be proved easily.

Lemma 1. *Assume that $n \ge 2$. Let (t_1, t_2) and (t_2, t_3) be two consecutive subintervals ($t_1 < t_2 < t_3$).*

(i) *If $t_2 = D_i$ for some i then $V(t)$ is continuous at t_2 and $V'(t_2+) - V'(t_2-) = 0$ or 1 according to whether $L(t_2-)$ is greater than s or less than or equal to s.*

(ii) *If $t_2 = A_i$ for some i then $V(t_2+) - V(t_2-) = S_i$ (the service time of C_i) and $V'(t_2+) - V'(t_2-) = -1$ or 0 according to whether $L(t_2-)$ is less than s or greater than or equal to s.*

(iii) *$V(t) = \bar{t} - t$ in the last subinterval $(D_{\pi(n-1)}, \bar{t})$.*

Let $W_{a,i}(t)$ denote the attained waiting time of C_i at t, which is defined as $t - A_i$ if C_i is in service at t or 0 otherwise. Let $W_a(t) = \sum_{i=1}^{n} W_{a,i}(t)$, the total attained waiting time of all customers in service at t, which is linear in each of the $2n - 1$ subintervals with a slope belonging to $\{1, 2, \dots, s\}$ (depending on the number of customers in service). The following lemma can also be proved easily.

Lemma 2. *Assume that $n \ge 2$. Let (t_1, t_2) and (t_2, t_3) be two consecutive subintervals ($t_1 < t_2 < t_3$).*

(i) *Suppose that $t_2 = A_i$ for some i. Then $W_a(t)$ is continuous at t_2 and $W_a'(t_2+) - W_a'(t_2-) = 1$ or 0 according to whether $L(t_2-)$ is less than s or greater than or equal to s.*

(ii) *Suppose that $t_2 = D_i$ for some i. If $L(t_2-) > s$, some customer C_j begins service at t_2, and then $W_a(t_2+) - W_a(t_2-) = W_{a,j}(t_2+) - W_{a,i}(t_2-) = A_i - A_j$ and $W_a'(t_2+) - W_a'(t_2-) = 0$. If $L(t_2-) \le s$ then $W_a(t_2+) - W_a(t_2-) = -W_{a,i}(t_2-) = A_i - D_i = A_i - t_2$ and $W_a'(t_2+) - W_a'(t_2-) = -1$.*

(iii) *$W_a(t) = t - \underline{t}$ in the first subinterval $(\underline{t}, A_2)$.*

We now construct a (time-reversed) sample path ω^* of a dual FCFS G/G/s queue in reverse time which has the same busy periods as ω. In the busy period $[\underline{t}, \bar{t}]$ of ω^*, n customers $C_1^*, \dots, C_n^*$ enter the system with corresponding arrival and departure times $A_i^* > D_i^*$, $i = 1, \dots, n$, where $A_i^* = D_i$ and $D_i^* = A_i$. Since $\underline{t} < D_{\pi(1)} < \cdots < D_{\pi(n)} = \bar{t}$, we have $\underline{t} < A_{\pi(1)}^* < \cdots < A_{\pi(n)}^* = \bar{t}$, implying that the n customers arrive in the order $C_{\pi(n)}^*, C_{\pi(n-1)}^*, \dots, C_{\pi(1)}^*$ (since these customers arrive in reverse time order). Since $\underline{t} = A_1 < A_2 < \cdots < A_n < \bar{t}$, we have $\underline{t} = D_1^* < D_2^* < \cdots < D_n^* < \bar{t}$, implying that the n customers leave the system in the order $C_n^*, C_{n-1}^*, \dots, C_1^*$. By (3), we have $j \in \{\pi(n), \pi(n-1), \dots, \pi(j-s+1)\}$, i.e. the $(n-j+1)$th departing customer must be one of the first $n-j+s$ arriving customers, implying that the arrival and departure times A_i^*, D_i^*, $i = 1, \dots, n$, are consistent with the requirement of FCFS service discipline. (Note that the jth arriving customer in forward time corresponds to the $(n-j+1)$th departing customer in reverse time.) So ω^* is a valid sample path of a (dual) FCFS G/G/s queue in reverse time.

Let $W_{a,i}^*(t)$ be the attained waiting time of C_i^* at t, i.e. $W_{a,i}^*(t) = A_i^* - t$ if C_i^* is in service at t and 0 otherwise. Let $W_a^*(t) = \sum_{i=1}^n W_{a,i}^*(t)$, the total attained waiting of all customers in service at t for ω^*. Note that the $2n-2$ points in

$$\{A_i^*, D_i^*, i = 1, \dots, n\} \setminus \{D_1^*, A_{\pi(n)}^*\} = \{A_i, D_i, i = 1, \dots, n\} \setminus \{A_1, D_{\pi(n)}\}$$

divide $[\underline{t}, \bar{t}]$ into $2n-1$ subintervals. In each (open) subinterval, $W_a^*(t)$ is linear with a slope belonging to $\{-1, \dots, -s\}$. Since the definitions of V, W_a, and W_a^* at arrival/departure times may be ambiguous, we take the convention that V and W_a^* are right continuous and W_a is left continuous. Note also that

$$L(t) = |\{i : A_i < t < D_i\}| = |\{i : D_i^* < t < A_i^*\}|$$

so that the number of customers in the system at t for ω is the same as that for ω^*.

Lemma 2 is stated in terms of ω. To facilitate the proof of Theorem 1 below, it is convenient to rewrite Lemma 2 in terms of ω^* as follows.

Lemma 2′. *Assume that $n \ge 2$. Let (t_1, t_2) and (t_2, t_3) be two consecutive subintervals $(t_1 < t_2 < t_3)$.*

(i) *Suppose that $t_2 = A_i^*$ for some i. Then $W_a^*(t)$ is continuous at t_2 and $W_a^{*\prime}(t_2-) - W_a^{*\prime}(t_2+) = -1$ or 0 according to whether $L(t_2+)$ is less than s or greater than or equal to s.*

(ii) *Suppose that $t_2 = D_i^*$ for some i. If $L(t_2+) > s$, some customer C_j^* begins service at t_2, and then $W_a^*(t_2-) - W_a^*(t_2+) = W_{a,j}^*(t_2-) - W_{a,i}^*(t_2+) = (A_j^* - t_2) - (A_i^* - t_2) = A_j^* - A_i^*$ and $W_a^{*\prime}(t_2-) - W_a^{*\prime}(t_2+) = 0$. If $L(t_2+) \le s$ then $W_a^*(t_2-) - W_a^*(t_2+) = -W_{a,i}^*(t_2+) = t_2 - A_i^* = D_i^* - A_i^*$ and $W_a^{*\prime}(t_2-) - W_a^{*\prime}(t_2+) = 1$.*

(iii) *$W_a^*(t) = \bar{t} - t$ in the 'first' subinterval $(A_{\pi(n-1)}^*, \bar{t})$.*

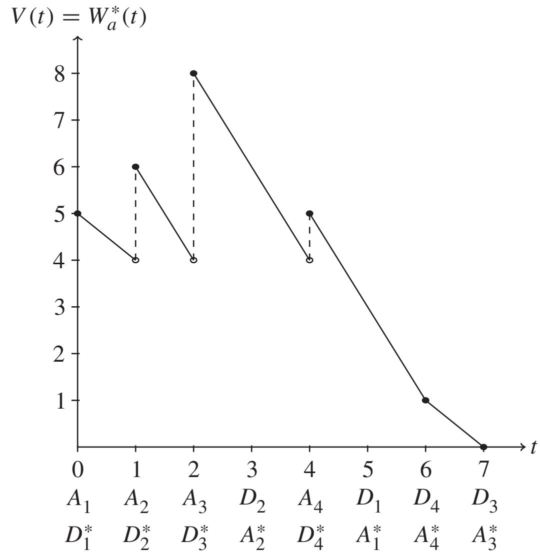

FIGURE 1.

Theorem 1. *In the busy period* $[\underline{t}, \bar{t}]$, *we have* $V(t) = W_a^*(t)$ *for all* t.

Before proving the theorem, we illustrate the result with Figure 1. We assume that there are two servers ($s = 2$) and that four customers enter the system during the busy period $[\underline{t}, \bar{t}] = [0, 7]$ with

$$\begin{aligned} &A_1 = D_1^* = 0, \qquad A_2 = D_2^* = 1, \qquad A_3 = D_3^* = 2, \qquad A_4 = D_4^* = 4,\\ &D_1 = A_1^* = 5, \qquad D_2 = A_2^* = 3, \qquad D_3 = A_3^* = 7, \qquad D_4 = A_4^* = 6. \end{aligned}$$

Note that this implies that

$$\begin{aligned} S_1 = D_1 - A_1 = 5, \qquad S_2 = D_2 - A_2 = 2,\\ S_3 = D_3 - D_2 = 4, \qquad S_4 = D_4 - D_1 = 1, \end{aligned}$$

and that customers $C_3^*, C_4^*, C_1^*, C_2^*$ enter the system at times 7, 6, 5, 3 (in reverse time) and begin service at times 7, 6, 4, 2, respectively.

Proof of Theorem 1. The case $n = 1$ is trivial. Assume that $n \geq 2$. The $2n - 2$ points in $\{A_i, D_i, i = 1, \dots, n\} \setminus \{A_1, D_{\pi(n)}\} = \{A_i^*, D_i^*, i = 1, \dots, n\} \setminus \{D_1^*, A_{\pi(n)}^*\}$ divide $[\underline{t}, \bar{t}]$ into $2n - 1$ subintervals. In the subinterval $(D_{\pi(n-1)}, \bar{t}) = (A_{\pi(n-1)}^*, \bar{t})$, which is the last subinterval for ω and the 'first' subinterval for ω^*, we have $V(t) = \bar{t} - t = W_a^*(t)$ by Lemmas 1(iii) and 2′(iii). Consider two consecutive subintervals (t_1, t_2) and (t_2, t_3), $t_1 < t_2 < t_3$. Suppose that $V(t) = W_a^*(t)$ for $t \in (t_2, t_3)$. We will show that $V(t) = W_a^*(t)$ for $t \in (t_1, t_2)$, which by induction yields $V(t) = W_a^*(t)$ in each of the $2n - 1$ open subintervals. It then follows from the right continuity of V and W_a^* that $V(t) = W_a^*(t)$ for $t \in [\underline{t}, \bar{t}]$.

It remains to show that $V(t) = W_a^*(t)$ for $t \in (t_1, t_2)$. We need to consider the following cases separately.

Case (i). Suppose that $t_2 = D_i = A_i^*$ for some i. By Lemmas 1(i) and 2′(i), we have $V(t_2-) = V(t_2+) = W_a^*(t_2+) = W_a^*(t_2-)$, where the second equality follows from the

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induction hypothesis that $V(t) = W_a^*(t)$ for $t \in (t_2, t_3)$. We consider the following two subcases.

Subcase (i.1). Suppose that $L(t_2-) \le s$ (implying that $L(t_2+) < s$). By Lemmas 1(i) and 2′(i), $V'(t_2+) - V'(t_2-) = 1$ and $W_a^{*\prime}(t_2-) - W_a^{*\prime}(t_2+) = -1$, implying that $V'(t_2-) = W_a^{*\prime}(t_2-)$ since $V'(t_2+) = W_a^{*\prime}(t_2+)$ by the induction hypothesis. It follows from the linearity of $V(t)$ and $W_a^*(t)$ in (t_1, t_2) that $V(t) = W_a^*(t)$ for $t \in (t_1, t_2)$.

Subcase (i.2). Suppose that $L(t_2-) > s$ (implying that $L(t_2+) \ge s$). By Lemmas 1(i) and 2′(i) and the induction hypothesis, $V_a'(t_2-) = V_a'(t_2+) = W_a^{*\prime}(t_2+) = W_a^{*\prime}(t_2-)$, implying that $V(t) = W_a^*(t)$ for $t \in (t_1, t_2)$.

Case (ii). Suppose that $t_2 = A_i = D_i^*$ for some i. Then $V(t_2+) - V(t_2-) = S_i$, the service time of C_i. We consider the following two subcases.

Subcase (ii.1). Suppose that $L(t_2-) < s$ (implying that $L(t_2+) \le s$). Then C_i begins service upon arrival, so $S_i = D_i - A_i$ and $V(t_2+) - V(t_2-) = D_i - A_i$. Also, $W_a^*(t_2+) - W_a^*(t_2-) = W_{a,i}^*(t_2+) = A_i^* - D_i^* = D_i - A_i$, implying that $V(t_2-) = W_a^*(t_2-)$. Furthermore, by Lemmas 1(ii) and 2′(ii), $V'(t_2+) - V'(t_2-) = -1$ and $W_a^{*\prime}(t_2-) - W_a^{*\prime}(t_2+) = 1$, implying that $V'(t_2-) = W_a^{*\prime}(t_2-)$ since $V'(t_2+) = W_a^{*\prime}(t_2+)$ by the induction hypothesis. So $V(t) = W_a^*(t)$ for $t \in (t_1, t_2)$.

Subcase (ii.2). Suppose that $L(t_2-) \ge s$ (implying that $L(t_2+) > s$). By (4) we have

$$V(t_2+) - V(t_2-) = S_i = D_i - \max\{A_i, D_{\pi(i-s)}\} = D_i - D_{\pi(i-s)}, \tag{5}$$

since C_i begins service at $\max\{A_i, D_{\pi(i-s)}\} = D_{\pi(i-s)}$. As for ω^*, C_i^* completes service at $t_2 = D_i^*$ and another customer begins service at t_2 since $L(t_2+) > s$. By time $t_2 = D_i^*(= A_i)$, the number of customers who have completed service is $n - i + 1$, since the departure times before t_2 (in reverse time) are $D_{i+1}^* = A_{i+1}, \dots, D_n^* = A_n$. Thus, the customer who begins service at t_2 must be the $(n - i + 1 + s)$th customer entering the system, i.e. $C_{\pi(i-s)}^*$. So

$$\begin{aligned} W_a^*(t_2-) - W_a^*(t_2+) &= W_{a,\pi(i-s)}^*(t_2-) - W_{a,i}^*(t_2+) \\ &= (A_{\pi(i-s)}^* - t_2) - (A_i^* - t_2) \\ &= D_{\pi(i-s)} - D_i, \end{aligned}$$

which, together with (5) and the induction hypothesis, yields $V(t_2-) = W_a^*(t_2-)$. Finally, by Lemmas 1(ii) and 2′(ii) and the induction hypothesis, $V'(t_2-) = V'(t_2+) = W_a^{*\prime}(t_2+) = W_a^{*\prime}(t_2-)$, implying that $V(t) = W_a^*(t)$ for $t \in (t_1, t_2)$.

This completes the proof.

Remark 1. In the arguments above, we have implicitly assumed that $A_j \ne D_i$ for all i, j. This restriction can easily be relaxed as follows. Let $I = \{i : A_i = D_j \text{ for some } j\}$. For (sufficiently small) $\varepsilon > 0$, define $A_{k,\varepsilon} = A_k - \varepsilon \mathbf{1}_I(k)$ and $D_{k,\varepsilon} = D_k$, $k = 1, \dots, n$, where $\mathbf{1}_I(k) = 1$ or 0 according to whether $k \in I$ or $k \notin I$. Then, by applying Theorem 1 to the arrival and departure times $\{A_{k,\varepsilon}, D_{k,\varepsilon}, k = 1, \dots, n\}$, we have $V_\varepsilon(t) = W_{a,\varepsilon}^*(t)$ for $t \in [\underline{t}, \overline{t}]$. Letting $\varepsilon \searrow 0$ yields $V(t) = W_a^*(t)$.

Remark 2. A stochastic FCFS G/G/s queueing system gives rise to a probability measure on the space of sample paths ω (endowed with a suitably specified σ-field), which induces a probability measure on the space of time-reversed sample paths ω^*, which in turn defines a dual FCFS G/G/s queue in reverse time. We may also think of both queues as coupled (i.e. defined on the same probability space). Then, by Theorem 1, $V(t) = W_a^*(t)$ for all t with probability 1, where V is the workload process for the original queue and W_a^* is the total attained waiting time process for the dual queue. It would be of interest to characterize the dual queues for some classes of queueing systems. For an FCFS M/M/s/k queue (k being the system capacity), it can readily be argued that the dual queue is also FCFS M/M/s/k in reverse time, which together with Theorem 1 implies that the workload process is equal in distribution to the total attained waiting process in reverse time. This is summarized in the following theorem.

Theorem 2. *For a time-stationary FCFS M/M/s/k queue with $1 \le s \le k \le \infty$, the workload process $V(t)$ is equal in distribution to the time-reversed total attained waiting time process $W_a(t)$, i.e.*

$$(V(t_1), \dots, V(t_k)) \stackrel{\mathrm{D}}{=} (W_a(-t_1), \dots, W_a(-t_k)) \quad \textit{for all } t_1, \dots, t_k,\ k \ge 1.$$

Consequently, $V(t)$ and $W_a(t)$ have the same stationary distribution and the same autocovariance function, i.e. $\operatorname{cov}(V(0), V(t)) = \operatorname{cov}(W_a(0), W_a(t))$ *for all t.*

Proof. For a time-stationary FCFS M/M/s/k queue, let $L(t)$ denote the queue length (number of customers in the system) at time t. As a birth-and-death process, $L(t)$ is time reversible. Let $L(\cdot) = \{L(t)\colon -\infty < t < \infty\}$, which is a random sample path such that the value of $L(\cdot)$ at time t is $L(t)$. Fix a realization ω_L of $L(\cdot)$. Let $[\underline{t}, \bar{t}]$ be a busy period of ω_L during which n customers $C_1, \dots, C_n$ enter the system with corresponding arrival times $\underline{t} = A_1 < A_2 < \cdots < A_n < \bar{t}$. Denote the ordered departure times by $\underline{t} < \widetilde{D}_1 < \widetilde{D}_2 < \cdots < \widetilde{D}_n = \bar{t}$. Necessarily, $|\{i : A_i < \widetilde{D}_j\}| > j$ for $j < n$. While the information of the A_is and $\widetilde{D}_i$s is contained in ω_L, the departure time D_i of C_i may not be observable given ω_L. Since, by (1), the (unobservable) permutation π satisfies $D_{\pi(1)} < \cdots < D_{\pi(n)}$, we have $D_{\pi(i)} = \widetilde{D}_i$ for all i, i.e. $D_i = \widetilde{D}_{\pi^{-1}(i)}$ for all i. A permutation ϕ of $(1\,2 \cdots n)$ is said to be an admissible matching if the paired arrival and departure times $(A_i, \widetilde{D}_{\phi^{-1}(i)})$, $i = 1, \dots, n$, satisfy the requirement of FCFS service discipline (cf. (2)). Since $(A_i, \widetilde{D}_{\pi^{-1}(i)}) = (A_i, D_i)$, the (unobservable) permutation π is (necessarily) admissible, which will be referred to as the true matching. Note that the identity permutation is always an admissible matching. It is not difficult to show that

(i) given $L(\cdot) = \omega_L$, the true matching π for the busy period $[\underline{t}, \bar{t}]$ is (*conditionally*) equally likely to be any one of the admissible matchings,

(ii) given $L(\cdot) = \omega_L$, the true matchings in different busy periods are (*conditionally*) independent.

(See Remark 3 below for further discussion.)

While ω_L consists only of information about the arrival and departure times, let ω be an expanded sample path version of ω_L which also includes the information of paired arrival and departure times. More precisely, write $\omega = (\omega_L, \Phi(\omega_L))$, where the second component $\Phi(\omega_L)$ (a rule matching arrival times with departure times) specifies an admissible matching in each of the busy periods of ω_L. Let $\omega^* = (\omega_L, \Phi(\omega_L))^* = (\omega_L^*, \Phi(\omega_L^*))$ be the corresponding (time-reversed) sample path of a dual queue as defined earlier, where ω_L^* is ω_L with the roles of arrival times and departure times interchanged, and the matching rule $\Phi(\omega_L^*)$ is such that an arrival time t' is matched with a departure time t'' under $\Phi(\omega_L^*)$ if and only if t' (departure time) and t'' (arrival time) are matched under $\Phi(\omega_L)$. Furthermore, let $L_\Pi = (L(\cdot), \Pi(L(\cdot)))$ be the random (expanded) sample path, where the second component $\Pi(L(\cdot))$ specifies the (unobservable) true matching in each of the busy periods of $L(\cdot)$. By (i) and (ii), it is instructive to think of the second component $\Pi(L(\cdot))$ as if it picks an admissible matching *at random* in each of the busy periods of $L(\cdot)$. Consequently, for the random time-reversed (expanded) sample path $L_\Pi^* = (L(\cdot), \Pi(L(\cdot)))^* = (L^*(\cdot), \Pi(L^*(\cdot)))$, we may also think of the second component $\Pi(L^*(\cdot))$ as if it picks an admissible matching *at random* in each of the busy periods of the first component $L^*(\cdot)$. Since $L(\cdot)$ is time reversible (i.e. $L^*(\cdot)$ is a time-reversed copy of $L(\cdot)$), it follows that the random (expanded) sample path L_Π^* is a time-reversed copy of the random (expanded) sample path L_Π.

Now let $V(\cdot)$ be the workload process on the random (expanded) sample path L_Π, and let $W_a(\cdot)$ and $W_a^*(\cdot)$ be the total attained waiting time processes on L_Π and L_Π^*, respectively. Since L_Π^* is a time-reversed copy of L_Π, $W_a^*(\cdot)$ is a time-reversed copy of $W_a(\cdot)$, i.e. for all $t_1, \dots, t_k, u_1, \dots, u_k$, $k \geq 1$,

$$\mathbb{P}(W_a^*(t_i) \leq u_i,\ i = 1, \dots, k) = \mathbb{P}(W_a(-t_i) \leq u_i,\ i = 1, \dots, k).$$

By Theorem 1,

$$\mathbb{P}(V(t_i) \leq u_i,\ i = 1, \dots, k) = \mathbb{P}(W_a^*(t_i) \leq u_i,\ i = 1, \dots, k),$$

showing that $(V(t_1), \dots, V(t_k))$ and $(W_a(-t_1), \dots, W_a(-t_k))$ have the same joint distribution. This completes the proof.

Remark 3. Denote by $(a_1\, d_1\, \cdots\, a_r\, d_r)$ the arrival–departure run pattern determined by the (ordered) arrival and depature times, A_i and $\widetilde{D}_i$, in the busy period $[\underline{t}, \bar{t}]$, i.e.

$$\begin{aligned} A_1 < \cdots < A_{a_1} < \widetilde{D}_1 < \cdots < \widetilde{D}_{d_1} < A_{a_1+1} < \cdots < A_{a_1+a_2} < \widetilde{D}_{d_1+1} < \cdots \\ < \widetilde{D}_{d_1+d_2} < \cdots < A_{a_1+\cdots+a_{r-1}+1} < \cdots < A_n < \widetilde{D}_{d_1+\cdots+d_{r-1}+1} < \cdots < \widetilde{D}_n. \end{aligned}$$

The a_is and d_is need to satisfy

$$\sum_{i=1}^{j} a_i > \sum_{i=1}^{j} d_i, \quad j = 1, \dots, r-1; \qquad \sum_{i=1}^{r} a_i = \sum_{i=1}^{r} d_i = n.$$

The number of admissible matchings is a function of $(a_1\, d_1\, \cdots\, a_r\, d_r)$, which will be denoted by $N_r(a_1\, d_1\, \cdots\, a_r\, d_r)$ and which can be computed recursively as follows. The first $\min\{a_1, s\}$ arriving customers are equally likely to leave the system first at $\widetilde{D}_1$ since the (common) service time distribution is memoryless. When one of $A_1, \dots, A_{\min\{a_1,s\}}$ is matched with $\widetilde{D}_1$, the run pattern reduces to $((a_1 - 1)\,(d_1 - 1)\,\cdots\, a_r\, d_r)$ if $d_1 > 1$ or $((a_1 + a_2 - 1)\, d_2\, \cdots\, a_r\, d_r)$ if $d_1 = 1$. This leads to the recursion

$$N_r(a_1\, d_1\, a_2\, d_2\, \cdots\, a_r\, d_r) = \begin{cases} \min\{a_1, s\} N_r((a_1 - 1)\,(d_1 - 1)\, a_2\, d_2\, \cdots\, a_r\, d_r) & \text{if } d_1 > 1, \\ \min\{a_1, s\} N_{r-1}((a_1 + a_2 - 1)\, d_2\, \cdots\, a_r\, d_r) & \text{if } d_1 = 1. \end{cases}$$

Alternatively, the departure time D_n of the last arriving customer C_n is equally likely to be any one of $\widetilde{D}_i$, $n-\min\{d_r, s\}+1 \leq i \leq n$. When A_n is matched with one of these $\widetilde{D}_i$s, the run pattern reduces to $(a_1\, d_1\, \cdots\, (a_r-1)\,(d_r-1))$ if $a_r > 1$ or $(a_1\, d_1\, \cdots\, a_{r-1}\,(d_{r-1}+d_r-1))$ if $a_r = 1$. This leads to a different but equivalent recursion for $N_r(a_1\, d_1\, \cdots\, a_r\, d_r)$.

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