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CONTINUOUS BIJECTIONS ON MANIFOLDS

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Abstract

The main results of the paper give necessary and sufficient conditions as well as sufficient conditions that continuous bijections of a manifold onto itself be homeomorphisms. Such conditions include the embedding of manifolds, preserving ends, preserving closed half-rays and restrictions on boundary components. A number of counterexamples are given to likely conjectures.

A topological space is reversible if each bijection of X is a homeomorphism. This concept is looked at for connected metric manifolds. Numerous examples are given along with sufficient conditions as well as necessary and sufficient conditions that a manifold be reversible.

Very shortly after defining "homeomorphism" one provides an example of a continuous bijection that is *not* a homeomorphism. The map $f:[0,1)\to S^1$ given by $f(t)=(\cos 2\pi t, \sin 2\pi t)$ is often used. (Notice that f is a map from one manifold to another.) The student has known since his first cálculus course that any continuous bijection of an interval *to itself* has a continuous inverse (is a homeomorphism). Later in the topology course he learns that a continuous bijection on any compact Hausdorff space is necessarily a homeomorphism and, perhaps, he hears of Brouwer's theorem on invariance of domain which implies the same statement for a manifold without boundary. Is every continuous bijection of a manifold to itself a homeomorphism? If not, what conditions on the manifold ensure an affirmative answer?

Let M be a finite-dimensional, connected, separable metric manifold. Its interior and boundary will be denoted by Int M and Bd M, respectively. We denote by $E(M)$ the set of all continuous bijections of M to itself and by $H(M)$ the set of all homeomorphisms of M onto itself. If $E(M)=H(M)$, we say that M is reversible and, if $E(M)-H(M)\neq\emptyset$, we say M is non-reversible.

Rajagopalan and Wilansky (1966) define a space (x, τ) to be "reversible" if it has no strictly stronger (weaker) topology τ' such that (x, τ) and (x, τ') are homeomorphic. Their first lemma shows that "reversible" is equivalent to the condition that every continuous bijection of X to itself is a homeomorphism.

As was remarked above, compact manifolds and manifolds without boundary are reversible. Each of the four different 1-dimensional manifolds, S^1, $[0,1]$, $[0,1)$, and $(0,1)$ is also reversible but for higher dimensions the first question above must be answered negatively.

THEOREM 1. *For each $n \geqq 2$ there exist non-reversible n-manifolds.*

PROOF. For $n = 2$ let M be the union of the following sets in $\mathbb{R}^2$:

1) $\{(x, y) : y \leqq 0\}$
2) $\{(x, y) : 3k - 1 \leqq x \leqq 3k,\ 0 \leqq y \leqq 2\},\ k \in Z$
3) $\{(x, y) : 3k \leqq x < 3k + 1,\ 1 \leqq y \leqq 2\},\ k \in Z$
4) $\{(x, y) : 3k + 1 \leqq x \leqq 3k + 3,\ 1 \leqq y \leqq 2\},\ k \in Z^+$

Then M is as in Figure 1. If one defines

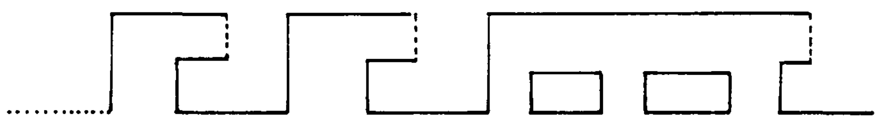

Figure 1

$$g(x, y) = (2x, y) \text{ when } 0 \leqq x < 1,\ 1 \leqq y \leqq 2$$
$$= (x, y) \quad \text{otherwise}$$

and then sets

$$h(x, y) = (x + 3, y),$$

the map $f = h \circ g$ is in $E(M) - H(M)$. Thus M is non-reversible. To complete the proof we note that $M \times \mathbb{R}^k$ is non-reversible if M is. □

The following lemmas will be used several times.

LEMMA 1. *For any manifold M and any map $f \in E(M)$, the restriction $f \mid \operatorname{Int} M$ is a homeomorphism and $f(\operatorname{Int} M)$ is a dense open subset of $\operatorname{Int} M$.*

PROOF. Everything except the density of $f(\operatorname{Int} M)$ follows directly from invariance of domain. If $f(\operatorname{Int} M)$ were not dense in M, then some closed cell e (of maximum dimension) would have to be covered by $f(\operatorname{Bd} M)$. Each component of Bd M is expressible as a monotone union of compact sets of codimension one. Via an appropriate diagonalization we can express Bd M as a monotone union $\bigcup_i e_i$ of such compact sets. Each $f \mid e_i$ is a homeomorphism and, since f is bijective, we may write

$$e = e \cap f(\mathrm{Bd}\, M) = e \cap f(\bigcup_i e_i) = e \cap \bigcup_i f(e_i)$$
$$= \bigcup_i \{e \cap f(e_i)\}.$$

This expresses e as a countable union of compact sets of codimension one, violating the dimensional sum theorem. □

LEMMA 2. *For any manifold M and any map $f \in E(M)$, if $f(Bd\, M) = Bd\, M$, then $f \in H(M)$.*

This is Theorem 3.4 of Pettey (1970).

THEOREM 2. *If $Bd\, M$ is compact, then M is reversible.*

PROOF. Let $f \in E(M)$. For any component C of Bd M we know from Lemma 2 that $f^{-1}(C) \subset \mathrm{Bd}\, M$. Since C is closed in M, $f^{-1}(C)$ is closed in Bd M and hence is compact. It follows that $f \mid f^{-1}(C)$ is a homeomorphism of $f^{-1}(C)$ to C. Each component of Bd M is the homeomorphic image of a component of Bd M. There being only finitely many components of Bd M, $f \mid \mathrm{Bd}\, M$ is a homeomorphism of Bd M onto itself and Lemma 3 applies. □

The argument above can be carried further. Suppose each component of Bd M is compact but that, perhaps, Bd M is not compact. Let f be any map in $E(M)$ and C_0 be any component of Bd M. Enumerate the remaining components of Bd M, $C_1, C_2, \cdots, C_k, \cdots$. Since C_0 is closed in M and Bd M is closed in M, $f^{-1}(C)$ is closed in Bd M. Then for each $k \geqq 0$, $C_k \cap f^{-1}(C_0)$ is compact. Thus $f \mid [C_k \cap f^{-1}(C_0)]$ is a homeomorphism for each $k \geqq 0$ and surely

$$C_0 = \bigcup_k f[C_k \cap f^{-1}(C_0)].$$

This expresses the connected set C_0 as a countable union of compact pairwise disjoint sets which is impossible (Theorem 3, p. 173, Kuratowski (1968)) unless $C_k \cap f^{-1}(C_0) = \varnothing$ for all but one value of k. Again then each component of Bd M must be covered by precisely one component of Bd M. Hence if C is a component of BdM we must have either $f(C) \subset \mathrm{Bd}\, M$ or $f(C) \subset \mathrm{Int}\, M$. This proves the next result.

THEOREM 3. *Suppose every component of $Bd\, M$ is compact. For any $f \in E(M)$, each component C of $Bd\, M$ satisfies either $f(C) \subset Bd\, M$ or $f(C) \subset Int\, M$.*

COROLLARY 1. *If every component of* Bd *M is compact, then a necessary condition for M to be non-reversible is that some component of* Bd *M be repeated infinitely often.*

PROOF. If M is non-reversible, then by Lemma 3 there exists $f \in E(M)$ such that $f(\operatorname{Bd} M) \cap \operatorname{Int} M \neq \varnothing$. From Theorem 3, then, some component C of Bd M is "swallowed", i.e., $f(C) \subset \operatorname{Int} M$. The distinct components $f^{-1}(C)$, $f^{-1}[f^{-1}(C)], \cdots$ suffice to prove this corollary. □

An example illustrating Theorem 3 and its Corollary 1 is illustrated in Figure 2. (This was actually the first non-reversible manifold we knew.) We construct M by removing from $\mathbb{R}^2$ the open disks $\{(x, y) : (x-k)^2 + y^2 < 1/16\}$ with circular boundary C_k, $k \in Z$. On each pair of circles C_{2k-1}, C_{2k}, $k > 0$, we construct a handle and on each C_{2k+1}, $k < 0$, we erect a cylinder $C_{2k+1} \times [0, 1)$. This provides a manifold M with countably many circles in its boundary. A map $f \in E(M) - H(M)$ can be defined by bending over the cylinder $C_{-1} \times [0, 1)$ until its (missing) upper boundary coincides with C_0, thus creating a new handle, and then translating to the right by two units.

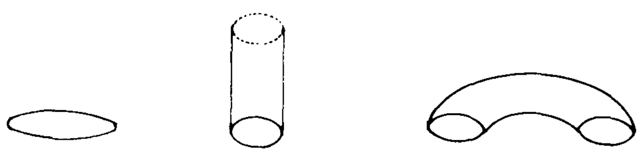

Figure 2

In the example above the boundary component C_0 is "swallowed". We know from the example in Theorem 1 that non-compact boundary components need *not* be swallowed but that example does have infinitely many identical boundary components. Before the reader is tempted into making a false conjecture we present the following modification of the first example: Let M be the union of the following sets in $\mathbb{R}^3$:

1) $\{(x, y, z) : z \leqq 0\}$
2) $\{(x, y, z) : -\frac{1}{2} \leqq x \leqq \frac{1}{2},\ 3k-1 \leqq y \leqq 3k,\ 0 \leqq z \leqq 2\},\ k \in Z$
3) $\{(x, y, z) : -\frac{1}{2} \leqq x \leqq \frac{1}{2},\ 3k \leqq y < 3k+1,\ 1 \leqq z \leqq 2\},\ k \in Z$
4) $\{(x, y, z) : -\frac{1}{2} \leqq x \leqq \frac{1}{2},\ 3k+1 \leqq y \leqq 3k+3,\ 1 \leqq z \leqq 2\},\ k \in Z^+$.

Then M is as pictured in Figure 3. This is a non-reversible 3-manifold with connected boundary. (A map $f \in E(M) - H(M)$ can be defined by setting

$$g(x, y, z) = (x, 2y, z) \qquad \text{if } -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2},\ 0 \leqq y \leqq 1,\ 1 \leqq z \leqq 2,$$
$$= (x, y, z) \qquad \text{otherwise}$$

and $h(x, y, z) = (x, y+3, z)$, then $f = h \circ g$.)

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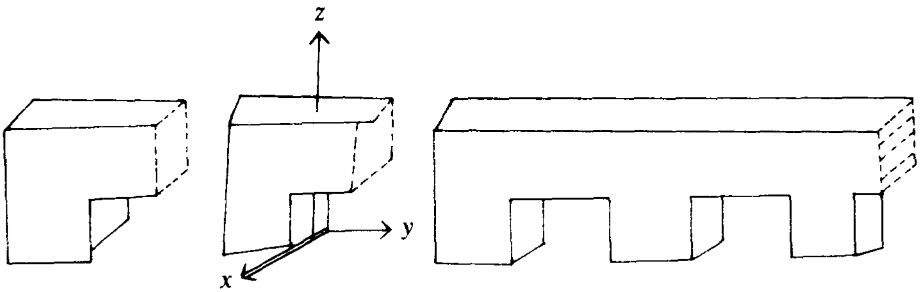

Figure 3

Another erroneous impression could have been imparted by our examples and results above. It is not true that a compact boundary component must either be swallowed entirely or left in the boundary. In the presence of non-compact boundary components, compact ones can be just partially swallowed. The example shown in Figure 4 is easily seen to have this property, of course.

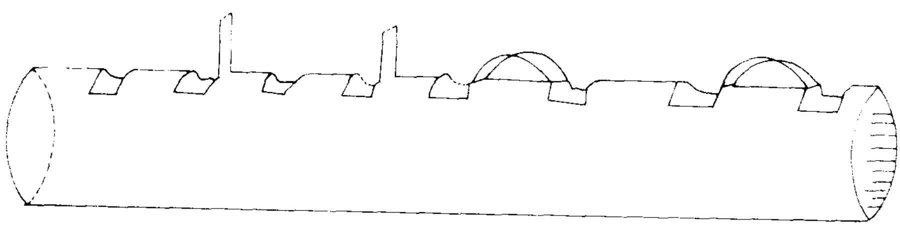

Figure 4

Let M be a non-compact manifold. A *generating sequence for* M is a sequence $\{D_i\}$ of compact submanifolds of M such that (i) for each $i \in Z^+$, $D_i \subset \operatorname{Int} D_{i+1}$ and (ii) $\bigcup_i D_i = N$. (Any *PL* manifold has such generating sequences, for example.) If $\{D_i\}$ is such a generating sequence for M, consider the manifold $N_i = M - \operatorname{Int} D_i$. We let C_{ij} denote the components of N_i. Each sequence $\{C_{ij(i)}\}$ such that $C_{ij(i)}$ is *non-compact* and $C_{i+1\,j(i+1)} \subset C_{ij(i)}$ for each i determines an *end* E of M. To avoid subscript problem let $C_k(E)$ denote the manifold $C_{ij(i)}$ in the sequence defining the end E.

Now consider a map $f \in E(M)$. Each $f \mid D_i$ is a homeomorphism of D_i into M and hence $f(D_i) \subset \operatorname{Int} f(D_{i+1})$ and, f being bijective, $\bigcup_i f(D_i) = M$. That is, $\{f(D_i)\}$ is another generating sequence for M. We shall say that *f carries the end E to the end E'* if, for each $i \in Z^+$, there exist $j, k \in Z^+$ such that

$$f(C_i(E)) \subset C_j(E') \quad \text{and} \quad C_i(E') \subset f(C_k(E)).$$

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THEOREM 4. *Let M be a non-compact manifold and $f \in E(M)$. If f carries each end of M to an end of M, then $f \in H(M)$.*

PROOF. Such a map f clearly extends to a continuous bijection $\tilde{f}: \tilde{M} \to \tilde{M}$ on the Freudenthal compactification $\tilde{M}$ of M. Since $\tilde{M}$ is Hausdorff, $\tilde{f}$ is a homeomorphism and so is its restriction $\tilde{f} \mid M = f$. □

Again suppose M is a non-reversible and $f \in E(M) - H(M)$. By Lemma 3 some point $a \in \operatorname{Bd} M$ has $f(a)$ in $\operatorname{Int} M$. Let U be an open cell neighborhood of $f(a)$ such that $\bar{U}$ is compact and lies in $\operatorname{Int} M$. For any generating sequence $\{D_i\}$ for M, we must have the set $f^{-1}(\bar{U}) - D_i \neq \varnothing$ for all i. For if $f^{-1}(\bar{U}) \subset D_i$ for any i, then $f^{-1}(\bar{U})$ would be compact. Then $f \mid f^{-1}(\bar{U})$ would be a homeomorphism whence the point a would be interior to the open cell $f^{-1}(U)$. It follows, as Rajagopalan and Wilansky (1966) pointed out that if $f \in E(M)$ preserves non-compactness, then $f \in H(M)$. We can carry this a bit further in our setting.

Let $U = f(\operatorname{Int} M)$ and $U = \bigcup^M C_i$ where the C_i's are compact. The sets $J_k = U - \bigcup_i^k C_i$ exhibit the ends of U. The unbounded components of J_k having compact frontiers yield decreasing sequences called "ends". Let $\{\bar{D}_k\}$ be the usual sequence determining an end for U. Then $\{f^{-1}(\bar{D}_k)\}$ determines an end for interior M since $f \mid \operatorname{Int} M$ is a homeomorphism. So the map $f: \operatorname{Int} M \to U$ permutes ends merely.

For M ends are in general different. The very fact that $f \in E(M) - H(M)$ takes some discrete sequence D to a convergent one shows that ends are not preserved. But by continuity f^{-1} does preserve discrete closed sets. It may be noted that by passing a half-ray through D that f fails to preserve contractible sets.

The following is a restatement of Theorem 5.1, Pettey (1970)

THEOREM 5. *A 2-manifold with a trivial 1-dimensional homology group is reversible.*

The proof of Theorem 5 depends on some rather detailed separation arguments and establishes a fact about continua in the plane that is of some interest.

COROLLARY. *If X is the space composed of a single isolated point and a countable number of disjoint copies of the line, then any 1-1 mapping of X into the two-sphere having a continuum image C has an image C that separates the two-sphere.*

THEOREM 6. *Let M be an n-manifold that embeds in euclidean n-space. Then if each boundary component of M is compact, then M is reversible.*

PROOF. By Theorem 3 one notes that any mapping violating reversibility would result in carrying a separating $(n-1)$-manifold into interior M. □

From Doyle and Hocking (1962) an n-manifold $M = U \cup R$ where $R \cap M = \varnothing$, U is topologically euclidean n-space and R has a dimension at most $n-1$. R is called "a residual set".

THEOREM 7. *M is a connected n-manifold. M is reversible if and only if for each residual set R of* Int *M and each bijection f in $E(M)$ it is true that $f(R)$ is residual for* Int *(M).*

PROOF. The necessity of the condition is clear enough.

Suppose that $f(R)$ always meets the residual condition. If M is not reversible select f in $E(M) - H(M)$. Then in Int M we note that $f(R)$ and $B = f(\text{Bd } M) \cap \text{Int } M$ are disjoint relatively closed sets. One selects $f(r)$ in $f(R)$ and $f(b)$ in B. In Int M there is a flat arc A joining $f(r)$ to $f(b)$. In any case A contains a first point proceeding from $f(r)$ to $f(b)$ at which it strikes B. Let this be $f(b)$. Similarly A contains a last point at which it strikes $f(R)$. Again call this $f(r)$. Whence A is an arc such $f(\mathring{M}) \supset A - f(b)$. We may then consider the set $f(R) \cup [A - f(b)] \subset f(\text{Int} M)$. The set $f(R) \subset f(R) \cup [A - f(b)]$ is in $f(\text{Int} M)$ and has a complement that is $(E^{n-1} - a \text{ point}) \times E^1$. Thus one may add to it another point set in $f(\text{Int } M)$ so as to obtain a slightly larger residual set in $f(\text{Int } M)$. Call this $f(R'')$. Consider, then $\text{Int } M - R''$. $f \mid \text{Int } M$ is a homeomorphism to $f(\text{Int } M)$. Whence we have R'' is a residual set in Int M. Clearly $f(R'')$ is not a residual set of Int M since it is not even closed in Int(M).

For f in $E(M)$ in general one has that for each R for $\mathring{M}$, $f(R) \cup [f(\text{Bd } M) \cap \text{Int } M]$ is a residual set for Int(M). □

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