

## CENTRE AND NORM

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We establish a correlation between the structures of a group of power automorphisms of some group and their mutual commutator subgroup and consider the consequences for the norm of a group, and for its capability.

### INTRODUCTION.

The norm of a group is the intersection of all normalisers of the subgroups of a group. It is known that the norm contains the centre of the group and is contained in its second centre  $Z_2(G)$  (see Schenkman [13] and Kappe [9]). The purpose of this paper is a more specified description. Before we are more precise about this we begin with a more general situation and look at power automorphisms of a group. In what follows, we define the commutator  $[a, b]$  of two elements to be  $a^{-1}b^{-1}ab$  while  $[a, \gamma]$  for a group element  $a$  and an automorphism  $\gamma$  stands for  $a^{-1}a^\gamma$ . Moreover, by a result of Cooper [4],  $a^{-1}a^\gamma \in Z(G)$  whenever  $\gamma$  is a power automorphism of  $G$ . If a position of the commutator bracket is filled by a set, the subgroup generated by the corresponding bracket elements is meant. Now we are able to formulate the more general result.

**THEOREM 1.** *Let  $G$  be a  $p$ -group, where  $p$  is a prime. Let  $\Gamma$  be a group of power automorphisms of  $G$  and assume that  $[G, \Gamma]$  has finite exponent.*

- (i) *If  $p$  is odd, then  $\Gamma$  is cyclic or  $[G, \Gamma]$  is cyclic.*
- (ii) *If  $p = 2$ , then  $\Gamma$  is a subgroup of a direct product of a cyclic group and a group of order 2 or  $[G, \Gamma]$  is cyclic.*

**REMARK 1.** Divisible Abelian  $p$ -groups show that the condition on the finite exponent is indispensable here.

Now the following result is a consequence.

**THEOREM 2.** *Let  $G$  be a  $p$ -group and denote by  $N(G), Z(G)$  its norm, centre respectively. At least one of the following two statements is true:*

- (A)  *$N(G)/Z(G)$  is cyclic,*
- (B)  *$[G, N(G)]$  is cyclic.*

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REMARK 2. If  $G$  is a regular  $p$ -group, then  $N(G)/Z(G)$  is cyclic. If  $x \in N(G)$ , for some  $p$ -power  $m$  we have  $x^m \in Z(G)$  and if  $x^{-1}yx = y^k$ , then  $y$  is of order dividing  $k^m - 1$ . By Cooper [4, p. 349] power automorphisms are universal in regular  $p$ -groups, in other words, if  $y, z \in G$  there is an integer  $r$  such that  $x^{-1}yx = y^r$  and  $x^{-1}zx = z^r$ , and the highest power of  $p$  dividing  $r - 1$  is independent of the choice of  $r$ . We obtain that  $G$  is of finite exponent and that there is  $r \in \mathcal{N}$  such that  $x^{-1}gx = g^r$  for all  $g \in G$ . So  $N(G)/Z(G)$  is mapped onto the multiplicative group  $Z_t^*$ , where  $t = \exp(G)$ . Notice that  $r = -1$  is impossible since the  $p$ -group is nonabelian if  $N(G) \neq Z(G)$ , and for  $p = 2$  we have that  $r - 1$  must be divisible by 4.

Examples of groups satisfying (A) and those satisfying (B) of Theorem 2 are given at the end of this paper.

A group  $G$  is called *capable*, if there is a group  $H$  such that  $H/Z(H) \cong G$ ; such groups are considered in detail in [1, 2, 3, 5, 6, 7]. The normal subgroup of  $Z^*(G)$  of  $G$  called the *epicentre* of  $G$ , plays a significant role in determining whether or not  $G$  is capable: it is the intersection of all normal subgroups  $N$  of  $G$  such that  $G/N$  is capable. Thus  $Z^*(G)$  is the smallest normal subgroup of  $G$  such that  $G/Z^*(G)$  is capable (see [3, Corollary 2.2]). Therefore  $G$  is capable if and only if  $Z^*(G) = 1$ . Another definition is

$$Z^*(G) = \cap \left\{ \phi(Z(E)) \mid (E, \phi) \text{ is a central extension of } G \right\}$$

(see [1, 2, 3, 5]). In Theorems 3 and 4 we find some groups  $G$  satisfying  $N(G) \neq Z(G)$  which are not capable. We use the subgroup  $Z^*(G)$  to prove these results.

**THEOREM 3.** *Assume that  $G$  is a finite group. If  $[G, N(G)]^m$  is cyclic and non-trivial for some  $m$ , then  $G$  is not capable.*

We need a statement on the exponent of  $[G, N(G)]$  for the next statement.

**THEOREM 4.** *Assume that  $G$  is a  $p$ -group. If  $C(N(G))$  possesses a supplement  $S$  in  $G$  such that  $Z(S) \not\subseteq C(N(G))$  and  $[G, N(G)]^2 \neq 1$ , then  $G$  is not capable.*

**COROLLARY.** *Assume that  $G$  is a  $p$ -group, where  $p \neq 2$ , and  $G$  is a split extension of an Abelian group  $A$  by a group  $B$ . If  $[G, N(G)]^2 \neq 1$  and one of the following conditions is satisfied:*

- (i)  $B$  is Abelian,
- (ii)  $C(A) = A$ ,

then  $G$  is not capable.

Example (VI) shows that the condition  $[G, N(G)]^2 \neq 1$  can not be strengthened.

#### PROOFS.

**PROOF OF THEOREM 1:** By a result of Cooper ([4, Theorem 2.2.1], see also [14, Theorem 1.5.2]),  $[G, \Gamma] \subseteq Z(G)$ . Let  $x \in G$ . Then  $[x, \Gamma]$  is contained in  $\langle x \rangle$  and thus

cyclic. Choose  $g \in G$  such that  $[g, \Gamma]$  has maximal order. Then  $[g, \Gamma] = \langle [g, \alpha] \rangle$  for some  $\alpha \in \Gamma$ . We claim that  $[G, \Gamma] = \langle [g, \alpha] \rangle$  when  $\Gamma$  is not cyclic, or (for  $p=2$ ) is not a subgroup of a direct product of a group of order 2 and a cyclic group.

Consider now the group  $[G, \alpha]$  and suppose that it is not cyclic. Then (by [12, 4.27 p. 102])  $[G, \alpha] = \langle [g, \alpha] \rangle \times V$  for some subgroup  $V$ . From this direct factor decomposition it follows that  $\langle x \rangle \cap \langle y \rangle = 1$  whenever  $[v, \alpha] \in \langle x \rangle$  and  $[gv, \alpha] \in \langle y \rangle$ , with  $1 \neq [v, \alpha] \in V$ . In particular,  $\langle v \rangle \cap \langle gv \rangle = 1$ .

Let  $F(\alpha) = \{h \in G \mid h^\alpha = h\}$ . Then  $F(\alpha)$  is a  $\Gamma$ -invariant normal subgroup of  $G$  and  $G/F(\alpha) \cong [G, \alpha]$  and  $G = \langle F(\alpha), g, M \rangle$  with  $[M, \alpha] = V$ .

Let  $\Phi(g) = \{\delta \in \Gamma \mid g^\delta = g\}$ . For  $\beta \in \Phi(g)$  and  $v \in M$  we have

$$[gv, \beta] = [v, \beta] \in \langle gv \rangle \cap \langle v \rangle = 1.$$

Therefore  $\Phi(g) \subseteq \Phi(v)$ . Let  $h \in F(\alpha)$  and  $\beta \in \Phi(g)$ . Then  $[vh, \alpha] = [v, \alpha] \in \langle vh \rangle$  and  $[gvh, \alpha] = [gv, \alpha] \in \langle gv \rangle$ . Consequently  $\langle vh \rangle \cap \langle gv \rangle = 1$ . But  $[vh, \beta] = [h, \beta] = [gvh, \beta]$  and so  $[h, \beta] = 1$ . Therefore  $\beta$  fixes  $\langle F(\alpha), g, M \rangle = G$  and  $\Phi(g) = 1$ . Since  $\Gamma/\Phi(g)$  is isomorphic to a subgroup of  $\text{Aut}(g)$  it is cyclic (for  $p$  odd) or a subgroup of a direct product of a group of order 2 and a cyclic group. Hence it follows that  $[G, \alpha] = \langle [g, \alpha] \rangle = [g, \Gamma]$  when  $\Gamma$  is not of the form just mentioned.

We still have to show  $\langle [g, \alpha] \rangle = [G, \Gamma]$ . As a first step, assume the existence of  $w \in G$  and  $\delta \in \Gamma$  such that  $[w, \delta] \neq 1$  and  $\langle [w, \delta] \rangle \cap [g, \Gamma] = 1$ . We have  $[w, \alpha] \in \langle w \rangle \cap [G, \alpha] = 1$  and  $[wg, \alpha] = [g, \alpha]$  while  $[wg, \delta] = [w, \delta][g, \delta] = [w, \delta][g, \alpha]^m$  for suitable  $m$ . These two elements generate a noncyclic group and not both of them are contained in  $\langle wg \rangle$ , a contradiction. We deduce  $[G, \delta]$  is cyclic for every  $\delta \in \Gamma$  and  $[G, \delta] \cap [G, \alpha] \neq 1$ . Assume now the existence of  $w$  and  $\delta$  as before but with the less stringent condition  $[w, \delta] \notin [G, \alpha]$ . Then  $\langle [w, \alpha] \rangle$  is a proper subgroup of  $\langle [w, \delta] \rangle$  and of  $\langle [g, \alpha] \rangle$ , likewise  $\langle [g, \delta] \rangle$  is a proper subgroup of  $\langle [g, \alpha] \rangle$  and of  $\langle [w, \delta] \rangle$ . We obtain  $\langle [wg, \alpha] \rangle = \langle [g, \alpha] \rangle$  and  $\langle [wg, \delta] \rangle = \langle [w, \delta] \rangle$  contrary to the automorphisms being power automorphisms. We have obtained  $[w, \delta] \in [G, \alpha] = \langle [g, \alpha] \rangle$  for all  $w \in G$  and  $\delta \in \Gamma$ , in other words,  $[G, \Gamma]$  is cyclic. This concludes the proof.  $\square$

**PROOF OF THEOREM 2:** As indicated in the introduction,  $N(G) \subseteq Z_2(G)$ . We shall apply Theorem 1 for subgroups  $W$  with  $Z(G) \subseteq W \subseteq N(G)$ .

As a first step we notice two things: (I) there are no nontrivial divisible subgroups in  $Z_2(G)/Z(G)$  if  $G$  is a  $p$ -group; (II) there are no nontrivial divisible subgroups in  $Z(G)$  if  $G$  is a  $p$ -group and  $N(G) \neq Z(G)$ .

For statement (I) assume otherwise and consider an element  $y \in Z_2(G) \setminus Z(G)$  such that there are elements  $y_i \in Z_2(G)$  such that  $y_i^{p^i} Z(G) = y Z(G)$  for all  $i \in \mathbb{N}$ . Let  $h$  be any element of order  $p^i$  in  $G$ . Then  $[h, y] = [h, y_i^{p^i}] = [h, y_i]^{p^i} = [h^{p^i}, y_i] = 1$ . We have shown  $y \in Z(G)$ , contrary to assumption. So statement (I) is true.

For statement (II), assume the existence of a Prüfer subgroup  $P$  in  $Z(G)$  and choose

an element  $x \in N(G) \setminus Z(G)$ . There is an element  $y \in G$  which is not centralised by  $x$ . Now  $\langle P, y \rangle$  is Abelian, and there is an element  $z$  such that  $\langle P, y \rangle = \langle P, z \rangle = P \times \langle z \rangle$ . The element  $z$  does not centralise  $x$  and the same is true for  $zw$  where  $w \in P$  is of the same order as  $z$ . Now  $x^{-1}zx = z^k$ ,  $x^{-1}zwx = (zw)^m$  and  $x^{-1}wx = w$ . By the direct product description this is only possible for  $k = m = 1$  contrary to the construction of  $z$ . This shows statement (II).

As a consequence we state, that  $N(G)/Z(G)$  and  $[G, N(G)]$  do not possess divisible subgroups other than the trivial group.

Let  $W_i/Z(G)$  be the maximal subgroup of exponent  $p^i$  of  $N(G)/Z(G)$ . If  $N(G)/Z(G)$  is non-cyclic, all  $W_i/Z(G)$  are non-cyclic. We apply Theorem 1 for  $G$  and  $W_i/Z(G)$  considered as a group of (inner) power automorphisms. We have to show that the prime 2 does no longer play an extra role here. In other words, we have to show that the quotient group  $N(G)/(N(G) \cap C(T))$  is cyclic for every cyclic subgroup  $T$  of  $G$  regardless of the prime. Assume that  $T = \langle y \rangle$  is of order  $2^n$  with  $n > 2$ . Then there is no  $x \in N(G)$  such that  $x^{-1}yx = y^{-1}$  since otherwise  $[x, [x, y]] = y^4 \neq 1$  and  $x \in N(G) \subseteq Z_2(G)$ . So for  $n > 2$ ,  $N(G)/(N(G) \cap C(T))$  is isomorphic to a subgroup of  $Z_{2^n}^*$  and does not contain  $-1$ , so it must be cyclic. - For  $n < 3$  we have that  $Z_{2^n}^*$  is itself cyclic.

Using Theorem 1 for  $G$  and  $W_i/Z(G)$  with the above modification we obtain that  $[G, W_i]$  is cyclic if  $W_i/Z(G)$  is non-cyclic, its order is  $p^i$ . Since  $[G, N(G)]$  can not be locally cyclic we obtain that  $[G, N(G)] = \cup [G, W_i]$  is cyclic (and finite) and  $N(G)/Z(G) = \cup W_i/Z(G)$  must have finite exponent. □

**PROOF OF THEOREM 3:** By hypothesis,

$$T = [G, N(G)]^m = [G, (N(G))^m]$$

is cyclic and nontrivial. We may assume that  $T$  is of order a prime  $p$  and that  $T = \langle [h, d] \rangle$  with  $h \in G$  and  $d \in (N(G))^m$ , so  $d^p \in Z(G)$  and  $T = [d, G]$ . Now  $C(d)$  is a normal subgroup of  $G$  and  $G/C(d)$  is cyclic of order  $p$ . There is a generating set  $\{g_1, \dots, g_k\}$  for  $G$  such that  $\langle g_i, C(d) \rangle = G$  for all  $i$ . But then  $\langle [g_i, d] \rangle = T \subseteq \langle g_i \rangle$  for all  $i$ , and  $T \subseteq Z^*(G)$  (see [3, Proposition 3.3]). So  $G$  is not capable. □

**REMARK.** A somewhat different proof of Theorem 3 can be given using [10, Theorem 2 and Corollary 1(2)].

**PROOF OF THEOREM 4:** We shall show that  $Z^*(G) \neq 1$  is true by  $[N(G), Z(S)] \subseteq Z^*(G)$ . Let  $G \cong H/M$  where  $M \subseteq Z(H)$ . For every  $g \in G$  we choose some  $g^* \in H$  as a pre-image of  $g$  - since we argue by commutators the actual choice of this pre-image does not matter. Choose  $d \in N(G)$  and  $t \in Z(S)$  such that  $[d, t] \neq 1$ . We shall show  $[d, t]^2 \in Z^*(G)$ . Since  $d \in N(G)$ , we have  $[d, g] \in \langle g \rangle$  for all  $g \in G$ , and so  $[[d^*, g^*], g^*] = 1$ , also  $d^* \in Z_3(H)$ . So  $d^*$  is a right-2-Engel element of  $H$ . Let  $s \in S$  and  $c \in C(N(G))$ . By Robinson [11, Theorem 7.13](see also Kappe [9]) we have

$$1 = [[d^*, c^*], t^*] = [[d^*, t^*], c^*]^{-1}$$

and

$$1 = [d^*, [t^*, s^*]] = [[d^*, t^*], s^*]^2.$$

We have shown  $[d^*, t^*]^2 \in Z(H)$  since  $G = SC(N(G))$ , and this proves Theorem 4. □

PROOF OF THE COROLLARY: It suffices to show that Theorem 4 may be applied: In both cases we take  $S = A$  and we shall show  $B \subseteq C(N(G))$ . Notice  $G' \subseteq A$  in the first case and  $Z(G) \subseteq A$  in the second, so we have for both cases  $G' \cap Z(G) \subseteq A$ . Now  $[B, N(G)] \subseteq B \cap G' \cap Z(G) = 1$ ; so the hypotheses of Theorem 4 are satisfied, and the Corollary is true. □

1. EXAMPLES.

(I)  $N(G)/Z(G)$  is noncyclic.

The most prominent example is  $Q_8$  for the prime 2. This is the member of smallest order of a family of examples which we shall now construct.

Let  $\Gamma$  be a field of  $p^n$  elements where  $n = p^k$  and let  $\gamma \in \Gamma$  be such that  $\gamma^p - \gamma = \delta \neq 0$ . We define the unitriangular matrices  $M(\beta_1, \beta_2, \dots, \beta_p) = (m_{ij})$  such that

$$\begin{aligned} 1 &\leq i, j \leq p + 1, \\ m_{ij} &= 0 \quad \text{for } j < i, \\ m_{ii} &= 1 \quad \text{for all } i, \\ m_{i, k+i} &= \beta_k^{p^{i-1}} \quad \text{for } k > 0. \end{aligned}$$

This set of  $(p + 1) \times (p + 1)$ -matrices forms a group with respect to usual matrix multiplication; we denote it by  $T$ . Let  $\Phi$  be a maximal subgroup of  $\Gamma^+$  which contains neither 1 nor  $\delta$ . The set  $\{M(0, \dots, \mu) \mid \mu \in \Phi\}$  is a proper subgroup of  $Z(T)$ , we shall denote it by  $F$ . The group  $G = T/F$  is our example. We notice that  $M(\beta_1, \dots)^n = M(0, \dots, 0, \beta_1^a)$  where  $a = 1 + p + \dots + p^{n-1}$ . We see that  $\beta_1^a$  is nonzero whenever  $\beta_1$  is nonzero, and that it belongs to the prime field of  $\Gamma$ , so it does not belong to  $\Phi$ .

On the other hand we have  $[M(\beta_1, \dots), M(0, \dots, 0, \lambda, \rho)] = M(0, \dots, 0, \tau)$  where  $\tau = \beta_1 \lambda^p - \beta_1^{p^{n-1}} \lambda$ , so for  $\gamma^p = \beta_1 \lambda^p$  we have  $\tau = \delta \notin \Phi$ . With the canonical epimorphism of  $T$  onto  $T/F = G$  we obtain that  $N(G) = Z_2(T)/F = Z_2(G)$ .

By Theorem 3, all of these groups are not capable. - Notice that the nilpotency class must be at least  $p$  by [4]. In the second case we do not have such a restriction.

(II)  $[G, N(G)]$  is noncyclic.

Let  $p$  be a prime. Again we begin with a field  $\Gamma$ , this time of order  $p^n$  where  $2 < n + 1 < p^k$ . Form the unitriangular  $(n + 2) \times (n + 2)$  - matrices over  $\Gamma$  analogous to the examples in the previous section; notice that  $\beta_1 = \beta_1^{p^n}$  so  $m_{12} = m_{n+1, n+2}$ . Choose  $\gamma \in \Gamma$  such that  $\delta = \gamma^p - \gamma \neq 0$ ; for  $n = 2; p \neq 2$  take  $\gamma$  such that  $\gamma^p = -\gamma$ . In particular,  $[M(\beta_1, \dots), M(0, \dots, 0, \gamma, \mu)] = M(0, \dots, 0, \delta \beta_1)$ . We call the group of matrices  $T$  as in the previous case.

Let  $A$  be the direct product of  $n$  cyclic groups of order  $p^{k+1}$ . Then  $A/A^p$  and  $A^{p^k}$  are isomorphic to  $\Gamma^+$ , and we define these isomorphisms  $\phi, \psi$  such that  $\phi(aA^p) = \psi(a^{p^k})$ . In the direct product  $T \times A$  consider the subgroup generated by the elements  $(M(\phi(aA^p), 0, \dots, 0), a)$  and call it  $U$ . By construction,  $U^p = \{(E, a^p)\}$  and  $U_{n+1} = \{(T_{n+1}, 1)\}$ . Let

$$V = \{(M(0, \dots, 0, \delta\psi(a^{p^k})), a^{p^k})\}.$$

Now  $U/V = G$  is the desired group, and

$$N(G) = \langle (M(0, \dots, 0, \gamma, 0), 1)V, Z(G) \rangle.$$

For  $n = 2$  and  $\Gamma$  a prime field of odd order,  $T$  contains a subgroup which is free (of exponent  $p \neq 2$ , of nilpotency class 3 and generated by two elements). This leads to a group  $K$  of order  $p^5$  with  $N(K) = Z_2(K)$ , see also (V).

Groups of class 2 we find for instance in the next family of examples (with  $A$  of exponent  $p^2$  and  $B$  of order  $p$ ).

(III) Iwasawa groups.

An Iwasawa group is a group in which all of its subgroups are permutable. For a complete description of finite Iwasawa  $p$ -groups see [14, p. 55]. Choose an Abelian group  $A$  of exponent  $p^n$  for  $p \neq 2$  and  $n > 1$ . Then form the split extension by a cyclic group  $\langle b \rangle$  of order  $p^{n-1}$  such that  $b^{-1}ab = a^{1+p}$  for all  $a \in A$ . Then the element  $c$  of order  $p$  in  $\langle b \rangle$  belongs to the norm of this extension. By the Corollary following Theorem 4, this group is not capable. For  $p = 2$  we choose  $n > 3$  and assume that the element  $b$  of order  $2^{n-2}$  maps every  $a \in A$  onto  $a^5$ . In this case the element  $c$  of order 4 in  $\langle b \rangle$  is contained in the norm and again  $[N(G), G]^2 \neq 1$ . Again this group is not capable. But if  $p = 2$ ;  $n = 3$ ;  $b = c$ , then this group is in fact capable, see (VI).

(IV) Groups of maximal class.

The quotient group  $(C_{p^n} \text{ wr } C_p) / Z(C_{p^n} \text{ wr } C_p)$  is a group of maximal class with unique Abelian normal subgroup of index  $p$  which we call  $N$ . The subgroup of order  $p$  operating on  $N$  we denote by  $\langle x \rangle$ . Now extend  $N$  by a element  $y$  of order  $p^2$  satisfying the following conditions:  $y^{-1}ay = x^{-1}ax$  for all  $a \in N$ , and  $y^p \in Z(\langle N, y \rangle)$ . Then  $Z_2(\langle N, y \rangle) = N(\langle N, y \rangle)$  and Wielandt series and central series coincide in all other places. Again, this group is not capable by Theorem 3. The same applies for all of its nonabelian subgroups. Notice that we have the family of (generalised) quaternion groups here if  $p = 2$ .

(V) Two-generator groups of class 3.

Choose a prime  $p > 3$  and define

$$D = \langle x, y \mid x^p = [[x, y], x]; y^p = [[x, y], y]; [[x, y], x], y \rangle = 1 \rangle.$$

The relations yield

$$[[x, y], x], x \rangle = 1 = [[x, y], y], y \rangle$$

and

$$\left[ \left[ [x, y], y \right], x \right] = 1,$$

so  $D$  is nilpotent of class 3 and  $N(D) = D'$ , also  $[D, N(D)]$  is of rank 2. Here  $D$  is not capable: If  $D \cong H/Z(H)$  and  $x^*, y^* \in H$  are the pre-images of  $x, y$ , then again

$$\left[ \left[ [x^*, y^*], x^* \right], x^* \right] = 1 = \left[ \left[ [x^*, y^*], y^* \right], y^* \right]$$

and

$$\left[ \left[ [x^*, y^*], x^* y^* \right], x^* y^* \right] = 1$$

and so

$$\left[ \left[ [x^*, y^*], x^* \right], y^* \right] \left[ \left[ [x^*, y^*], y^* \right], x^* \right] = 1$$

while

$$1 = [x^*, y^*], [x^*, y^*] = \left[ \left[ [x^*, y^*], x^* \right], y^* \right] \left[ \left[ [x^*, y^*], y^* \right], x^* \right]^{-1}.$$

This shows that all commutators of length 4 are trivial and  $D \cong H$  contrary to assumption.- For the prime 3, the powers  $x^p, y^p$  in the relations should be replaced by  $x^9, y^9$  to obtain the same result.- An argument like that in the proof of Theorem 4 shows that these groups are not capable.

(VI) Capable 2- groups.

Let  $s = 2^n > 2$  and  $1 \leq i, j, k \leq m$  with  $m \neq 1$ . Consider

$$T = \langle w_1, w_2, \dots, w_m, v \mid 1 = [[w_i, w_j], w_k] = w_i^{2^s} = v^2 = w_i^{-s-1} v^{-1} w_i v \rangle.$$

The centre  $Z(T)$  is generated by all commutators  $[w_i, w_j]$ , and

$$N(T/Z(T)) = \langle w_i^2, v, Z(T) \rangle / Z(T).$$

This shows that  $G$  may be capable if  $[G, N(G)]^2 = 1$ .

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