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CONTIGUITY AND REMOTE CONTIGUITY OF SOME RANDOM GRAPHS

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Abstract

Asymptotic properties of random graph sequences, like the occurrence of a giant component or full connectivity in Erdös–Rényi graphs, are usually derived with very specific choices for the defining parameters. The question arises as to what extent those parameter choices may be perturbed without losing the asymptotic property. For two sequences of graph distributions, asymptotic equivalence (convergence in total variation) and contiguity have been considered by Janson (2010) and others; here we use so-called remote contiguity to show that connectivity properties are preserved in more heavily perturbed Erdös–Rényi graphs. The techniques we demonstrate here with random graphs also extend to general asymptotic properties, e.g. in more complex large-graph limits, scaling limits, large-sample limits, etc.

Keywords: Random graphs; contiguity; remote contiguity; graph connectivity; giant component

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1. Asymptotic properties of random graphs

Most asymptotic properties of random graph sequences are derived in a highly regular setting, based on certain precise choices for the defining parameters. This specificity raises the question of to what extent perturbations of parameters leave said properties intact. For perturbations of the Erdös–Rényi (ER) graph (e.g. [2]), Janson's seminal paper [6] discusses how an asymptotic property of one ER graph sequence can be related to that of another, based on asymptotic equivalence (convergence in total variation) and contiguity [4, 8, 9, 12].

In this paper we propose a more general form of asymptotic congruence called *remote contiguity*, introduced in [7]. While the conditions inducing asymptotic equivalence or contiguity can sometimes be too stringent, remote contiguity is applicable more widely. To demonstrate this, we consider two sequences of random graphs (X^n) and (Y^n), with distributions denoted by (P_n) and (Q_n) respectively. The subscript n denotes the number of vertices in the graph: we look at graphs that grow to infinite size and study conditions under which some or all of the asymptotic properties of (X^n) also apply to (Y^n), based on uniform tightness of rescaled log-likelihood ratios of (Q_n) with respect to (P_n).

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By an *asymptotic graph property* we mean any sort of limit in probability: the property is expressed in the form $P_n(A_n) \to 0$, with some sequence of n-vertex graph events (A_n). In the case at hand we consider well-known connectivity properties of ER graphs with asymptotically bounded expected degrees, such as the occurrence of a giant component in the supercritical regime and $O(\log(n))$-fragmentation in the subcritical regime, as well as asymptotic connectedness in the less sparse regime of ER graphs with expected degrees that diverge logarithmically. We also consider the so-called critical window, in which the largest component displays asymptotic growth of order $O(n^{2/3})$. (For an extensive review of random graph asymptotics and asymptotic properties of ER graphs in particular, see [13]).

1.1. Perturbations of graph sequences

To find laws Q_n for random graphs Y^n that share asymptotic properties with the $X^n \sim P_n$, we can impose the (sufficient) condition that the Hellinger (or total variational) distance between P_n and Q_n goes to zero in the limit (asymptotic equivalence, see [6, Definition 1.1]); then any property of (X^n) is shared by the graph sequence (Y^n) in the sense that, for any sequence of n-vertex graph events (A_n), $P_n(A_n) - Q_n(A_n) \to 0$. Asymptotic equivalence occurs if and only if there exists a coupling of X_n and Y_n such that $P(X_n \neq Y_n)$ tends to zero as $n \to \infty$ [6, Theorem 4.2].

Janson argues that *asymptotic equivalence is too strong* as a condition for the sharing of asymptotic properties: for example, the giant component occurs in a large family of inhomogeneous versions of the ER graph [1], much larger than the subset of all asymptotically equivalent graphs. Le Cam's notion of contiguity [8, 9, 10] is a weaker, more appropriate condition for the sharing of asymptotic properties: (Q_n) is said to be *contiguous with respect to* (P_n) (notation $Q_n \lhd P_n$) if $P_n(A_n) = o(1) \;\Rightarrow\; Q_n(A_n) = o(1)$ for any sequence of n-vertex events (A_n). Janson applies contiguity to sequences of perturbed ER graphs and demonstrates its wider applicability. (We discuss Janson's condition for contiguity of inhomogeneous ER graphs in Section 3.)

The main point of this paper is that *contiguity is still too strong* if the $P_n(A_n)$ are known to converge to zero faster than a certain rate (a_n) (rather than just being $o(1)$). Given a sequence $a_n \downarrow 0$, we say that (Q_n) is *a_n-remotely contiguous with respect to* (P_n) (notation $Q_n \lhd a_n^{-1} P_n$) if

$$P_n(A_n) = o(a_n) \;\Rightarrow\; Q_n(A_n) = o(1) \tag{1}$$

for any sequence of n-vertex events (A_n). Remote contiguity was introduced in [7] for the frequentist analysis of Bayesian, posterior-based limits, and argued to offer generalization of contiguity-based statistical arguments, which typically apply in smooth-parametric (e.g. local-asymptotically normal, see [8]) models, to a much more general (e.g. non-parametric) setting; see [7, Subsection 3.3]. [3] and [11] use remote contiguity to generalize a consistency conclusion reached for an idealized sequence of data distributions to the general class of sequences that are realistic for the data in the problem.

To use remote contiguity with sequences of random graphs, we look for sequential models (P_n) for n-vertex graphs with asymptotic properties (A_n) and *known* (a_n), as in (1), and analyse the family of those sequential graph distributions (Q_n) that satisfy $Q_n \lhd a_n^{-1} P_n$ to conclude that random graphs distributed according to (Q_n) share the asymptotic property reflected by the events (A_n). It is noted that for many asymptotic graph properties, *sharp* rates (a_n) are known (see, e.g., [13]).

In Section 2 we briefly review ER random graphs to fix the notation, and we recall Janson's condition for contiguity of ER graph sequences. In Section 3 we apply remote contiguity to

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sequences of inhomogeneous ER graphs, compare with Janson's condition, and formulate a weaker, Lindeberg-type condition that involves the rate sequence (a_n) (which, in most cases, is not only sufficient but also necessary). In Section 4 we define so-called *remotely contiguous domains of attraction* for all the aforementioned asymptotic connectivity properties of ER graphs. Section 5 summarizes conclusions and discusses possible directions of further research. Details on remote contiguity are discussed in [7] and summarized in Appendix A.

The ultimate goal of this paper is to convince the reader that remote contiguity provides a meaningful generalization of the notion of contiguity, applicable to a much wider range of problems. We remark that, like asymptotic equivalence and contiguity, remote contiguity compares *any* pair of sequences of probability distributions, including (but not limited to) the comparison of ER graph distributions.

2. Homogeneous and inhomogeneous ER graphs

Let $G_n = ([n], E_n)$ denote the complete graph with n vertices, with vertex labels from $[n] := \{1, 2, \ldots, n\}$ and edge set E_n (which does not include self-loops; the edge between vertices i and j is denoted (ij)). Denote the space of all subgraphs of G_n by $\mathscr{X}_n$. The homogeneous ER random graph with edge probability $0 < p < 1$, a random element of $\mathscr{X}_n$ denoted X^n, is $([n], E'_n)$ where E'_n contains any $e \in E_n$ independently with probability p. The presence or absence of an edge $e = (ij)$ from E_n in the graph X^n is expressed in terms of (independent) random variables, $X^n_{ij} = 1$ or $X^n_{ij} = 0$ respectively. The degree of vertex i is denoted by $D_i(X^n)$. We denote the distribution of X^n with edge probability p by $P_{p,n}$. When we speak of an 'ER graph (with edge probability p)', we refer to the *sequence of distributions* $(P_{p,n})$ as an element of $\prod_n M^1(\mathscr{X}_n)$, and we denote the class of all ER graphs as $\mathscr{E}$. We generalize from $\mathscr{E}$ in two stages: we distinguish the class $\mathscr{H} \subset \prod_n M^1(\mathscr{X}_n)$ of all *homogeneous ER graphs*, containing all sequences (Q_n) of the form $Q_n = P_{p_n,n}$, for some n-dependent $0 \leq p_n \leq 1$; we generalize further by considering the class $\mathscr{I} \subset \prod_n M^1(\mathscr{X}_n)$ of all *inhomogeneous ER graphs*, in which the probability for occurrence of an edge may depend on the vertices it connects (see below).

The n-dependence of the (p_n) plays a central role for the asymptotic properties of homogeneous ER graphs: a prime example is the sequence of ER graphs with edge probabilities $p_n = \lambda/n$ ($\lambda < 1$, $\lambda = 1$ and $\lambda > 1$ characterize the so-called *subcritical, critical, and supercritical regimes*). With a slight abuse of notation, we denote the distributions of these graphs with $P_{\lambda,n}$. In some cases we also leave room for further n-dependence, e.g. ER graphs $Y^n \sim P_{\lambda_n,n}$.

The inhomogeneous ER random graph with edge probabilities $0 < p_{n,ij} < 1$ for $1 \leq i < j \leq n$ is $([n], F_n)$, where F_n contains any $e = (ij) \in E_n$ independently with probabilities $p_{n,ij}$; we denote the distribution of the inhomogeneous ER graph Y^n with edge probabilities $(p_{n,ij} \colon 1 \leq i < j \leq n)$ by $P_{(p_{n,ij}),n}$. The presence or absence of an edge $e = (ij)$ from E_n in F_n is expressed in terms of (independent) random variables, $Y^n_{ij} = 1$ or $Y^n_{ij} = 0$ respectively; write $Y^n_{ij} \sim P_{p,n,ij}$ and $P_{(p_{n,ij}),n} = \prod_{i<j} P_{p,n,ij}$. Of foremost interest to this work are edge probabilities of the form $p_{n,ij} = \mu_{n,ij/n}$, for which it is assumed throughout that

$$\lim_{n\to\infty} \sup_{i<j} \frac{\mu_{n,ij}}{n} < 1, \tag{2}$$

i.e. that edge probabilities are uniformly bounded away from 1 (compare with the bound $p_{ij,n} < 0.9$ in Janson's theorem, see Theorem 1). For later reference, we define the parameter spaces

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$\Lambda_n = \mathbb{R}^{n(n-1)/2}$ for all $n \geq 1$, with inner product norm

$$\|\lambda_n - \mu_n\|_{2,n}^2 = \sum_{i<j} (\lambda_{n,ij} - \mu_{n,ij})^2.$$

In what follows, we examine to what extent known properties of ER graphs, like the occurrence of a *giant component* or a *critical window*, are shared in the classes of homogeneous and inhomogeneous ER graphs (like the *stochastic block model* that is central in network theory). For a general review, see [1]; for more in relation to contiguity, see [6, Examples 3.1, 3.5 and 3.6]; for other possibilities, see [6, Remark 1.6] and [5]. For the following theorem and lemma, consider two inhomogeneous ER graphs distributed according to $P_n = P_{(p_{n,ij}),n}$ and $Q_n = P_{(q_{n,ij}),n}$. Restricted to contiguity and transcribed into our notation, Janson's Corollary 2.12 says the following.

Theorem 1. ([6]) *Assume that* $\sup_{i<j} p_{n,ij} < 0.9$. *If*

$$\sum_{i,j} \frac{(p_{n,ij} - q_{n,ij})^2}{p_{n,ij}} = O(1), \tag{3}$$

then $Q_n \lhd P_n$.

(Janson's Corollary 2.12 also specifies that P_n and Q_n are asymptotically equivalent if the sum in (3) is $o(1)$ rather than $O(1)$, but that fact plays no role in what follows.)

3. Remote contiguity of inhomogeneous ER graphs

The application of remote contiguity to the ER graph involves sufficient conditions (considered in Subsection 3.1) and necessary conditions (considered in Subsection 3.2).

3.1. Sufficient conditions for remote contiguity of ER graphs

To extend the results of [6] to remotely contiguous random graphs, consider the likelihood ratio with observation Y^n, which is that of $\frac{1}{2}n(n-1)$ independent Bernoulli experiments:

$$\begin{aligned}
\frac{dP_n}{dQ_n}(Y^n) &= \prod_{1\leq i<j\leq n} \left(\frac{p_{n,ij}}{q_{n,ij}}\right)^{Y_{ij}^n} \left(\frac{1-p_{n,ij}}{1-q_{n,ij}}\right)^{1-Y_{ij}^n} \\
&= \prod_{1\leq i<j\leq n} \left(\frac{p_{n,ij}}{1-p_{n,ij}} \frac{1-q_{n,ij}}{q_{n,ij}}\right)^{Y_{ij}^n} \left(\frac{1-p_{n,ij}}{1-q_{n,ij}}\right) \\
&= \exp\left(-\sum_{i<j} \left(k_{n,ij} Y_{ij}^n + l_{n,ij}\right)\right),
\end{aligned} \tag{4}$$

where

$$k_{n,ij} = \log\left(\frac{q_{n,ij}(1-p_{n,ij})}{p_{n,ij}(1-q_{n,ij})}\right), \quad l_{n,ij} = \log\left(\frac{1-q_{n,ij}}{1-p_{n,ij}}\right).$$

Before stating the lemma, we note that the Kullback–Leibler divergences of P_n with respect to Q_n equal

$$-\mathbb{E}_{Q_n} \log \frac{dP_n}{dQ_n} = \sum_{i<j} \left(k_{n,ij} q_{n,ij} + l_{n,ij}\right) \geq 0.$$

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Lemma 1. *If we write* $\Delta_n = -\mathbb{E}_{Q_n} \log(\mathrm{d}P_n/\mathrm{d}Q_n) + \log(a_n)$*, and for every* $\epsilon > 0$ *there is an* $M > 0$ *such that, for large enough* n*,*

$$Q_n\left(\sum_{i<j} k_{n,ij}(Y^n_{ij} - q_{n,ij}) > \log(M) - \Delta_n \right) < \epsilon, \tag{5}$$

then $Q_n \lhd a_n^{-1} P_n$*.*

Proof. Consider the application of Lemma 8(ii) to the likelihood ratios (4), with $Y^n \sim Q_n$. For given $\epsilon > 0$, let $M > 0$ be as in (5); then

$$Q_n\left(\sum_{i<j} \big(k_{n,ij} Y^n_{ij} + l_{n,ij}\big) + \log(a_n) > \log(M) \right) < \epsilon,$$

which we may rewrite as $Q_n(a_n((\mathrm{d}P_n/\mathrm{d}Q_n)(Y^n))^{-1} > M) < \epsilon$ for large enough $n \geq 1$. □

Clearly, of primary concern is the connection with [6]: the following proposition illustrates how Janson's Corollary 2.12 (represented in abridged form in Theorem 1) relates to remote contiguity.

Proposition 1. *Assume that* $\sup_{i<j} p_{n,ij} < C/(1+C)$ *for some* $C > 0$*, and that Janson's condition (3) holds. Then* $Q_n \lhd a_n^{-1} P_n$ *for any* $a_n \to 0$*.*

Remark 1. The assertion of Proposition 1, $Q_n \lhd a_n^{-1} P_n$ for *any rate* a_n, is equivalent to contiguity, $Q_n \lhd P_n$, the assertion of Theorem 1.

Proof. According to (5), remote contiguity revolves around the control of tail probabilities for the sequence $\sum_{i<j} k_{n,ij}(Y^n_{ij} - q_{n,ij})$. A sufficient condition for uniform tightness is

$$\mathbb{E}_{Q_n}\left(\sum_{i<j} k_{n,ij}(Y^n_{ij} - q_{n,ij}) \right)^2 = O(1),$$

which we prove below. Note that $\mathbb{E}_{Q_n}\big(\sum_{i<j} k_{n,ij}(Y^n_{ij} - q_{n,ij})\big)^2 = \sum_{i<j} k^2_{n,ij} q_{n,ij}(1-q_{n,ij})$, and that

$$\begin{aligned} k^2_{n,ij} &\leq 4 \max\left\{ \left| \log\left(\frac{q_{n,ij}}{p_{n,ij}}\right)\right|, \left| \log\left(\frac{1-q_{n,ij}}{1-p_{n,ij}}\right)\right| \right\}^2 \\ &\leq 4\left(\log\left(\frac{q_{n,ij}}{p_{n,ij}}\right)\right)^2 + 4\left(\log\left(\frac{1-q_{n,ij}}{1-p_{n,ij}}\right)\right)^2; \end{aligned}$$

using the inequality $(\log(y))^2 \leq (y-1)^2/y$ for $y > 0$ either with $y = q_{n,ij}/p_{n,ij}$ or with $y = (1-q_{n,ij})/(1-p_{n,ij})$, we obtain

$$\begin{aligned} k^2_{n,ij} &\leq 4\left(\frac{q_{n,ij}}{p_{n,ij}} - 1 \right)^2 \frac{p_{n,ij}}{q_{n,ij}} + 4\left(\frac{1-q_{n,ij}}{1-p_{n,ij}} - 1 \right)^2 \frac{1-p_{n,ij}}{1-q_{n,ij}} \\ &= 4\frac{(p_{n,ij} - q_{n,ij})^2}{p_{n,ij}} \frac{1}{q_{n,ij}} + 4\frac{(p_{n,ij} - q_{n,ij})^2}{1-p_{n,ij}} \frac{1}{1-q_{n,ij}}. \end{aligned}$$

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Substituting and using that $\sup_{i<j} p_{n,ij} < C/(1+C)$, we find

$$\begin{aligned}\sum_{i<j} k_{n,ij}^2 q_{n,ij}(1-q_{n,ij}) &\le 4\sum_{i<j}\frac{(p_{n,ij}-q_{n,ij})^2}{p_{n,ij}}(1-q_{n,ij}) + 4\sum_{i<j}\frac{(p_{n,ij}-q_{n,ij})^2}{1-p_{n,ij}}q_{n,ij}\\ &\le 4(1+C)\sum_{i<j}\frac{(p_{n,ij}-q_{n,ij})^2}{p_{n,ij}},\end{aligned}$$

and under Janson's condition (3) the term on the right-hand side is $O(1)$. In addition, in view of the inequality $\log(y) \le (y-1)$ for $y>0$, we have

$$\begin{aligned}-\mathbb{E}_{Q_n}\log\frac{dP_n}{dQ_n} &= \sum_{i<j}\log\left(\frac{q_{n,ij}}{p_{n,ij}}\right)q_{n,ij} + \sum_{i<j}\log\left(\frac{1-q_{n,ij}}{1-p_{n,ij}}\right)(1-q_{n,ij})\\ &\le \sum_{i<j}\left(\frac{q_{n,ij}}{p_{n,ij}}-1\right)q_{n,ij} + \sum_{i<j}\left(\frac{1-q_{n,ij}}{1-p_{n,ij}}-1\right)(1-q_{n,ij})\\ &= \sum_{i<j}\frac{(q_{n,ij}-p_{n,ij})^2}{p_{n,ij}(1-p_{n,ij})} \le (1+C)\sum_{i<j}\frac{(q_{n,ij}-p_{n,ij})^2}{p_{n,ij}}.\end{aligned}$$

Under Janson's condition (3) the term on the right-hand side is $O(1)$. Consequently, $-\Delta_n \to \infty$ *for any* $a_n \to 0$. Together with the uniform tightness of the sequence of sums $\sum_{i<j}(k_{n,ij}Y_{ij}^n + l_{n,ij})$, this leads to remote contiguity $Q_n \lhd a_n^{-1}P_n$ for any $a_n \to 0$. □

This contiguity proof, however, does not exploit the presence of a sum of independent components to the full extent. To sharpen the argument, we normalize the sums appropriately and impose sufficient conditions for remote contiguity. In this case the contributions to sums are independent, and the appropriate normalization constants are

$$s_n^2 = \sum_{i<j} k_{n,ij}^2 q_{n,ij}(1-q_{n,ij})$$

for all $n \ge 1$, as per Lindeberg's theorem. To illustrate how conditions for remote contiguity weaken those for contiguity, we note that Janson's condition (3) implies that s_n^2 remains bounded, whereas in Lemma 2 s_n^2 may diverge.

Lemma 2. *Assume that* $s_n < \infty$ *for every* $n \ge 1$*, and* $s_n \to \infty$*. Suppose that, for every* $\epsilon > 0$*,*

$$\frac{1}{s_n^2}\sum_{i<j}\mathbb{E}_{Q_n}\big(k_{n,ij}^2(Y_{ij}^n - q_{n,ij})^2 \mathbf{1}_{\{k_{n,ij}|Y_{ij}^n - q_{n,ij}|>\epsilon s_n\}}\big) \to 0. \tag{6}$$

Then $Q_n \lhd a_n^{-1}P_n$ *for any* (a_n)*,* $a_n \downarrow 0$*, such that*

$$\frac{1}{s_n}\left(\sum_{i<j}\big(k_{n,ij}q_{n,ij} + l_{n,ij}\big) + \log(a_n)\right) \to -\infty. \tag{7}$$

Proof. Apply the Lindeberg–Feller condition of Theorem 5 to (5), s_n-normalized sums converge weakly to the standard normal distribution if (6) holds. Note that, under condition (7), we have $-\Delta_n/s_n \to \infty$, and we conclude that, for every $\epsilon > 0$ and any choice for $M > 0$, condition (5) holds for large enough n. □

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3.2. Necessary conditions for remote contiguity of ER graphs

Next, we argue that, when (6) holds and $s_n \to \infty$, then (7) is also a necessary condition for remote contiguity at rate a_n.

Lemma 3. *Assume that $s_n < \infty$ for every $n \geq 1$ and that $s_n \to \infty$, and suppose that (6) is satisfied for every $\epsilon > 0$. Then (7) is necessary for $Q_n \triangleleft a_n^{-1} P_n$ to hold.*

Proof. For every $n \geq 1$, let ν_n be a measure that dominates both P_n and Q_n (e.g. $\nu_n = (P_n + Q_n)/2$) and define $p_n = \mathrm{d}P_n/\mathrm{d}\nu_n$ and $q_n = \mathrm{d}Q_n/\mathrm{d}\nu_n$. Let (a_n), (b_n) be such that $a_n, b_n > 0$, $a_n, b_n \downarrow 0$, and $s_n^{-1} \log(b_n) \to 0$. Define $C_n = \{y^n \in \mathscr{X}_n : q_n(y^n) > p_n(y^n)/(a_n b_n)\}$. It is immediate that $P_n(C_n) \leq \int_{C_n} a_n b_n q_n(y^n)\, \mathrm{d}\nu_n(y^n) \leq a_n b_n Q_n(C_n) = o(a_n)$. For any $\eta > 0$ and n large enough, we find

$$
\begin{aligned}
Q_n(C_n) &= Q_n\left(a_n b_n \left(\frac{\mathrm{d}P_n}{\mathrm{d}Q_n}(Y_n) \right)^{-1} > 1 \right) \\
&= Q_n\left(\frac{\sum_{i<j} k_{n,ij}(Y_{ij}^n - q_{n,ij})}{s_n} > \frac{-\log(b_n)}{s_n} - \frac{\Delta_n}{s_n} \right) \\
&\geq Q_n\left(\frac{\sum_{i<j} k_{n,ij}(Y_{ij}^n - q_{n,ij})}{s_n} > \eta - \frac{\Delta_n}{s_n} \right),
\end{aligned}
$$

since $-\log(b_n)/s_n \to 0$. If (7) does not hold, $\lim_{n\to\infty} \Delta_n/s_n > -\infty$, so that, for some $M > 0$ and all n large enough,

$$
Q_n(C_n) \geq Q_n\left(\frac{\sum_{i<j} k_{n,ij}(Y_{ij}^n - q_{n,ij})}{s_n} > \eta + M \right).
$$

By Theorem 5, $\sum_{i<j} k_{n,ij}(Y_{ij}^n - q_{n,ij})/s_n$ converges weakly to a standard normal distribution. Then $\liminf_{n\to\infty} Q_n(C_n) > 0$ and (Q_n) is not a_n-remotely contiguous with respect to (P_n). We conclude that (7) is necessary for remote contiguity at rate (a_n). □

To estimate whether sequential data (Y^n), $Y^n \in \mathscr{X}_n$, was generated by (P_n) or by (Q_n), statisticians use (randomized tests based on) test functions $\phi_n : \mathscr{X}_n \to [0, 1]$. A test function is considered (minimax-)optimal if it minimizes the sum of type-I and type-II errors:

$$
\pi_n(\phi_n) = \mathbb{E}_{P_n} \phi_n(Y^n) + \mathbb{E}_{Q_n}(1 - \phi_n(Y^n)).
$$

As it turns out, there is a general upper bound for π_n in terms of the Hellinger affinity $\alpha(P_n, Q_n)$ between P_n and Q_n:

$$
\inf_{\phi} \pi_n(\phi) \leq \int_{\mathscr{X}_n} \sqrt{p_n(y^n) q_n(y^n)}\, \mathrm{d}\nu_n(y^n) \quad (=: \alpha(P_n, Q_n)),
$$

and the likelihood ratio test function minimizes π_n (see also [9, Section 16.4]). Clearly, if there exists a sequence (ϕ_n) such that $\pi_n(\phi_n) = o(1)$, then (P_n) and (Q_n) are not contiguous.

To reason likewise regarding a_n-remote contiguity, consider an a_n-weighted version of π_n, $\pi_n'(\phi_n) = a_n^{-1} \mathbb{E}_{P_n} \phi_n(Y^n) + \mathbb{E}_{Q_n}(1 - \phi_n(Y^n))$. Reasoning the same as in the contiguous case, we find the following correspondence between testability and remote contiguity.

Lemma 4. *If there exists a sequence (ϕ_n) such that $\pi_n'(\phi_n) = o(1)$, then the sequences (P_n) and (Q_n) are not a_n-remotely contiguous. This is the case whenever $\alpha(P_n, Q_n) = o(a_n^{1/2})$.*

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Proof. For every $n \geq 1$, the likelihood ratio test function, defined for all $y^n \in \mathscr{X}_n$ by $\psi_n(y^n) = \mathbf{1}_{\{q_n > a_n\, p_n\}}(y^n)$, minimizes π_n'. Suppose that there exists a sequence (ϕ_n) such that $\pi_n'(\phi_n) = o(1)$. Then $\pi_n'(\psi_n) = o(1)$, so there are events $A_n = \{y^n : q_n(y^n) > a_n\, p_n(y^n)\}$ such that $P_n(A_n) = o(a_n)$, but $Q_n(A_n) \to 1$, showing that (P_n) and (Q_n) are not a_n-remotely contiguous. We note the following upper bound for π_n':

$$\begin{aligned}\pi_n'(\psi_n) = \inf_{\phi} \pi_n'(\phi) &= \int_{\{q_n > a_n\, p_n\}} p_n(y^n)\, \mathrm{d}\nu_n(y^n) + \int_{\{q_n \leq a_n\, p_n\}} q_n(y^n)\, \mathrm{d}\nu_n(y^n) \\ &\leq \int_{\{q_n > a_n\, p_n\}} \sqrt{a_n^{-1}\, p_n(y^n) q_n(y^n)}\, \mathrm{d}\nu_n(y^n) \\ &\quad + \int_{\{q_n \leq a_n\, p_n\}} \sqrt{a_n\, p_n(y^n) q_n(y^n)}\, \mathrm{d}\nu_n(y^n) \leq a_n^{-1/2}\, \alpha(P_n, Q_n),\end{aligned}$$

where the last bound holds for large enough n. □

In the proof of Lemma 3 we change the argument of Lemma 4 slightly (through the inclusion of a separate sequence $b_n \downarrow 0$), but the essence is the same: the existence of certain test sequences precludes remote contiguity.

In the case of n-vertex ER graph distributions, the Hellinger affinity is equal to the product of the Hellinger affinities for each of the independent, Bernoulli-distributed random variables Y_{ij}^n:

$$\alpha(P_n, Q_n) = \prod_{i<j} \left(\sqrt{p_{n,ij}\, q_{n,ij}} + \sqrt{(1 - p_{n,ij})(1 - q_{n,ij})} \right). \tag{8}$$

3.3. Perturbations of ER graphs

Lemma 2 formulates a condition that delimits the range of applicability for remote contiguity in terms of a Lindeberg-type condition involving the sequence of Kullback–Leibler divergences. In this section we simplify that condition with sufficient conditions formulated directly in terms of the defining parameters of the ER graphs.

Lemma 5. *Choose $q_{n,ij} = \lambda_{n,ij}/n$, $p_{n,ij} = \mu_{n,ij}/n$ with $0 \leq \lambda_{n,ij}, \mu_{n,ij} \leq n$, and define $P_n = P_{(p_{n,ij}),n}$, $Q_n = P_{(q_{n,ij}),n}$ for all $n \geq 1$ and all $1 \leq i, j \leq n$. Assume that*

$$r_n := \sup_{i<j} \frac{|\mu_{n,ij} - \lambda_{n,ij}|}{\mu_{n,ij}} \to 0, \qquad R_n := \sum_{i<j} \frac{(\lambda_{n,ij} - \mu_{n,ij})^2}{\mu_{n,ij}(n - \mu_{n,ij})} \to \infty. \tag{9}$$

Then $Q_n \triangleleft a_n^{-1} P_n$ if and only if $a_n = o(\exp(-R_n))$. If, instead, $R_n = O(1)$, then $Q_n \triangleleft P_n$.

Proof. Let $n \geq 1$ be given. Using the inequality $\log(1 + x) \leq x$, valid for any $x > -1$, we estimate the Kullback–Leibler divergence as follows:

$$\begin{aligned}-\mathbb{E}_{Q_n} \log \frac{\mathrm{d}P_n}{\mathrm{d}Q_n}(Y^n) &= \sum_{i<j} \log\left(\frac{q_{n,ij}}{p_{n,ij}}\right) q_{n,ij} + \sum_{i<j} \log\left(\frac{1 - q_{n,ij}}{1 - p_{n,ij}}\right)(1 - q_{n,ij}) \\ &= \sum_{i<j} \left[\frac{\lambda_{n,ij}}{n} \log\left(1 + \frac{\lambda_{n,ij} - \mu_{n,ij}}{\mu_{n,ij}}\right) \right. \\ &\quad \left. + \sum_{i<j} \log\left(1 - \frac{(\lambda_{n,ij} - \mu_{n,ij})/n}{1 - \mu_{n,ij}/n}\right)\left(1 - \frac{\lambda_{n,ij}}{n}\right) \right]\end{aligned}$$

$$\leq \sum_{i<j} \frac{\lambda_{n,ij} - \mu_{n,ij}}{n} \left(\frac{\lambda_{n,ij}}{\mu_{n,ij}} - \frac{1 - \lambda_{n,ij}/n}{1 - \mu_{n,ij}/n} \right)$$
$$= \sum_{i<j} \left(1 + \frac{\mu_{n,ij}/n}{1 - \mu_{n,ij}/n} \right) \frac{(\lambda_{n,ij} - \mu_{n,ij})^2}{n\mu_{n,ij}} = \sum_{i<j} \frac{(\lambda_{n,ij} - \mu_{n,ij})^2}{\mu_{n,ij}(n - \mu_{n,ij})}.$$

Note that, by (2), (9), and the expansion $\log(1+x) = x + O(x^2)$ $(x \to 0)$, we have

$$\begin{aligned} k_{n,ij}^2 &= \left(\log \left(\frac{\lambda_{n,ij}}{\mu_{n,ij}} \frac{1 - \mu_{n,ij}/n}{1 - \lambda_{n,ij}/n} \right) \right)^2 \\ &= \left(\log \left(1 + \frac{\lambda_{n,ij} - \mu_{n,ij}}{\mu_{n,ij}} \right) - \log \left(1 - \frac{(\lambda_{n,ij} - \mu_{n,ij})/n}{1 - \mu_{n,ij}/n} \right) \right)^2 \\ &= \left(\frac{\lambda_{n,ij} - \mu_{n,ij}}{\mu_{n,ij}} \right)^2 \left(\frac{1}{1 - \mu_{n,ij}/n} + O(r_n) \right)^2 \end{aligned}$$

for all $1 \leq i < j \leq n$, and

$$\begin{aligned} q_{n,ij}(1 - q_{n,ij}) &= \frac{\lambda_{n,ij}}{n} \left(1 - \frac{\lambda_{n,ij}}{n} \right) \\ &= \frac{\mu_{n,ij}}{n} \left(1 + \frac{\lambda_{n,ij} - \mu_{n,ij}}{\mu_{n,ij}} \right) \left(1 - \frac{\mu_{n,ij}}{n} \left(1 + \frac{\lambda_{n,ij} - \mu_{n,ij}}{\mu_{n,ij}} \right) \right) \\ &= \frac{\mu_{n,ij}}{n} \left(1 - \frac{\mu_{n,ij}}{n} + O(r_n) \right). \end{aligned}$$

As a consequence of these two displays, we have

$$\begin{aligned} s_n^2 &= \sum_{i<j} k_{n,ij}^2 q_{n,ij}(1 - q_{n,ij}) \\ &= \sum_{i<j} \left(\frac{\lambda_{n,ij} - \mu_{n,ij}}{\mu_{n,ij}} \right)^2 \left(\frac{1}{1 - \mu_{n,ij}/n} + O(r_n) \right)^2 \frac{\mu_{n,ij}}{n} \left(1 - \frac{\mu_{n,ij}}{n} + O(r_n) \right) \\ &= \sum_{i<j} \frac{(\lambda_{n,ij} - \mu_{n,ij})^2}{\mu_{n,ij}(n - \mu_{n,ij})} \left(1 + \left(1 - \frac{\mu_{n,ij}}{n} \right) O(r_n) \right)^2 (1 + O(r_n)) \\ &= (1 + o(1)) \sum_{i<j} \frac{(\lambda_{n,ij} - \mu_{n,ij})^2}{\mu_{n,ij}(n - \mu_{n,ij})}. \end{aligned}$$

Condition (7) is then satisfied for any a_n such that $\limsup_{n\to\infty} R_n^{-1} \log(a_n) < -1$. It remains to verify condition (6). To that end, consider, for any ϵ, $\delta > 0$,

$$\begin{aligned} &\frac{1}{s_n^2} \sum_{i<j} \mathbb{E}_{Q_n} \left(k_{n,ij}^2 (Y_{ij}^n - q_{n,ij})^2 \mathbf{1}_{\{k_{n,ij}|Y_{ij}^n - q_{n,ij}| > \epsilon s_n\}} \right) \\ &\qquad \leq \frac{1}{\epsilon^\delta s_n^{2+\delta}} \sum_{i<j} \mathbb{E}_{Q_n} |k_{n,ij}|^{2+\delta} |Y_{ij}^n - q_{n,ij}|^{2+\delta} \\ &\qquad = \frac{1}{\epsilon^\delta s_n^{2+\delta}} \sum_{i<j} |k_{n,ij}|^{2+\delta} q_{n,ij}(1 - q_{n,ij}) \left((1 - q_{n,ij})^{1+\delta} + q_{n,ij}^{1+\delta} \right) \end{aligned}$$

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$$\leq \frac{1}{\epsilon^\delta s_n^{2+\delta}} \|k_n\|_{\infty,n}^\delta \sum_{i<j} k_{n,ij}^2 q_{n,ij}(1-q_{n,ij})$$
$$= \frac{1}{\epsilon^\delta}\left(\frac{\|k_n\|_{\infty,n}}{s_n}\right)^\delta \leq O\left(\frac{r_n}{R_n^{1/2}}\right)^\delta \to 0,$$

implying that condition (6) is satisfied. All the assumptions of Lemma 2 are thus fulfilled and an application of the latter, along with Lemma 3, yields the first result.

If we replace the hypothesis $R_n \to \infty$ with $R_n = O(1)$, we have

$$-\mathbb{E}_{Q_n} \log \frac{\mathrm{d}P_n}{\mathrm{d}Q_n}(Y^n) = O(R_n) = O(1).$$

The hypothesis $R_n = O(1)$ also implies that $s_n^2 = (1+o(1))R_n = O(1)$ and, in turn, uniform tightness of the sequence $\sum_{i<j} k_{n,ij}(Y_{ij}^n - q_{n,ij})$, so that $Q_n \triangleleft P_n$. □

Remark 2. The uniform convergence assumption $r_n \to 0$ in (9) is not strictly necessary to have $a_n^{-1}P_n \triangleright Q_n$ for all rate sequences $a_n = o(\exp(-R_n))$. At the cost of a more involved proof, the results of Lemma 5 can be extended to more general cases such as where r_n is suitably bounded from above but not necessarily decaying to 0. However, assuming $r_n \to 0$ does not appear overly restrictive, since it does not preclude, for example, increasing variance of log-likelihood ratios $\log(\mathrm{d}P_n/\mathrm{d}Q_n)$ (unlike (3)) and diverging L_p distances $\|\mu_n - \lambda_n\|_{n,p}$, with $1 \leq p < \infty$.

The following results give examples of applications of the previous lemma to inhomogeneous and homogeneous perturbations of a homogeneous ER graph.

Corollary 1. (Homogeneous perturbation of the ER graph.) *Choose $q_{n,ij} = \lambda_n/n$, $p_{n,ij} = \lambda/n$ with $0 \leq \lambda_n \leq n$, $1 \leq i, j \leq n$, and define $P_n = P_{(p_{n,ij}),n}$, $Q_n = P_{(q_{n,ij}),n}$ for all $n \geq 1$. Assume that $\lambda_n \to \lambda$. If $n(\lambda - \lambda_n)^2 \to \infty$, then $Q_n \triangleleft a_n^{-1}P_n$ if and only if*

$$a_n = o\left(\exp\left(-\frac{n}{2\lambda}(\lambda_n - \lambda)^2\right)\right).$$

If instead $n(\lambda_n - \lambda)^2 = O(1)$, then $Q_n \triangleleft P_n$.

Proof. We have $r_n = |\lambda_n - \lambda|/\lambda$ and, as $n \to \infty$,

$$R_n = \binom{n}{2}\frac{(\lambda_n - \lambda)^2}{\lambda(n-\lambda)} = \frac{1}{2}(1 + O(n^{-1}))\frac{n(\lambda_n - \lambda)^2}{\lambda}.$$

Then the result follows immediately from Lemma 5. □

Corollary 2. *(Inhomogeneous perturbation of the ER graph.) Choose $q_{n,ij} = \lambda_{n,ij}/n$, $p_{n,ij} = \lambda/n$ with $0 \leq \lambda_n \leq n$, $1 \leq i, j \leq n$, and define $P_n = P_{(p_{n,ij}),n}$, $Q_n = P_{(q_{n,ij}),n}$ for all $n \geq 1$. Assume that $\|\lambda_n - \lambda\|_{\infty,n} = o(1)$. If also $n^{-1}\|\lambda_n - \lambda\|_{2,n}^2 \to \infty$, then $Q_n \triangleleft a_n^{-1}P_n$ if and only if*

$$a_n = o\left(\exp\left(-\frac{\|\lambda_n - \lambda\|_{2,n}^2}{n\lambda}\right)\right).$$

If instead $n^{-1}\|\lambda_n - \lambda\|_{2,n}^2 = O(1)$, then $Q_n \triangleleft P_n$.

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Proof. The result follows immediately from Lemma 5 when we note that $r_n = \|\lambda_n - \lambda\|_{n,\infty}/\lambda$ and that

$$R_n = \frac{\|\lambda_n - \lambda\|_{2,n}^2}{\lambda(n-\lambda)} = (1 + O(n^{-1}))\frac{\|\lambda_n - \lambda\|_{2,n}^2}{n\lambda} \quad \text{as } n \to \infty. \qquad \square$$

4. Connectivity properties of ER graphs

In this section we collect several well-known properties of ER graphs and examine their so-called *remotely contiguous domains of attraction*, the families of perturbed homogeneous and inhomogeneous ER graphs that maintain connectivity properties like the occurrence of a giant component, $n^{2/3}$-scaling of the giant component at criticality, $O(\log(n))$-fragmentation, or full asymptotic connectedness through remote contiguity.

4.1. The giant component in the supercritical ER graph

As was first shown in [2], every supercritical ER graph contains a *giant component*, i.e. a sequence of connected components in X^n containing a non-vanishing fraction of all vertices, with probability growing to 1. More precisely, we have the following theorem.

Theorem 2. *For every $\lambda > 1$, $\nu \in \left(\frac{1}{2}, 1\right)$ there is a $\delta(\lambda, \nu) > 0$ such that*

$$P_{\lambda,n}\left(\big||\mathscr{C}_{\max}| - \zeta_\lambda n\big| > n^\nu\right) = O\left(n^{-\delta(\lambda,\nu)}\right), \tag{10}$$

where ζ_λ is the survival probability of a Poisson branching process with mean offspring λ.

For a proof of this classical result (and the specific way in which $\delta(\lambda, \nu)$ depends on λ and ν) see, for example, [13, Theorem 4.8].

To generalize the occurrence of a giant component, we set conditions for sequences of laws (Q_n) of ER graphs such that, for some $0 < \delta < \delta(\lambda, \nu)$,

$$Q_n \lhd n^\delta P_{\lambda,n}. \tag{11}$$

Let $\mathscr{Q}(\lambda, \delta)$ denote the collection of all sequences (Q_n) in $\prod_n M^1(\mathscr{X}_n)$ satisfying (11). In all those cases, a giant component containing an asymptotic fraction ζ_λ of all vertices occurs (to within order n^ν vertices). For given $\lambda > 1$ and $\nu \in \left(\frac{1}{2}, 1\right)$, the sequences (Q_n) for which (11) holds for some $0 < \delta < \delta(\lambda, \nu)$ is the union $\mathscr{Q}(\lambda, \nu) = \bigcup\{\mathscr{Q}(\lambda, \delta) : 0 < \delta < \delta(\lambda, \nu)\}$ and, for given $\lambda > 1$, the union over all ν contains the cases in which a giant component containing a fraction ζ_λ occurs (to within some negligible n^ν-fraction of the vertices). We call $\mathscr{Q}(\lambda) = \cup\left\{\mathscr{Q}(\lambda, \nu) : \nu \in \left(\frac{1}{2}, 1\right)\right\}$ the *remotely contiguous domain of attraction for the occurrence of a giant component containing $\zeta_\lambda n$ vertices*. Ultimately, the union $\mathscr{Q} = \cup\{\mathscr{Q}(\lambda) : \lambda > 1\}$ forms a class in which a giant component (containing *some* asymptotically non-vanishing fraction of all vertices) will form with probability growing to 1. We refer to that class as the *remotely contiguous domain of attraction for the occurrence of a giant component*.

Example 1. (*Homogeneous perturbation of supercritical ER graphs.*) For some $\lambda > 1$ choose $P_n = P_{\lambda,n}$ and $Q_n = P_{\lambda_n,n}$, with $\lambda_n \to \lambda$. For any choice $\frac{1}{2} < \nu < 1$, let $0 < \delta < \delta(\lambda, \nu)$ as in Theorem 2 be given.

As we have seen in Corollary 1, $R_n = \frac{1}{2}n\lambda^{-1}(1 + O(n^{-1}))(\lambda_n - \lambda)^2$, so to render the rate a_n in definition (1) high enough to cover the probabilities for occurrence of a giant component,

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cf. Theorem 2, we choose

$$\lambda_n = \left(1 + \sqrt{\frac{2\delta}{\lambda}\frac{\log(n)}{n}}\right)\lambda, \tag{12}$$

so that $R_n = (1 + O(n^{-1}))\delta \log(n)$. Then $n(\lambda_n - \lambda)^2 \to \infty$ and $n^{-\delta(\lambda,\nu)} = o(\exp(-R_n))$. Therefore, by Corollary 1, we can conclude that, for all homogeneous λ_n-perturbations of the ER graph sequence of the form (12), a giant component containing an asymptotic fraction ζ_λ of the vertices occurs to within order-n^ν vertices, $P_{\lambda_n,n}\big(\big||\mathscr{C}_{\max}| - \zeta_\lambda n\big| > n^\nu\big) = o(1)$, since (10) ensures the occurrence of such a giant component at $\lambda > 1$. We may therefore characterize the class of homogeneous ER graphs in the remotely contiguous domain of attraction for the occurrence of a giant component containing $\zeta_\lambda n$ vertices as

$$\mathscr{H} \cap \mathscr{Q}(\lambda) = \bigcup_{\nu\in(1/2,1)} \left\{ (P_{\lambda_n,n}) \in \mathscr{H}\colon (\lambda_n - \lambda)^2 < 2\lambda\,\delta(\lambda,\nu)\frac{\log(n)}{n},\ n \geq 1 \right\},$$

and the class of homogeneous ER graphs in the remotely contiguous domain of attraction for the occurrence of a giant component as $\mathscr{H} \cap \mathscr{Q} = \bigcup\{\mathscr{H} \cap \mathscr{Q}(\lambda)\colon \lambda > 1\}$.

Example 2. (*Inhomogeneous perturbation of supercritical ER graphs.*) Denote by $\mathscr{I}_\infty$ the class of inhomogeneous ER graphs obtained via *uniform* perturbations of a homogeneous ER graph $P_{\lambda,n}$, with $\lambda > 0$, i.e. the class of ER graphs with edge probabilities $q_{n,ij} = \lambda_{n,ij}/n$ satisfying $\sup_{i<j}|\lambda_{n,ij} - \lambda| \to 0$ for some $\lambda > 0$. In an analogous fashion, resorting to Corollary 2, we can characterize the class of uniformly perturbed inhomogeneous ER graphs in $\mathscr{Q}(\lambda)$, with $\lambda > 1$, as

$$\mathscr{I}_\infty \cap \mathscr{Q}(\lambda) = \bigcup_{\nu\in(1/2,1)} \left\{ (P_{(\lambda_{n,ij}),n}) \in \mathscr{I}_\infty\colon \sum_{i<j}(\lambda_{n,ij} - \lambda)^2 < \lambda\,\delta(\lambda,\nu)\,n\log(n),\ n \geq 1 \right\},$$

and the class of uniformly perturbed inhomogeneous ER graphs in the remotely contiguous domain of attraction for the occurrence of a giant component as

$$\mathscr{I}_\infty \cap \mathscr{Q} = \bigcup \big\{\mathscr{I}_\infty \cap \mathscr{Q}(\lambda)\colon \lambda > 1\big\}.$$

4.2. Fragmentation in subcritical ER graphs

Define $I_\lambda = \lambda - 1 - \log(\lambda)$ for $0 < \lambda < 1$, that is, for the subcritical regime of the ER graph.

Theorem 3. *For given $0 < \lambda < 1$ and every $a > I_\lambda^{-1}$, there exists a $\delta = \delta(a, \lambda) > 0$ such that $P_{\lambda,n}\big(|\mathscr{C}_{\max}| \geq a\log(n)\big) = O(n^{-\delta})$. Moreover, for any $a < I_\lambda^{-1}$, there exists an $\eta = \eta(a, \lambda) > 0$ such that $P_{\lambda,n}\big(|\mathscr{C}_{\max}| \leq a\log(n)\big) = O(n^{-\eta})$.*

For a proof of Theorem 3, see, for example, [13, Theorems 4.4 and 4.5]. To prove that the largest connected component in other random graph sequences has cardinality lying between two multiples of $\log(n)$, we again require (11) for some $0 < \delta < \min(\delta(a,\lambda), \eta(a',\lambda)) =: \zeta(\lambda, a, a')$ and $0 < a < I_\lambda^{-1} < a'$. We thus define the remotely contiguous domain of attraction for fragmentation into clusters of maximal cardinality $I_\lambda^{-1}\log(n)$:

$$\mathscr{Q}(\lambda, a, a') = \bigcup \big\{\mathscr{Q}(\lambda,\delta)\colon 0 < \delta < \zeta(\lambda, a, a')\big\}.$$

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The union over all $0<\lambda<1$ and $0<a<I_\lambda^{<}a'<\infty$ forms the remotely contiguous domain of attraction for fragmentation into clusters of maximal cardinality of order $\log(n)$:

$$\mathscr{L}=\bigcup\left\{\mathscr{Q}(\lambda,a,a')\colon 0<\lambda<1,\ 0<a<I_\lambda^{-1}<a'<\infty\right\}.$$

Example 3. (*Homogeneous perturbation of subcritical ER graphs.*) Following reasoning similar to Example 1 and applying Corollary 1, we characterize the class of ER graphs with maximal connected component of order $\log(n)$, which are obtained by homogeneous perturbations of a subcritical graphs with $0<\lambda<1$, as

$$\mathscr{L}(\lambda)=\bigcup\left\{(P_{\lambda_n,n})\in\mathscr{H}\colon 0<a<I_\lambda^{-1}<a'<\infty,\right.$$
$$\left.(\lambda_n-\lambda)^2<2\lambda\zeta(\lambda,a,a')\frac{\log(n)}{n},\ n\geq 1\right\}.$$

And we define the homogeneous part of the remotely contiguous domain of attraction for fragmentation into clusters of maximal cardinality of order $\log(n)$ by $\mathscr{L}\cap\mathscr{H}=\cup_{0<\lambda<1}\mathscr{L}(\lambda)$.

Remark 3. Example 3 can be extended to inhomogeneous perturbations by application of Corollary 2. We leave the details to the reader.

4.3. Maximal connected components in the critical ER graph

It is well known that the largest connected components in a sequence of ER graphs at criticality ($\lambda=1$) have cardinalities of order $O(n^{2/3})$. In fact, there exists a so-called *critical window* of $O(n^{-1/3})$ homogeneous perturbations around $\lambda=1$ for which this critical behaviour of the largest connected component remains valid.

Theorem 4. *For some $\theta\in\mathbb{R}$, every $n\geq 1$, and all $1\leq i<j\leq n$, define $\lambda_{n,ij}=\lambda_n=1+\theta n^{-1/3}$. There exists a constant $b=b(\theta)$ such that $P_{\lambda_n,n}\big(a\,n^{2/3}\leq|\mathscr{C}_{\max}|\leq a^{-1}\,n^{2/3}\big)\geq 1-b\,a$ for all $a<1$.*

For a proof, see, for example, [13, Theorem 5.1].

We now examine to what extent the remotely contiguous domain of attraction for occurrence of a maximal connected component of order (approximating) $n^{2/3}$ around the critical point $\lambda=1$ coincides with the perturbations of order $n^{-1/3}$ in the parameter λ that Theorem 4 guarantees.

To reformulate the question: for some $\lambda_n\to 1$, define the homogeneous ER graphs Y_n distributed according to $Q_n=P_{\lambda_n,n}$, $P_n=P_{1,n}$, and analyse the requirement $Q_n\lhd\omega_nP_n$ for any rate $a_n=1/\omega_n$.

To render the assertion of Theorem 4 at $\lambda=1$ amenable to extension by remote contiguity, we have to make a choice for a sequence $a_n\to 0$: applied to $\lambda=1$, Theorem 4 guarantees that there exists a constant $b=b(0)>0$ such that

$$P_{1,n}\big(|\mathscr{C}_{\max}|<a_nn^{2/3}\text{ or }|\mathscr{C}_{\max}|>a_n^{-1}n^{2/3}\big)\leq b\,a_n. \tag{13}$$

We examine the family of perturbed ER graphs that displays the same a_n-adjusted critical maximal cluster size of order $n^{2/3}$.

The choice for (a_n) is of great influence on the maximal permitted perturbation $|\lambda_n-1|$.

Lemma 6. *Let $\lambda_n\to 1$ as $n\to\infty$, such that $\lambda_n-1=O(n^{-1/3})$; then there exists a constant $A>0$ such that $\alpha(P_{\lambda_n,n},P_{1,n})=A\exp\big(-\frac{1}{16}n(\lambda_n-1)^2\big)+o(1)$.*

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Proof. For $Q_n = P_{\lambda_n,n}$, $P_n = P_{1,n}$, the Hellinger affinity (cf. (8)) is given by

$$
\begin{aligned}
\alpha(P_n, Q_n) &= \left(\frac{\lambda_n^{1/2}}{n} + \left(1 - \frac{1}{n}(\lambda_n + 1) + \frac{\lambda_n}{n^2}\right)^{1/2}\right)^{\binom{n}{2}} \\
&= \left(1 + \frac{(1 + (\lambda_n - 1))^{1/2}}{n} - \frac{\lambda_n + 1}{2n} + O(n^{-2})\right)^{\binom{n}{2}} \\
&= \left(1 + \frac{1 + \frac{1}{2}(\lambda_n - 1) - \frac{1}{8}(\lambda_n - 1)^2}{n} - \frac{\lambda_n + 1}{2n} + O(n^{-1}(\lambda_n - 1)^3) + O(n^{-2})\right)^{\binom{n}{2}} \\
&= \left(1 - \frac{(\lambda_n - 1)^2}{8n} + O(n^{-1}(\lambda_n - 1)^3) + O(n^{-2})\right)^{\binom{n}{2}} \\
&= \exp\left(-\frac{1}{16}n(\lambda_n - 1)^2 + O(n(\lambda_n - 1)^3) + O(1)\right) \\
&= A\exp\left(-\frac{1}{16}n(\lambda_n - 1)^2\right) + o(1)
\end{aligned}
$$

for some constant $A > 0$. □

A slightly more detailed version of this proof shows that, whenever $\lambda_n - 1 = o(n^{-1/2})$, the representation of $\alpha(P_{\lambda_n,n}, P_{1,n})$ in Lemma 6 holds with $A = 1$ and therefore $P_{\lambda_n,1} \lhd P_{1,n}$, in which case remote contiguity applies with any a_n decaying to 0. For example, for some small $\epsilon > 0$ and the choice $a_n = n^{-\epsilon}$, we find that

$$
P_{\lambda_n,n}\left(|\mathscr{C}_{\max}| < n^{2/3-\epsilon} \text{ or } |\mathscr{C}_{\max}| > n^{2/3+\epsilon}\right) \to 0. \tag{14}
$$

On the other hand, in light of Lemma 3, if $n|\lambda_n - 1|^2$ goes to ∞ fast enough, that is, if $\alpha_n(P_{\lambda_n,1}, P_{1,n}) = o(a_n^{1/2})$, (P_{λ_n}) is not a_n-remotely contiguous with respect to $(P_{1,n})$. For example, for some small $\epsilon > 0$ and the choice $a_n = n^{-\epsilon}$, we find that if

$$
\liminf_{n\to\infty} \frac{n(\lambda_n - 1)^2}{\log(n)} > 8\epsilon
$$

then $(P_{\lambda_n,n})$ is not $n^{-\epsilon}$-remotely contiguous with respect to $(P_{1,n})$. An application of Corollary 1 allows us to further refine the requirement by imposing $a_n = o(\exp(-n(\lambda_n - 1)^2/2))$. For example, if we choose a_n to decrease as $\log(n)^{-1}$, then the above shows that remote contiguity limits the perturbation to be of smaller order than $\sqrt{\log(\log(n))/n}$.

Unfortunately, remotely contiguous domains of attraction for near-critical maximal cluster sizes (for example, those intended in (14)) have an extent of order $(n^{-1}\log(n))^{1/2}$, not the order $n^{-1/3}$ that occurs in Theorem 4. So remote contiguity does not cover the entire range of possible perturbations that preserve near-critical maximal cluster sizes. If we impose perturbations proportional to $n^{-1/3}$, requiring remote contiguity leads to exponential rates $a_n \sim \exp(-n^{1/3})$, which overwhelms the polynomial factor in assertion (13).

This illustrates a limitation that is important to point out: remote contiguity makes no distinction between asymptotic assertions, other than by rate: as long as the probabilities $P_n(A_n)$ converge to zero fast enough, cf. (1), remote contiguity asserts $Q_n(A_n) = o(1)$ without regard for the further details involved in the definition of the events A_n. In the case at hand, when we ask questions regarding the size of the maximal cluster, there are properties very specific to

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homogeneous ER graphs at criticality that enable $n^{-1/3}$-proportionality of the critical window. Lemmas 4 and 6 demonstrate that there are other asymptotic assertions (B_n) with probabilities $P_{1,n}(B_n)$ of order $o(a_n)$ but with probabilities $P_{1+O(n^{-1/3}),n}(B_n)$ that do not go to zero.

4.4. Asymptotic connectedness in ER graphs

Recall that any homogeneous ER graph with edge probability λ_n/n is disconnected with high probability if $\limsup_{n\to\infty} \lambda_n < \infty$ (see, e.g. [13, Section 5.3]). The results of this subsection apply to ER graphs with diverging (λ_n), typically of $O(\log(n))$.

Lemma 7. *Let $\lambda_n \to \infty$ as $n \to \infty$. If $\lambda_n - \log(n) \to -\infty$, then $P_{\lambda_n,n}(\mathscr{C}_{\max}$ is connected$) = O(\lambda_n/(n-\lambda_n)) = o(1)$. If, instead, $\lambda_n - \log(n) \to \infty$, then $P_{\lambda_n,n}(\mathscr{C}_{\max}$ is disconnected$) = O(n^{-1/4})$.*

Proof. The first result is a direct consequence of the first inequality in [13, Proposition 5.10, (5.3.25), (5.3.26)]. As for the second result, with $\lambda_n^* = \min(\lambda_n, 2\log(n))$,

$$P_{\lambda_n,n}(\mathscr{C}_{\max} \text{ is disconnected}) \le 1 - P_{\lambda_n^*,n}(\mathscr{C}_{\max} \text{ is connected}).$$

The conclusion follows by using [13, (5.3.14), (5.3.21)–(5.3.24), (5.3.27)] with $\lambda = \lambda_n^*$. □

Example 4. (*Connectivity in inhomogeneous ER graphs.*) Consider an inhomogeneous ER graph with edge probabilities $q_{n,ij} = c_{n,ij}\log(n)/n$. A sufficient condition for such a graph to be asymptotically connected is the existence of a suitable sequence (d_n) with $d_n > 0$, $\liminf_{n\to\infty} d_n > 1$, and $\lim_{n\to\infty} d_n \log(n)/n < 1$. To see this, also assume that

$$\sup_{i<j}\left|\frac{c_{n,ij}}{d_n} - 1\right| \to 0, \qquad \limsup_{n\to\infty} \frac{\sum_{i<j}(c_{n,ij} - d_n)^2}{d_n(n-\log(n))} < \frac{1}{4}.$$

Then the assumptions $r_n = o(1)$ and $a_n = o(\exp(-R_n))$ of Lemma 5 are satisfied, with $\mu_{n,ij} = d_n \log(n)$ and $a_n = n^{-\delta}$ for some $\delta < \frac{1}{4}$. Hence, $Q_n \lhd n^{\delta} P_n$, where P_n is the distribution of the homogeneous ER graph with edge probability $d_n \log(n)/n$. By Lemma 7, the latter has a probability of not being connected of order $O(n^{-1/4})$, thus entailing that

$$Q_n(\mathscr{C}_{\max} \text{ is disconnected}) = o(1)$$

as $n \to \infty$ by remote contiguity.

5. Conclusions and discussion

We have attempted to highlight how remote contiguity can be used to generalize asymptotic properties, much like asymptotic equivalence and contiguity, but with a wider range of applicability. In particular, we have shown that remote contiguity can be applied to the connectivity properties of ER graphs in various regimes of edge sparsity. Conditions are formulated for the defining parameters of the random graph enabling remote contiguity and the generalization of asymptotic properties.

It is expected that remote contiguity proves helpful for the generalization of other asymptotic random graph properties. For example, it is known that the degree sequence of ER graphs distributed according to $P_{\lambda,n}$, for some $\lambda > 0$ converges to a Poisson distribution. We now write $p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$, $k \ge 1$, and $P_k^{(n)}(X^n) = n^{-1}\sum_{i=1}^n \mathbf{1}_{D_i(X^n)=k}$ for the empirical degree distribution.

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Proposition 2. *For any* $\lambda > 0$ *and ER graphs* $X^n \sim P_{\lambda,n}$,

$$P_{\lambda,n}\Big(\max_{k\geq 1} |P_k^{(n)}(X_n) - p_k| > \varepsilon_n \Big) = O(1/(n\varepsilon_n^2)) \tag{15}$$

as $n \to \infty$, *for any* $\varepsilon_n \downarrow 0$ *such that* $n\varepsilon_n \to \infty$.

Proof. The result in (15) follows immediately from the inequality

$$\sum_{k\geq 0} \big| \mathbb{E}_{P_{\lambda,n}}\big(P_k^{(n)}(X^n)\big) - p_k \big| < 2\varepsilon_n,$$

valid for all large n, and [13, (5.4.17)]. □

Application of remote contiguity enables the following generalization.

Corollary 3. *For any ER graph* Y^n *of law* $Q_n = P_{(\lambda_{n,ij}),n}$ *satisfying* $\sup_{i<j} |\lambda_{n,ij} - \lambda| = o(1)$ *and* $\sum_{i<j} |\lambda_{n,ij} - \lambda|^2 < n\lambda\delta \log(n\varepsilon_n^2)$ *with* $0 < \delta < 1$ *as* $n \to \infty$,

$$Q_n\Big(\max_{k\geq 1} \big|P_k^{(n)}(Y^n) - p_k\big| > \varepsilon_n \Big) = o(1).$$

Proof. The result follows immediately from (15) and an application of Corollary 2. □

More ambitious forms of generalization are conceivable. For example, we could consider the so-called preferential attachment graph, which displays a degree distribution with heavy tails asymptotically (see [13, Theorem 8.3]). Dependence of edges makes the analysis more demanding technically, but the machinery of remote contiguity continues to apply. Thus we can study the extent to which the model of [13, (8.2.1)] may be perturbed without influencing the asymptotic tail behaviour of the degree distribution.

But the application of remote contiguity is not limited to random graphs; generalization of *any* asymptotic property in *any* sequence of probabilistic models can be analysed with remote contiguity. To illustrate this, we note that for two sequences (P_n) and (Q_n) on measurable spaces $(\mathscr{X}_n, \mathscr{B}_n)$, we have $Q_n \triangleleft a_n^{-1} P_n$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$Q_n\left(\frac{dP_n}{dQ_n} < \delta a_n \right) < \epsilon$$

(or, equivalently, if every subsequence of $(a_n(dP_n/dQ_n)^{-1})$ has a weakly converging subsequence). Lemma 8 gives a variety of general conditions to establish remote contiguity, analogous to Le Cam's First Lemma [9, Chapter 3, Section 3, Proposition 3]. Moreover, the arguments of Subsection 3.2 are fully general, so there are also general conditions to exclude remote contiguity, for example if the Hellinger affinity decreases to zero fast enough:

$$\alpha(P_n, Q_n) = o\big(a_n^{1/2}\big).$$

We therefore express the hope that remote contiguity can be applied in more general examples besides random graphs, in a role that generalizes the role of contiguity.

Appendix A. Remote contiguity

Remote contiguity was introduced in [7] to demonstrate that asymptotic properties of Bayesian posterior distributions can be lifted to frequentist statements of asymptotic consistency, hypothesis testing, model selection, and uncertainty quantification. For another

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statistical example in the setting of extreme value theory, [3, 11] use remote contiguity to prove consistency with respect to relatively complicated true data distributions by simpler, approximating sequences of max-stable distributions.

Here and elsewhere, $M^1(\mathscr{X})$ denotes the collection of all probability measures on a measurable space $(\mathscr{X}, \mathscr{B})$.

Definition 1. Given measurable spaces $(\mathscr{X}_n, \mathscr{B}_n)$ with two sequences of probability measures $P_n, Q_n \in M^1(\mathscr{X}_n)$ for all $n \geq 1$, and a sequence $\rho_n \downarrow 0$, we say that Q_n is ρ_n-remotely contiguous with respect to P_n, notation $Q_n \lhd \rho_n^{-1} P_n$, if $P_n \phi_n(X^n) = o(\rho_n) \Rightarrow Q_n \phi_n(X^n) = o(1)$ for every sequence of $\mathscr{B}_n$-measurable $\phi_n \colon \mathscr{X}_n \to [0, 1]$.

Given two sequences (P_n) and (Q_n), contiguity $P_n \lhd Q_n$ is equivalent to remote contiguity $P_n \lhd a_n^{-1} Q_n$ for all $a_n \downarrow 0$.

The following is the remotely contiguous analogue of Le Cam's First Lemma [9, Section 3.3, Proposition 3].

Lemma 8. *Let probability measures* (P_n), (Q_n) *on measurable spaces* $(\mathscr{X}_n, \mathscr{B}_n)$ *and* $a_n \downarrow 0$ *be given. Then* $Q_n \lhd a_n^{-1} P_n$ *if any of the following hold:*

(i) For any bounded, $\mathscr{B}_n$*-msb* $T_n \colon \mathscr{X}_n \to [0, 1]$, $a_n^{-1} T_n \xrightarrow{P_n} 0 \Rightarrow T_n \xrightarrow{Q_n} 0$.

(ii) For any $\epsilon > 0$, *there is a* $\delta > 0$ *such that* $Q_n(\mathrm{d}P_n/\mathrm{d}Q_n < \delta\, a_n) < \epsilon$ *for large enough* n.

(iii) There is a $b > 0$ *such that* $\liminf_n b\, a_n^{-1} P_n(\mathrm{d}Q_n/\mathrm{d}P_n > b\, a_n^{-1}) = 1$.

(iv) For any $\epsilon > 0$, *there is a constant* $c > 0$ *such that* $\|Q_n - Q_n \wedge c\, a_n^{-1} P_n\| < \epsilon$ *for large enough* n.

(v) Under Q_n, *every subsequence of* $(a_n(\mathrm{d}P_n/\mathrm{d}Q_n)^{-1})$ *has a weakly convergent subsequence.*

Remark 4. For any measurable space $(\mathscr{X}, \mathscr{B})$, the definition of $(\mathrm{d}P/\mathrm{d}Q)^{-1} \colon \mathscr{X} \to (0, \infty]$: $x \mapsto 1/(dP/dQ(x))$ is Q_n-almost sure: given a (sigma-finite) measure ν that dominates both P and Q (e.g. $\nu = P + Q$), write $\mathrm{d}P/\mathrm{d}\nu = p$ and $\mathrm{d}Q/\mathrm{d}\nu = q$. Then the measurable map $p/q\, \mathbf{1}_{\{q>0\}} \colon \mathscr{X} \to [0, \infty)$ is a ν-almost everywhere version of $\mathrm{d}P/\mathrm{d}Q$, and $q/p\, \mathbf{1}_{\{q>0\}} \colon \mathscr{X} \to [0, \infty]$ defines $(\mathrm{d}P/\mathrm{d}Q)^{-1}$ Q-almost surely.

Characterization (v) provides the most insightful formulation, relating remote contiguity to weak convergence of rescaled likelihood ratios, cf. [9]. In most applications, characterization (ii) is the most practical to demonstrate remote contiguity.

Appendix B. The Lindeberg–Feller theorem

In its most basic form, the Lindeberg–Feller theorem formulates a condition for the convergence of sums of independent (but not necessarily identically distributed) random variables to a central limit. There exist versions for dependent random variables too. A *triangular array* consists of a sequence $(k(n))$ that increases to infinity, and random variables $X_{n,k}$, where $n \geq 1$ and $1 \leq k \leq k(n)$.

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Theorem 5. *For each $n \geq 1$, let $X_{n,k}$, $1 \leq k \leq k(n)$, be independent, with expectations $\mu_{n,k} \in \mathbb{R}$ and variances $\sigma^2_{n,k} < \infty$. With $s_n^2 = \sum_{k=1}^{k(n)} \sigma^2_{n,k}$, assume that, for every $\epsilon > 0$,*

$$\frac{1}{s_n^2} \sum_{k=1}^{k(n)} \mathbb{E}\|X_{n,k} - \mu_{n,k}\|^2 \mathbf{1}_{\{\|X_{n,k}-\mu_{n,k}\|>\epsilon s_n\}} \to 0.$$

Then s_n-normalized, $\mu_{n,k}$-centred sums converge weakly to the standard normal distribution,

$$\frac{1}{s_n} \sum_{k=1}^{k(n)} (X_{n,k} - \mu_{n,k}) \xrightarrow{\text{w.}} N(0, 1).$$

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