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COMPOSITION OPERATORS ON WIENER AMALGAM SPACES

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Abstract. For a complex function F on $\mathbb{C}$, we study the associated composition operator $T_F(f) := F \circ f = F(f)$ on Wiener amalgam $W^{p,q}(\mathbb{R}^d)$ $(1 \leqslant p < \infty, 1 \leqslant q < 2)$. We have shown T_F maps $W^{p,1}(\mathbb{R}^d)$ to $W^{p,q}(\mathbb{R}^d)$ if and only if F is real analytic on $\mathbb{R}^2$ and $F(0) = 0$. Similar result is proved in the case of modulation spaces $M^{p,q}(\mathbb{R}^d)$. In particular, this gives an affirmative answer to the open question proposed in Bhimani and Ratnakumar (J. Funct. Anal. **270**(2) (2016), 621–648).

§1. Introduction

Let X and Y be normed spaces of complex functions on $\mathbb{R}^d$. For a given function $F : \mathbb{R}^2 (= \mathbb{C}) \to \mathbb{C}$, we associate it, with the composition operator $T_F : f \mapsto F(f)$, where $F(f) = F \circ f$ is the composition of functions F and $f : \mathbb{R}^d \to \mathbb{C}$. If $T_F(X) \subset Y$, we say the composition operator T_F maps X to Y. In particular, if $T_F(X) \subset X$, we say the composition operator T_F acts on X. Which functions F have the property that T_F maps X to Y? Of course, the properties of the operator T_F strongly depend on X and Y. The aim of this paper is to take a small step toward the answer in the case of modulation and Wiener amalgam spaces (see Section 2.2 for precise definitions).

In the last decade, modulation and Wiener amalgam spaces have turned out to be very fruitful within pure and applied mathematics. In fact, these spaces are nowadays present in investigations that concern problems on pseudodifferential/Fourier integral operators, Strichartz estimates, and so on (we refer the reader to recent survey [24] and the reference therein). For instance, the unimodular Fourier multiplier operator $e^{i|D|^\alpha}$ is not bounded on most of the Lebesgue spaces $L^p(\mathbb{R}^d)$ $(p \neq 2)$, in contrast it is bounded on $W^{p,q}(\mathbb{R}^d)(1 \leqslant p, q \leqslant \infty)$ for $\alpha \in [0, 1]$, and on $M^{p,q}(\mathbb{R}^d)(1 \leqslant p, q \leqslant \infty)$ for $\alpha \in [0, 2]$ (cf. [1, 3, 9]). The cases $\alpha = 1, 2$ are of particular interest

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because they occur in the time evolution of wave and Schrödinger equations, respectively.

Mathematicians [1, 3, 5, 8, 24, 26, 27] have been using these spaces as a regularity class of initial data of the Cauchy problem for nonlinear dispersive equations. But one of the underneath issues of the nonlinear dispersive equations in the realm of modulation and Wiener amalgam spaces is to determine, which is the most general nonlinearity one can take, which is not yet completely clear. Composition operators are simple examples of nonlinear mappings. And when we try to study local and global well-posedness results for nonlinear dispersive equations (Schrödinger/wave/Klein–Gordon etc.) in modulation and Wiener amalgam spaces, it is indispensable to study nonlinear mappings on it. Taking these considerations into our account, we are motivated to study composition operators on these spaces. Specifically, we prove the following theorem.

THEOREM 1.1. *Let* $1 \leqslant p \leqslant \infty, 1 \leqslant q < 2$, *and let*

$$(X, Y) = (M^{p,1}(\mathbb{R}^d), M^{p,q}(\mathbb{R}^d)) \qquad \text{or} \qquad (W^{p,1}(\mathbb{R}^d), W^{p,q}(\mathbb{R}^d)).$$

Suppose that T_F *is the composition operator associated to a complex function* F *on* $\mathbb{C}$. *Then*

(1) *If* T_F *maps* X *to* Y, *then* F *must be real analytic on* $\mathbb{R}^2$. *Moreover,* $F(0) = 0$ *if* $p < \infty$.
(2) *If* F *is real analytic on* $\mathbb{R}^2$ *which takes origin to itself, and* $p < \infty$, *then* T_F *acts on* X.

COROLLARY 1.2. *Let* $1 \leqslant p < \infty, 1 \leqslant q < 2$, *and let*

$$(X, Y) = (M^{p,1}(\mathbb{R}^d), M^{p,q}(\mathbb{R}^d)) \qquad \text{or} \qquad (W^{p,1}(\mathbb{R}^d), W^{p,q}(\mathbb{R}^d)).$$

Then F *is real analytic on* $\mathbb{R}^2$ *and* $F(0) = 0$ *if and only if* T_F *maps* X *to* Y. *In particular,* F *is real analytic on* $\mathbb{R}^2$ *and* $F(0) = 0$ *if and only if* T_F *acts on* X.

COROLLARY 1.3. (1) *There exists* $f \in W^{p,1}(\mathbb{R}^d)$ *such that* $f|f|^\alpha \notin W^{p,q}(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty, 1 \leqslant q < 2)$ *for any* $\alpha \in (0, \infty) \setminus 2\mathbb{N}$. (2) *There exists* $f \in M^{p,1}(\mathbb{R}^d)$ *such that* $f|f|^\alpha \notin M^{p,q}(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty, 1 \leqslant q < 2)$ *for any* $\alpha \in (0, \infty) \setminus 2\mathbb{N}$.

We note that recently Bhimani–Ratnakumar [6, Theorem 3.9] have proved if $F : \mathbb{R}^2 \to \mathbb{C}$ is real analytic and $F(0) = 0$, then T_F acts on

$X = M^{1,1}(\mathbb{R}^d)$, $d \geqslant 1$. And this result is generalized by Kobayashi–Sato [15, Theorem 1.1] for $X = M^{p,1}(\mathbb{R})$ or $W^{p,1}(\mathbb{R})$ for $1 < p < \infty$, although it is restricted to the case when $d = 1$. Thus we remark that our Theorem 1.1(2) settles the case when $d > 1$, and gives an affirmative answer to the open question proposed in [6, p. 646]. It is also worth noting the following.

Remark 1.4.

(1) In Theorem 1.1(1), conditions on the range of $q \in [1, 2)$ are sharp in the sense that if we take $q = 2$ then the same conclusion may not hold. (Since $W^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, if $T_F : W^{p,1}(\mathbb{R}^d) \to W^{p,2}(\mathbb{R}^d)$, then F may not be real analytic on $\mathbb{R}^2$.)

(2) Bhimani–Ratnakumar [6, Theorem 3.2] is a particular case of Theorem 1.2(1) as $M^{p,1}(\mathbb{R}^d)$ is a proper subclass of $M^{p,q}(\mathbb{R}^d)(1 < q < 2)$.

(3) In view of Corollary 1.3, we can point out that the standard method for evolving nonlinear dispersive equations, with the nonlinearity $f|f|^{\alpha}$ $(\alpha \in (0, \infty) \setminus 2\mathbb{N})$, which is of importance in applications, is ruled out.

The sequel contains required notations and preliminary in Section 2, proof for the necessary condition of Theorem 1.1(1) in Section 3, proof for the sufficient condition of Theorem 1.1(2) in Section 4, and concluding remarks in Section 5.

§2. Notations and preliminaries

2.1 Notations

The notation $A \lesssim B$ means $A \leqslant cB$ for some constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leqslant B \leqslant cA$ for some $c \geqslant 1$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space A_1 into A_2. The mixed $L^{p,q}(\mathbb{R}^d \times \mathbb{R}^d)$ norm is denoted by

$$\|f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x, w)|^p \, dx \right)^{q/p} dw \right)^{1/q} \quad (1 \leqslant p, q < \infty),$$

the $L^\infty(\mathbb{R}^d)$ norm is $\|f\|_{L^\infty} = \text{ess.sup}_{x\in\mathbb{R}^d}|f(x)|$, the $\ell^q(\mathbb{Z}^d)$ norm is $\|a_n\|_{\ell^q} = (\sum_{n\in\mathbb{Z}^d} |a_n|^q)^{1/q}$. We denote d-dimensional torus by $\mathbb{T}^d \equiv [0, 2\pi)^d$, and $L^p(\mathbb{T}^d)$-norm is denoted by

$$\|f\|_{L^p(\mathbb{T}^d)} = \left(\int_{[0,2\pi)^d} |f(t)|^p \, dt \right)^{1/p}.$$

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The space of smooth functions on $\mathbb{R}^d$ with compact support is denoted by $C_c^\infty(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$ (with its usual topology), the space of tempered distributions is $\mathcal{S}'(\mathbb{R}^d)$. For $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^d x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform defined by

$$\mathcal{F}f(w) = \widehat{f}(w) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot w}\, dt, \quad w \in \mathbb{R}^d. \tag{1}$$

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = f^{\vee}(x) = \int_{\mathbb{R}^d} f(w) e^{2\pi i x \cdot w}\, dw, \quad x \in \mathbb{R}^d, \tag{2}$$

and this Fourier transform can be uniquely extended to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. For $s \in \mathbb{R}, w \in \mathbb{R}^d$, we put $\langle w \rangle^s = (1 + |w|^2)^{s/2}$.

2.2 Modulation and Wiener amalgam spaces

Let $g \in \mathcal{S}(\mathbb{R}^d)$ be a nonzero window function. The short-time Fourier transform (STFT) of a function (tempered distribution) f with respect to a window g is

$$V_g f(x, w) := \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i w \cdot t}\, dt, \quad (x, w) \in \mathbb{R}^{2d} \tag{3}$$

whenever the integral exists.

In 1983, Feichtinger [10] introduced a class of Banach spaces, which allow a measurement of the space variable and the Fourier transform variable of a function or distribution f on $\mathbb{R}^d$ simultaneously using the STFT, the so-called modulation spaces.

DEFINITION 2.1. (Modulation spaces) For $1 \leqslant p, q \leqslant \infty$, and for given a nonzero smooth rapidly decreasing function $g \in \mathcal{S}(\mathbb{R}^d)$, the weighted modulation space $M_s^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which, the following norm

$$\|f\|_{M_s^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, w)|^p\, dx \right)^{q/p} \langle w \rangle^{sq}\, dw \right)^{1/q}$$

is finite, with the usual modification if p or q are infinite.

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This definition is independent of the choice of the window, in the sense that different window functions yield equivalent modulation space norms (cf. [12, Proposition 11.3.2(c), p. 233]). When $s=0$, we simply write $M_0^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$.

By reversing the order of integration we define the another family of spaces, so-called Wiener amalgam spaces.

Definition 2.2. (Wiener amalgam spaces) For $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$, the weighted Wiener amalgam space $W_s^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{W_s^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,w)|^q \langle w\rangle^{sq}\, dw\right)^{p/q} dx\right)^{1/p}$$

is finite, with usual modifications if p or $q=\infty$.

This definition is independent of the choice of the window g, in the sense that different window functions yield equivalent Wiener amalgam space norms. When $s=0$, we simply write $W_0^{p,q}(\mathbb{R}^d) = W^{p,q}(\mathbb{R}^d)$.

We note that there is another characterization [25] of Wiener amalgam and modulation spaces: let $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\text{supp}\phi \subset (-1,1)^d$$

and

$$\sum_{k\in\mathbb{Z}^d} \phi(w-k) = 1, \quad \forall w \in \mathbb{R}^d.$$

Then we have the equivalence

$$\|f\|_{W_s^{p,q}} \asymp \|\|\langle k\rangle^s \phi(D-k) f\|_{\ell^q}\|_{L^p}$$

and

$$\|f\|_{M_s^{p,q}} \asymp \|\|\langle k\rangle^s \phi(D-k) f\|_{L^p}\|_{\ell^q},$$

where $\phi(D-k)f = \mathcal{F}^{-1}(\widehat{f} \cdot T_k \phi)$.

2.3 Properties of modulation and Wiener amalgam spaces

We gather some basic properties of Wiener amalgam and modulation spaces which will be frequently used in the sequel.

Lemma 2.3. *Let* $p, q, p_i, q_i \in [1,\infty]$ $(i=1,2)$.

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(1) $W^{p,q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow W^{p,q_2}(\mathbb{R}^d)$ *and* $M^{p,q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,q_2}(\mathbb{R}^d)$ *hold for* $q_1 \leqslant \min\{p, p'\}$ *and* $q_2 \geqslant \max\{p, p'\}$ *with* $1/p + 1/p' = 1$.

(2) *If* $q_1 \leqslant q_2$ *and* $p_1 \leqslant p_2$, *then* $W^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow W^{p_2,q_2}(\mathbb{R}^d)$ *and* $M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d)$.

(3) $M^{p,q}(\mathbb{R}^d) \hookrightarrow W^{p,q}(\mathbb{R}^d)$ *when* $q \leqslant p$ *and* $W^{p,q}(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$ *when* $p \leqslant q$.

(4) *The space* $W^{p,p}(\mathbb{R}^d) = M^{p,p}(\mathbb{R}^d)(1 \leqslant p < \infty)$ *is invariant under Fourier transform.*

(5) *The spaces* $W^{p,q}(\mathbb{R}^d)$ *and* $M^{p,q}(\mathbb{R}^d)$ *are Banach spaces.*

(6) *The spaces* $W^{p,q}(\mathbb{R}^d)$ *and* $M^{p,q}(\mathbb{R}^d)$ *are invariant under complex conjugation.*

(7) $\mathcal{S}(\mathbb{R}^d)$ *is dense in* $M^{p,q}(\mathbb{R}^d)$ *and* $W^{p,q}(\mathbb{R}^d)$ *for* $p, q \in [1, \infty)$.

(8) *Let* $0 < \lambda \leqslant 1$, *and put* $f_\lambda(x) = f(\lambda x)$. *There exist constants* C *and* C' *such that* $\|f_\lambda\|_{M^{\infty,1}} \leqslant C\|f\|_{M^{\infty,1}}$ *for* $f \in M^{\infty,1}(\mathbb{R}^d)$.

Proof. The proof of statements (1) and (2) can be found in [23] and [21], respectively. For the proof of statement (8), see [22]. For the proof of statements (2), (5), and (7), see [12]. We only give the arguments for the statement (6) because it provides the reader with some insight about the fundamental identity of time–frequency analysis: in fact, easy computation gives the fundamental identity

$$V_g f(x, w) = e^{-2\pi i x\cdot w}\, V_{\widehat{g}}\widehat{f}(w, -x)$$

but this immediately gives a proof of (6). ∎

Proposition 2.4. (Algebra property) *Let* $p, q, p_i, q_i \in [1, \infty]$ $(i = 0, 1, 2)$ *satisfy* $1/p_1 + 1/p_2 = 1/p_0$ *and* $1/q_1 + 1/q_2 = 1 + 1/q_0$. *Then*

(1) $M^{p_1,q_1}(\mathbb{R}^d) \cdot M^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow M^{p_0,q_0}(\mathbb{R}^d)$ *with norm inequality*

$$\|fg\|_{M^{p_0,q_0}} \lesssim \|f\|_{M^{p_1,q_1}}\|g\|_{M^{p_2,q_2}}.$$

In particular, $M^{p,1}(\mathbb{R}^d)$ *is an algebra under pointwise multiplication with norm inequality*

$$\|fg\|_{M^{p,1}} \lesssim \|f\|_{M^{p,1}}\|g\|_{M^{p,1}}.$$

(2) $W^{p_1,q_1}(\mathbb{R}^d) \cdot W^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow W^{p_0,q_0}(\mathbb{R}^d)$ *with norm inequality*

$$\|fg\|_{W^{p_0,q_0}} \lesssim \|f\|_{W^{p_1,q_1}}\|g\|_{W^{p_2,q_2}}.$$

In particular, $W^{p,1}(\mathbb{R}^d)$ is an algebra under pointwise multiplication with norm inequality

$$\|fg\|_{W^{p,1}} \lesssim \|f\|_{W^{p,1}} \|g\|_{W^{p,1}}.$$

Proof. Cf. [26], [3, Corollary 2.7], and [8, Lemma 2.2]. ∎

We refer to [12] for a classical foundation of these spaces and [27] for some recent developments in nonlinear dispersive equations and connections to these spaces and the references therein.

§3. Necessary condition

In this section, we prove Theorem 1.1(1): if the composition operator T_F maps Wiener amalgam spaces $W^{p,1}(\mathbb{R}^d)$ to $W^{p,q}(\mathbb{R}^d)$, then, necessarily, F is real analytic on $\mathbb{R}^2$. And also a similar necessity condition for modulation spaces. We start with the following.

Definition 3.1. A complex valued function F, defined on an open set E in the plane $\mathbb{R}^2$, is said to be real analytic on E, if to every point $(s_0, t_0) \in E$, there exists an expansion of the form

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}(s-s_0)^m(t-t_0)^n, \quad a_{mn} \in \mathbb{C}$$

which converges absolutely for all (s,t) in some neighborhood of (s_0, t_0). If $E = \mathbb{R}^2$ and the above series converges absolutely for all $(s,t) \in \mathbb{R}^2$, then F is called real entire.

We let $A^q(\mathbb{T}^d)$ be the class of all complex functions f on the $d-$torus $\mathbb{T}^d$ whose Fourier coefficients

$$\widehat{f}(m) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i m\cdot x}\, dx, \quad (m \in \mathbb{Z}^d)$$

satisfy the condition

$$\|f\|_{A^q(\mathbb{T}^d)} := \|\widehat{f}\|_{\ell^q} < \infty.$$

We recall, the classical theorem of Katznelson [13, p. 156], see also, [20, Theorem 6.9.2] for $A^1(\mathbb{T})$ which have been proved in 1959, and later generalized by Rudin [19] in 1962 for $A^q(G)$, where G is infinite compact abelian group and $1 < q < 2$. We just rephrased it here by combining both of them as required in our context.

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THEOREM 3.2. (Katznelson–Rudin) *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$, and $1 \leqslant q < 2$. If T_F maps $A^1(\mathbb{T}^d)$ to $A^q(\mathbb{T}^d)$, then F is real analytic on $\mathbb{R}^2$.*

Now we introduce periodic Wiener amalgam and modulation spaces, and for this reason, first we recall some definitions, and introduce temporary notations. We are starting by noting that there is a one-to-one correspondence between functions on $\mathbb{R}^d$ that are 1-periodic in each of the coordinate directions and functions on torus $\mathbb{T}^d$, and we may identify $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with $[0,1)^d$. Let $\mathcal{D}(\mathbb{T}^d)$ be the vector space $C^\infty(\mathbb{T}^d)$ endowed with the usual test function topology, and let $\mathcal{D}'(\mathbb{T}^d)$ be its dual, the space of distributions on $\mathbb{T}^d$. Let $\mathcal{S}(\mathbb{Z}^d)$ denote the space of rapidly decaying functions $\mathbb{Z}^d \to \mathbb{C}$. Let $\mathcal{F}_T : \mathcal{D}(\mathbb{T}^d) \to \mathcal{S}(\mathbb{Z}^d)$ be the toroidal Fourier transform (hence the subscript T) defined by

$$(\mathcal{F}_T f)(\xi) := \hat{f}(\xi) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i \xi \cdot x}\, dx, \quad (\xi \in \mathbb{Z}^d).$$

Then $\mathcal{F}_T$ is a bijection and the inverse Fourier transform is given by

$$(\mathcal{F}_T^{-1} f)(x) := \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \quad (x \in \mathbb{T}^d),$$

and this Fourier transform is extended uniquely to $\mathcal{F}_T : \mathcal{D}'(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{Z}^d)$.

The Wiener amalgam spaces $W^{p,q}(\mathbb{T}^d)$ consist of all $f \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$\|f\|_{W^{p,q}(\mathbb{T}^d)} := \| \| \phi(D_T - k) f \|_{\ell^q} \|_{L^p(\mathbb{T}^d)} < \infty,$$

and modulation spaces $M^{p,q}(\mathbb{T}^d)$ consist of all $f \in \mathcal{D}'(\mathbb{T}^d)$ such that

$$\|f\|_{M^{p,q}(\mathbb{T}^d)} := \| \| \phi(D_T - k) f \|_{L^p(\mathbb{T}^d)} \|_{\ell^q} < \infty,$$

for some ϕ with compact support in the discrete topology of $\mathbb{Z}^d$, where $\phi(D_T - k) f = \mathcal{F}_T^{-1}(T_k \phi \cdot \mathcal{F}_T f)$.

The next result ensures that Wiener amalgam and modulation spaces coincide with the classical Fourier algebra. Specifically, we have the following proposition.

PROPOSITION 3.3. *Let $1 \leqslant p, q \leqslant \infty$. Then, we have*

$$M^{p,q}(\mathbb{T}^d) = W^{p,q}(\mathbb{T}^d) = A^q(\mathbb{T}^d),$$

with norm inequality

$$\|f\|_{M^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{W^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{A^q(\mathbb{T}^d)}.$$

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Proof. For the proof we refer to [21, Section 5] and [17, Lemma 1]. ∎

We now define the local-in-time versions of the Wiener amalgam and modulation spaces in the following way. Given an interval $I = [0, 1)^d$, we let $W^{p,q}(I)$ be the restriction of $W^{p,q}(\mathbb{R}^d)$ onto I via

$$\|f\|_{W^{p,q}(I)} := \inf\{\|g\|_{W^{p,q}(\mathbb{R}^d)} : g = f \text{ on } I\} \tag{4}$$

and $M^{p,q}(I)$ the restriction of $M^{p,q}(\mathbb{R}^d)$ onto I via

$$\|f\|_{M^{p,q}(I)} = \inf\{\|g\|_{M^{p,q}(\mathbb{R}^d)} : g = f \text{ on } I\}. \tag{5}$$

We note that Bényi–Oh has proved the "equivalence" of the periodic function spaces ($M^{p,q}(\mathbb{T}^d)$ and $W^{p,q}(\mathbb{T}^d)$) and their local-in-time versions (defined on a bounded interval $I = [0, 1)^d$, that is $M^{p,q}(I)$ and $W^{p,q}(I)$) in [2, Appendix B] (see also [2, Remark 3.3]) via establishing the equivalence of norms:

$$\|f\|_{M^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{M^{p,q}(I)} \quad \text{and} \quad \|f\|_{W^{p,q}(\mathbb{T}^d)} \asymp \|f\|_{W^{p,q}(I)}, \tag{6}$$

where $1 \leqslant p, q \leqslant \infty$.

PROPOSITION 3.4. *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$, and $1 \leqslant q < 2$. If T_F maps $W^{p,1}(\mathbb{R}^d)$ to $W^{p,q}(\mathbb{R}^d)$, then T_F maps $A^1(\mathbb{T}^d)$ to $A^q(\mathbb{T}^d)$.*

Proof. Let $f \in A^1(\mathbb{T}^d)$. Then $f^*(x) = f(e^{2\pi i x_1}, \ldots, e^{2\pi i x_d})$ is a periodic function on $\mathbb{R}^d$ with absolutely convergent Fourier series

$$f^*(x) = \sum_{m \in \mathbb{Z}^d} \widehat{f}(m) e^{2\pi i m \cdot x}.$$

Choose $g \in C_c^\infty(\mathbb{R}^d)$ such that $g \equiv 1$ on $Q_d = [0, 1)^d$. Then we claim that $gf^* \in W^{1,1}(\mathbb{R}^d) \subset W^{p,1}(\mathbb{R}^d)$. Once the claim is assumed, by hypothesis, we have

$$F(gf^*) \in W^{p,q}(\mathbb{R}^d). \tag{7}$$

Note that if $z \in \mathbb{T}^d$, then $z = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_d})$ with some unique $x = (x_1, \ldots, x_d) \in Q_d$, hence

$$F(f(z)) = F(f^*(x)) = F(gf^*(x)), \quad \text{for } x \in Q_d. \tag{8}$$

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Now if $\phi \in C_c^\infty(\mathbb{T}^d)$, then $g\phi^*$ is a compactly supported smooth function on $\mathbb{R}^d$. Also $\phi(z) = g(x)\phi^*(x)$ for every $x \in Q_d$, as per the notation above and hence

$$\phi(z)F(f)(z) = g(x)\phi^*(x)F(gf^*)(x), \tag{9}$$

for some $x \in Q_d$.

By (9), Proposition 3.3, (6), (4), and Proposition 2.4(2), we obtain

$$\begin{aligned}
\|\phi F(f)\|_{A^q(\mathbb{T}^d)} &= \|g\phi^* F(gf^*)\|_{A^q(\mathbb{T}^d)} \\
&\asymp \|g\phi^* F(gf^*)\|_{W^{p,q}(\mathbb{T}^d)} \\
&\asymp \|g\phi^* F(gf^*)\|_{W^{p,q}(Q_d)} \\
&\lesssim \|g\phi^* F(gf^*)\|_{W^{p,q}} \\
&\lesssim \|g\phi^*\|_{W^{\infty,1}} \|F(gf^*)\|_{W^{p,q}},
\end{aligned}$$

which is finite for every smooth cutoff function ϕ supported on Q_d in view of Lemma 2.3(1), and (7). Now by compactness of $\mathbb{T}^d$, a partition of unity argument shows that $F(f) \in A^q(\mathbb{T}^d)$.

To complete the proof, we need to prove the claim. By Lemma 2.3(4), it is enough to show that $\widehat{gf^*} = \widehat{g} * \widehat{f^*} \in W^{1,1}(\mathbb{R}^d)$. We put, $\mu = \sum_{k\in\mathbb{Z}^d} c_k\delta_k$, where $c_k = \widehat{f}(k)$ and δ_k is the unit Dirac mass at k. We note that, μ is a complex Borel measure on $\mathbb{R}^d$, and the total variation of μ, that is, $\|\mu\| = |\mu|(\mathbb{R}^d) = \sum_{k\in\mathbb{Z}^n} |c_k|$ is finite. We compute the Fourier–Stieltjes transform of μ:

$$\begin{aligned}
\widehat{\mu}(y) &= \int_{\mathbb{R}^d} e^{-2\pi i x\cdot y} d\mu(x) \\
&= \int_{\mathbb{R}^d} e^{-2\pi i x\cdot y} \left(\sum_{k\in\mathbb{Z}^d} c_k d\delta_k(x) \right) \\
&= \sum_{k\in\mathbb{Z}^d} c_k \int_{\mathbb{R}^d} e^{-2\pi i x\cdot y} d\delta_k(x) \\
&= f^*(-y).
\end{aligned}$$

So,

$$\widehat{f^*} = \mu = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)\delta_m.$$

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It follows that

$$\widehat{g} * \widehat{f^*} = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)\widehat{g} * \delta_m = \sum_{m\in\mathbb{Z}^d} \widehat{f}(m)T_m\widehat{g}.$$

Since the translation operator T_m is an isometry on $W^{1,1}(\mathbb{R}^d)$, it follows that the above series is absolutely convergent in $W^{1,1}(\mathbb{R}^d)$, and hence $\widehat{gf^*} \in W^{1,1}(\mathbb{R}^d)$ as claimed. ∎

Proof of Theorem 1.1(1). If T_F maps $W^{p,1}(\mathbb{R}^d)$ to $W^{p,q}(\mathbb{R}^d)$, then T_F maps $A^1(\mathbb{T}^d)$ to $A^q(\mathbb{T}^d)$ by Proposition 3.4. Hence the analyticity follows from Theorem 3.2.

The necessity of $F(0)=0$ is obvious if $p<\infty$ and can be obtained by taking Lemma 2.3(1) into our account and testing T_F for zero function. In fact, we can compute (see [1, Proof of Theorem 14]) the STFT of a constant function (say 1) with respect to the windowed function $g(\xi)=e^{-\pi|\xi|^2}$, which can be given by

$$|V_g 1(x,w)| = e^{-\pi|w|^2}.$$

From this it is clear that nonzero constant functions cannot be in $W^{p,q}(\mathbb{R}^d)$ if $p<\infty$. This completes the proof of Theorem 1.1(1) for the pair $(X,Y)=(W^{p,1}(\mathbb{R}^d), W^{p,q}(\mathbb{R}^d))$. Taking Proposition 3.3 into account and exploiting the method as before, the proof of Theorem 1.1(1) can be obtained for the pair $(X,Y)=(M^{p,1}(\mathbb{R}^d), M^{p,q}(\mathbb{R}^d))$. ∎

Proof of Corollary 1.3. The nonlinear mapping $F:\mathbb{C}\to\mathbb{C}: z\mapsto z|z|^\alpha$ is not real analytic on $\mathbb{R}^2$ for $\alpha\in(0,\infty)\setminus 2\mathbb{N}$. ∎

§4. Sufficient conditions

We recall that in 1932 Wiener proved that if $F(z)=1/z$, then T_F acts on $A^1(\mathbb{T})\setminus\{0\}$, and in 1935 Lévy generalizes this result: if F is real analytic on $\mathbb{R}^2$, then T_F acts on $A^1(\mathbb{T})$. This is called Wiener–Lévy [16, 28] theorem. Now in this section, we shall prove Theorem 1.1(2), and we note that our approach to the proof is inspired by the Wiener–Lévy theorem.

Unless explicitly mentioned, throughout this section we assume that $X=M^{p,1}(\mathbb{R}^d)$ or $W^{p,1}(\mathbb{R}^d)$ with $1\leqslant p<\infty$. First, we collect some technical results which should be regarded as the tool to proving Theorem 1.1(2). We start with the following.

DEFINITION 4.1. Let ϕ be a function defined on $\mathbb{R}^d$.

(1) We say that ϕ belongs to X locally at point $\gamma_0 \in \mathbb{R}^d$ if there is a neighborhood V of γ_0 and a function $h \in X$ such that $\phi(\gamma) = h(\gamma)$ for every $\gamma \in V$.
(2) We say that ϕ belongs to X locally at ∞ if there is a compact set $K \subset \mathbb{R}^d$ and a function $h \in X$ such that $\phi(\gamma) = h(\gamma)$ in the complement of K.

The next lemma gives the useful criterion for functions to be in X.

LEMMA 4.2. *If ϕ belongs to X locally at every point of $\mathbb{R}^d \cup \{\infty\}$, then $\phi \in X$.*

To prove Lemma 4.2 and for the sake of the convenience of the reader, first we recall the following two lemmas.

LEMMA 4.3. (The C^∞ Urysohn Lemma) *If $K \subset \mathbb{R}^d$ is compact and U is an open set containing K, there exists $f \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leqslant f \leqslant 1$, $f = 1$ on K, and $\operatorname{supp}(f) \subset U$. (For the proof, see [11, p. 245]).*

LEMMA 4.4. *Suppose $K \subset \mathbb{R}^d$ is compact and let $V_1, \ldots, V_n$ be open sets with $K \subset \bigcup_{j=1}^n V_j$. Then there exist open sets $W_1, W_2, \ldots, W_n$ with $\overline{W_j} \subset V_j$ and $K \subset \bigcup_{j=1}^n W_j$.*

Proof. For each $\epsilon > 0$ let V_j^ϵ be the set of points in V_j whose distance from $\mathbb{R}^d \setminus V_j$ is greater than ϵ. Clearly V_j^ϵ is open and $\overline{V_j^\epsilon} \subset V_j$. It follows that $K \subset \bigcup_1^n V_j^\epsilon$ if ϵ is sufficiently small. ∎

Proof of Lemma 4.2. Suppose first that ϕ has a compact support K. By hypothesis, it follows that, for any $\gamma \in K$, there is a neighborhood of γ, say V_γ, and $h_\gamma \in W^{p,1}(\mathbb{R}^d)$ such that, $\phi(x) = h_\gamma(x)$ for all $x \in V_\gamma$. Next, we observe that, $\{V_\gamma : \gamma \in K\}$ forms an open cover of K, since K is compact, there exist open sets $V_{\gamma_1}, \ldots, V_{\gamma_n}$ and functions $h_1, \ldots, h_n \in W^{p,1}(\mathbb{R}^d)$ such that $\phi = h_i$ in V_{γ_i} and $V_{\gamma_1} \cup V_{\gamma_2} \cup \cdots \cup V_{\gamma_n}$ covers K, that is, $K \subset \bigcup_{j=1}^n V_{\gamma_j}$. Then by Lemma 4.4, we have

(i) open sets $W_1, \ldots, W_n$ with compact closures $\overline{W_j} \subset V_{\gamma_j}$ such that $W_1 \cup \cdots \cup W_n$ covers K, that is, $K \subset \bigcup_{j=1}^n W_j$; and by Lemma 4.3, we get
(ii) functions $k_j \in W^{p,1}(\mathbb{R}^d)$ such that $k_j = 1$ on $\overline{W_j}$ and $k_j = 0$ outside V_{γ_j}.

Now, by using (i) and (ii), we have, $\phi(x)k_j(x) = h_j(x)k_j(x)$, for all $x \in \mathbb{R}^d$ and by Proposition 2.4, we get, $h_jk_j \in W^{p,1}(\mathbb{R}^d)$, and so $\phi k_j \in W^{p,1}(\mathbb{R}^d)$, for

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$1 \leqslant j \leqslant n$. Therefore, if we put

$$\psi = \phi\{1 - (1 - k_1)(1 - k_2) \ldots (1 - k_n)\} \tag{10}$$

it follows that $\psi \in W^{p,1}(\mathbb{R}^d)$.

The multiplier of ψ in (10) is 1 whenever one of k_i is 1, and this happens at every point of K; outside K, $\phi = 0$; hence $\psi = \phi$, and thus $\phi \in W^{p,1}(\mathbb{R}^d)$.

In the general case, ϕ belongs to $W^{p,1}(\mathbb{R}^d)$ locally at ∞, so that there is a function $g \in W^{p,1}(\mathbb{R}^d)$ which coincides with ϕ outside some compact subset of $\mathbb{R}^d$. Then $\phi - g$ has compact support and belongs to $W^{p,1}(\mathbb{R}^d)$ locally at every point of $\mathbb{R}^d$; by the first case, $\phi - g \in W^{p,1}(\mathbb{R}^d)$, and so $\phi \in W^{p,1}(\mathbb{R}^d)$. This completes the proof if $X = W^{p,1}(\mathbb{R}^d)$. The case $X = M^{p,1}(\mathbb{R}^d)$ can be obtained similarly. □

We denote by X_{loc} the space functions that are locally in X at each $\gamma_0 \in \mathbb{R}^d$.

LEMMA 4.5. [6, p. 634] *Let f be a function defined on $\mathbb{R}^d$.*

(1) *$f \in X_{\text{loc}}$ if and only if $\phi f \in X$ for all $\phi \in C_c^\infty(\mathbb{R}^d)$.*
(2) *f belongs to X locally at ∞ if and only if there exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $(1 - \phi)f \in X$.*

PROPOSITION 4.6. [6, Proposition 3.14] *Let $f \in M^{1,1}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and $\epsilon > 0$. Then there exists a $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\phi\,[f - f(x_0)]\,\|_{M^{1,1}} < \epsilon$. The function ϕ can be chosen so that $\phi \equiv 1$ in some neighborhood of x_0.*

PROPOSITION 4.7. *Let $f \in X, x_0 \in \mathbb{R}^d$ and $\epsilon > 0$. Then there exists a $\Phi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\Phi[f - f(x_0)]\|_X < \epsilon$. The function ϕ can be chosen so that $\Phi \equiv 1$ in some neighborhood of x_0.*

Proof. Let $f \in X$. Choose $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi \equiv 1$ in some neighborhood of x_0. In view of $M^{1,1}(\mathbb{R}^d) = W^{1,1}(\mathbb{R}^d)$ and Proposition 2.4, we have

$$\begin{aligned}\|\psi f\|_{M^{1,1}} &\leqslant \|f\|_Y \|\psi\|_{M^{1,1}} \\ &\lesssim \|f\|_X < \infty,\end{aligned}$$

where $Y = M^{\infty,1}$ if $X = M^{p,1}$ and $Y = W^{\infty,1}$ if $X = W^{p,1}$. Thus $h := \psi f \in M^{1,1}(\mathbb{R}^d)$. We can now apply Proposition 4.6 for $h \in M^{1,1}(\mathbb{R}^d)$: given $\epsilon' > 0$, there exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\|\phi(h - h(x_0))\|_{M^{1,1}} < \epsilon'$$

and $\phi \equiv 1$ on the support of ψ. Now define $\Phi(x) = \psi(x)\phi(x)$ for all $x \in \mathbb{R}^d$. Note that $\Phi \in C_c^\infty(\mathbb{R}^d)$ and $\Phi \equiv 1$ on some neighborhood of x_0. By definition of Φ and Lemma 2.3, we have

$$\begin{aligned}\|\Phi(f - f(x_0))\|_X &= \|\phi(h - h(x_0))\|_X \\ &\leqslant C\|\phi(h - h(x_0))\|_{M^{1,1}} < C\epsilon'.\end{aligned}$$

Taking $\epsilon' = \epsilon/C$, we get $\|\Phi(f - f(x_0))\|_X < \epsilon$. This completes the proof. □

PROPOSITION 4.8. *Let $f \in X$, and $\epsilon > 0$. There also exists a $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\|(1-\psi)f\|_X < \epsilon$.*

Proof. Let $f \in X$, and $\epsilon' > 0$. By Lemma 2.3(7), there exists $g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|f - g\|_X < \epsilon'. \tag{11}$$

We recall the fact that for any $g \in \mathcal{S}(\mathbb{R}^d)$, there exists $\lambda_0 \in (0, 1)$ such that

$$\|(1 - \phi_\lambda)g\|_{M^{1,1}} < \frac{\epsilon}{2} \tag{12}$$

for any $\lambda \in (0, \lambda_0)$, where $\phi_\lambda(x) = \phi(\lambda x) \in C_c^\infty(\mathbb{R}^d)$, (see for instance the proof of [6, Proposition 3.14]). We define $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(x) := \phi(\lambda x)$ where $\lambda \in (0, \lambda_0)$. By Lemma 2.3 and (12), we have

$$\begin{aligned}\|(1-\psi)f\|_X &\leqslant \|(1-\psi)(f-g)\|_X + \|(1-\psi)g\|_X \\ &= \|f - g - \psi(f-g)\|_X + \|(1-\phi_\lambda)g\|_X \\ &\leqslant \|f-g\|_X + C\|\psi\|_Y\|f-g\|_X + \|(1-\phi_\lambda)g\|_{M^{1,1}} \\ &\leqslant (1 + C\|\phi_\lambda\|_Y)\|f-g\|_X + \frac{\epsilon}{2},\end{aligned}$$

where $Y = M^{\infty,1}$ if $X = M^{p,1}$ and $Y = W^{\infty,1}$ if $X = M^{p,1}$. By Lemma 2.3(3) and 2.3(8), we have $\|\phi_\lambda\|_Y \lesssim \|\phi_\lambda\|_{M^{\infty,1}} \lesssim \|\phi\|_{M^{\infty,1}}$. Using this, we have

$$\|(1-\psi)f\|_X \leqslant (1 + C'\|\phi\|_{M^{\infty,1}})\|f-g\|_X + \frac{\epsilon}{2}. \tag{13}$$

Taking $\epsilon' = \epsilon/2(1 + C'\|\phi\|_{M^{\infty,1}})$, and using (11) and (12), we obtain that $\|(1-\psi)f\|_X < \epsilon$. □

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Proof of Theorem 1.1(2). Write $f = f_1 + if_2 \in X$, where f_1 and f_2 are real functions, and with an abuse of notation, we write $F(f) = F(f_1, f_2)$. To show that $F(f)$ is in X, enough to show, in view of Lemma 4.2 that $F(f) \in X_{\text{loc}}$ and $F(f)$ belongs to X locally at ∞. First we show that $F(f) \in X_{\text{loc}}$. Fix $x_0 \in \mathbb{R}^d$ and put $f(x_0) = s_0 + it_0$. Since F is real analytic at (s_0, t_0), there exists $\delta > 0$ such that F has the power series expansion

$$F(s,t) = F(s_0,t_0) + \sum_{m,n=0}^{\infty} a_{mn}(s-s_0)^m(t-t_0)^n, \quad (a_{00}=0) \tag{14}$$

which converges absolutely for $|s - s_0| \leqslant \delta$, $|t - t_0| \leqslant \delta$. Then

$$\begin{aligned} F(f_1(x), f_2(x)) &= F(s_0,t_0) \\ &\quad + \sum_{(m,n)\neq(0,0)} a_{mn}[f_1(x)-f_1(x_0)]^m[f_2(x)-f_2(x_0)]^n \end{aligned} \tag{15}$$

whenever the series converges.

Note that both f_1 and f_2 are in X, being the real and imaginary parts of f. Hence in view of Proposition 4.6, we can find $\phi \in C_c^\infty(\mathbb{R}^d)$, such that $\phi \equiv 1$ near x_0 and $\|\phi[f_i - f_i(x_0)]\|_X < \delta$, for $i = 1, 2$. Now consider the function G on $\mathbb{R}^d$ defined by

$$\begin{aligned} G(x) &= \phi(x)F(s_0,t_0) \\ &\quad + \sum_{(m,n)\neq(0,0)} a_{mn}(\phi(x)[f_1(x)-f_1(x_0)])^m(\phi(x)[f_2(x)-f_2(x_0)])^n. \end{aligned}$$

Since $\|\phi[f_i - f_i(x_0)]\|_X < \delta$, for $i = 1, 2$ and in view of Proposition 2.4, we see that the above series is absolutely convergent in X. Also since $\phi \equiv 1$ in some neighborhood of x_0, it follows that $G \equiv F(f)$ in some neighborhood of x_0. Since x_0 is arbitrary, this shows that $F(f) \in X_{\text{loc}}$.

To show that $F(f) \in X$ locally at infinity, we take $(s_0, t_0) = (0, 0)$ in equation (14). Since $F(0) = 0$, the expansion (15) now becomes

$$F(f_1(x), f_2(x)) = \sum_{(m,n)\neq(0,0)} a_{mn}[f_1(x)]^m[f_2(x)]^n,$$

whenever the series converges.

By Proposition 4.8, we have $\|(1-\psi)f_i\|_X < \delta$, for $i = 1, 2$ for some $\psi \in C_c^\infty(\mathbb{R}^d)$. Now consider the function H defined by

$$H(x) = \sum_{(m,n)\neq(0,0)} a_{mn}[(1-\psi(x))f_1(x)]^m[(1-\psi(x))f_2(x)]^n.$$

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The above series is absolutely convergent in X, in view of the above norm estimates, hence $H \in X$. Also since ψ is compactly supported, $1-\psi \equiv 1$ in the complement of a large ball centered at the origin, hence $H = F(f)$ in the compliment of a compact set. This shows that $F(f)$ belongs to X locally at infinity. This completes the proof of Theorem 1.1(2). □

THEOREM 4.9. *Suppose that T_F is the composition operator associated to a complex function F on $\mathbb{C}$, and $1 \leqslant p \leqslant \infty$. If F is a real entire given by $F(s,t) = \sum_{m,n=0}^{\infty} a_{mn} s^m t^n$ with $F(0) = 0$, then T_F acts on $W^{p,1}(\mathbb{R}^d)$. In particular, we have*

$$\|T_F(f)\|_{W^{p,1}} \leqslant \sum_{m,n=0}^{\infty} |a_{mn}| \|f\|_{W^{p,1}}^{m+n}, \quad (f = f_1 + if_2).$$

Proof. Let $f \in W^{p,1}(\mathbb{R}^d)$ with $f_1 = (f + \bar{f})/2$ and $f_2 = (f - \bar{f})/2i$. Then $f_1, f_2 \in W^{p,1}(\mathbb{R}^d)$ and so $f_1^m, f_2^n \in W^{p,1}(\mathbb{R}^d)$ by Proposition 2.4. Since the series $\sum_{m,n=0}^{\infty} a_{mn} s^m t^n$ converges absolutely for all (s,t), the series $\sum_{n,m=0}^{\infty} a_{mn} f_1^m f_2^n$ converges in the norm of $W^{p,1}(\mathbb{R}^d)$; and its sum is $F(f) = \sum_{n,m=0}^{\infty} a_{mn} f_1^m f_2^m$. □

§5. Concluding remarks

By the frequency-uniform localization (see [27, Chapter 6]) techniques, the modulation and Wiener amalgam spaces can be viewed as a Besov/Lizorkin–Triebel type space associated with a uniform decomposition (see [25, 26]). We note that in the last two decades, composition operators have been studied (by Bourdaud, Sickel *et al.*) extensively on Besov and Lizorkin–Triebel spaces. We refer to the enlightening survey article [7] by Bourdaud–Sickel and the references therein for more details. Recently [4, 14, 15, 18, 23] some progress has been made for a composition operator on weighted modulation spaces. But we believe, yet we have very little information for composition operators on modulation and Wiener amalgam spaces. Specifically, we note:

(1) Feichtinger [10] has established the basic properties of Wiener amalgam spaces and modulation spaces on locally compact groups. It would be interesting to investigate the analogue of Theorem 1.1 for locally compact groups.
(2) We have answered the problem stated in introductory paragraph in a few specific cases (Theorem 1.1). What about the remaining cases?

(3) It would be interesting to find necessary and sufficient conditions on $F:\mathbb{C}\to\mathbb{C}$ such that the composition operator is bounded on modulation/Wiener amalgam spaces, that is, $\|F\circ f\|_X \lesssim \|f\|_X$ for $X = M^{p,q}(\mathbb{R}^d)$ or $W^{p,q}(\mathbb{R}^d)$ $(p = q \neq 2)$.

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