Adv. Appl. Prob. **47,** 1212–1234 (2015)
Printed in Northern Ireland

OPTIMAL DOUBLE STOPPING OF A BROWNIAN BRIDGE

ERIK J. BAURDOUX,* *London School of Economics*
NAN CHEN,** *The Chinese University of Hong Kong*
BUDHI A. SURYA,*** *Victoria University of Wellington and Bandung Institute of Technology*
KAZUTOSHI YAMAZAKI,**** *Kansai University*

Abstract

We study optimal double stopping problems driven by a Brownian bridge. The objective is to maximize the expected spread between the payoffs achieved at the two stopping times. We study several cases where the solutions can be solved explicitly by strategies of a threshold type.

Keywords: Brownian bridge; optimal double stopping; buying–selling strategy

2010 Mathematics Subject Classification: Primary 60G40
Secondary 60H30

1. Introduction

In this paper we study several optimal double stopping problems for a Brownian bridge. Given a Brownian bridge $\{X_s\}_{t\le s\le 1}$ starting from x at time $0 \le t < 1$ and ending at 0 at time 1 or, equivalently, a Brownian motion conditioned to be at 0 at time 1, our objective is to choose a pair of stopping times, $t \le \tau_1 \le \tau_2 < 1$ such that the expected spread between the payoffs $f(X_{\tau_2})$ and $f(X_{\tau_1})$ is maximized for a given functional f.

The optimal double stopping problem has received much attention recently in the field of finance. In particular, this is used to derive a 'buy low and sell high' strategy so as to maximize the expected spread between the two payoffs. The strategy called *mean-reversion* typically uses the 'mean' computed from the historical data as a benchmark; an asset is bought if the price is lower and is sold when it is higher. Closely related is the trading strategy called *pairs trading*. Two assets of similar characteristics (e.g. in the same industry category) are considered. By longing one and shorting the other, one can construct a mean-reverting portfolio. An implementation of a pairs trading reduces to solving a single or double stopping problem where one wants to decide the time of (entry and) liquidation of the position so as to maximize the spread; see, e.g. [4], [7], and [13] among others.

There are several motivations to consider a Brownian bridge as an underlying process. We list here three examples where an asset process is expected to converge to a given value at a given time and, hence, a Brownian bridge is a suitable process to use in modeling.

Received 11 September 2014.
* Postal address: Department of Statistics, London School of Economics, London WC2A 2AE, UK.
** Postal address: Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong.
*** Postal address: School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand.
**** Postal address: Department of Mathematics, Faculty of Engineering Science, Kansai University, 3-3-35 Yamate-cho Suita, Osaka, 564-8680, Japan. Email address: kyamazak@kansai-u.ac.jp

The first example, known as the *stock pinning*, is a phenomenon where a stock price tends to end up in the vicinity of the strike of its option near its expiry. This is observed typically for heavily traded assets; within minutes before the expiration, the stock price experiences a strong mean-reversion to the strike; see [2] and [3] and references therein for a discussion on the mechanism of the stock pinning.

The second example is a sudden mispricing of assets due to the market's overreaction to news and rumors, which is followed by a rapid recovery to the original value. In the well-known 2010 *flash crash*, the Dow Jones Industry Average fell about nine percent and then recovered within minutes; see, e.g. [8, Chapter III]. While its cause is still in dispute, it is believed to have been first triggered by some newly disclosed information on the debt crisis in Greece, followed by a chain reaction of a large execution of sales by the automated algorithmic/high frequency trading. While the price may not recover completely to the original price, the difference is small in comparison to the magnitude of the large fall caused by these events.

The third example comes from the dynamic prices of goods in the existence of seasonality and/or fixed sales deadlines. Important examples include low cost carriers/high speed rails, hotel rooms, and theater tickets, where these goods become worthless after given deadlines. In the field of revenue/yield management, the price of such goods is chosen dynamically (and stochastically) over time so as to maximize the expected total yield; the problem reduces to striking the balance between maximizing the price per unit and minimizing the remaining stocks at the deadline. In typical models, the dynamic programming principle applies and the optimal price becomes a function of the remaining number of stocks and the remaining time until the deadline; see, among others, the seminal paper by Gallego and van Ryzin [5]. According to these models, the price converges to a given value on condition that the remaining inventory vanishes by the deadline; this is aimed by the manager and is indeed more than likely achieved when the demand is high (e.g. holiday seasons).

The optimal double stopping problem for a Brownian bridge considered in this paper is applicable in situations where one wants to buy and sell an asset to maximize the spread until it converges to the fixed value as in these examples.

There are a number of papers on the single optimal stopping problem for a Brownian bridge. In particular, Shepp [11] solved the problem of maximizing the first moment of the stopped Brownian bridge (under the assumption that it starts at 0) by rewriting the problem in terms of a time-changed Brownian motion. Ekström and Wanntorp [3] solved the problem for several payoff functionals with arbitrary starting values. Our findings heavily rely on the latter; we shall start with the results in [3] and extend to the optimal double stopping problem. Regarding the discrete-time analog (the urn problem), we refer the reader to [9] and [14] for single optimal stopping problems. For optimal double stopping problems, Ivashko [6] considered the problem of maximizing the spread of the first moment; Sofrenov *et al.* [12] considered a different but related buying–selling problem under independent observations.

1.1. Problems

Fix $0 \le t < 1$ and consider a Brownian bridge $\{X_s\}_{t \le s \le 1}$ satisfying

$$\mathrm{d}X_s = -\frac{X_s}{1-s}\,\mathrm{d}s + \mathrm{d}W_s, \qquad t \le s < 1 \tag{1}$$

with $X_t = x \in \mathbb{R}$ and where $\{W_s\}_{t \le s \le 1}$ denotes a standard Brownian motion. We let $\mathbb{P}_{t,x}$ and $\mathbb{E}_{t,x}$ be the conditional probability and expectation under which $X_t = x$ for any $0 \le t < 1$ and $x \in \mathbb{R}$.

We consider the following three problems.

Problem 1. Maximizing the expected spread of $\mathbb{E}_{t,x}[X_{\tau_2} - X_{\tau_1}]$.

Problem 2. Maximizing the expected spread of $\mathbb{E}_{t,x}[(X_{\tau_2}^{2n+1} - X_{\tau_1}^{2n+1})\mathbf{1}_{\{X_{\tau_1} \le 0\}} + (X_{\tau_1}^{2n+1} - X_{\tau_2}^{2n+1})\,\mathbf{1}_{\{X_{\tau_1} > 0\}}]$ for a given integer $n \ge 0$.

Problem 3. Maximizing the expected spread of $\mathbb{E}_{t,x}[|X_{\tau_2}|^q - |X_{\tau_1}|^q]$ for a given $q > 0$.

The supremum is taken over all pairs of stopping times $t \le \tau_1 \le \tau_2 < 1$ almost surely (a.s.) with respect to the filtration generated by X.

Problem 1 corresponds to the case where short-selling is not permitted, and an asset must be bought prior to being sold. Problem 2 is the case where it is allowed; if the price at the first exercise time is negative (respectively positive), the asset is bought (respectively sold) and then it is sold (respectively bought) at the second exercise time. Problem 3 models the case when the payoff function is v-shaped with respect to the underlying process; this is motivated by investing strategies such as a straddle.

For each problem, we shall show that the optimal stopping times are first hitting times of the time-changed process $\{X_s/\sqrt{1-s}\}_{t \le s \le 1}$.

To the best of the authors' knowledge, this is the first result on the finite-time horizon optimal double stopping problem where the solution is nontrivial and explicit. It is remarked that a finite-time horizon optimal stopping in general lacks an explicit solution even for a single stopping case. For other processes, we expect that the solutions are either trivial (e.g. buying immediately and selling at the maturity) or do not admit analytical solutions. It is also noted that thanks to the a.s. fixed end point of a Brownian bridge, the two stoppings are always exercised; for other processes, one needs to take care of a scenario where the first and/or second stoppings never occur during the time horizon.

1.2. Outline of the paper

The rest of the paper is organized as follows. In Section 2 we review the single optimal stopping problem of a Brownian bridge as obtained in [3] with some complements that will be needed for our analysis in later sections. In Sections 3, 4, and 5 we solve Problems 1, 2, and 3, respectively. Some proofs are deferred to Appendix A.

2. Preliminaries

In this section we review the results of Ekström and Wanntorp [3] for the optimal single stopping problem of a Brownian bridge. As there are a few details omitted in [3], which will be important in our analysis, we complement these results here. Throughout, let us define, for all $q > 0$,

$$F_q(y) := \int_0^\infty u^{q-1} \mathrm{e}^{yu - u^2/2}\, \mathrm{d}u, \qquad G_q(y) := F_q(-y), \qquad y \in \mathbb{R}. \tag{2}$$

(We remark that [3, Equation (3.5)] contains a typographical error in the definitions of F_q and G_q. We suggest the reader refer to Ekström *et al.* [4, Section 4] for a correct version.) These functions can be written in terms of the confluent hypergeometric/parabolic cylinder functions; see, e.g. [1]. Consider the partial differential equation $\mathcal{L}\xi(t,x) = 0$ for $\xi \in C^1 \times C^2$ on some open set E with the infinitesimal generator $\mathcal{L}$ for a Brownian bridge (1),

$$\mathcal{L}\xi(t,x) := \frac{\partial}{\partial t}\xi - \frac{x}{1-t}\frac{\partial}{\partial x}\xi + \frac{1}{2}\frac{\partial^2}{\partial x^2}\xi, \qquad (t,x) \in E.$$

This can be simplified by setting $\xi(t, x) = (1-t)^{q/2}\zeta(x/\sqrt{1-t})$ to an ordinary differential equation (ODE),

$$\zeta''(y) - y\zeta'(y) - q\zeta(y) = 0. \tag{3}$$

A general solution of (3) can be written as a linear combination of F_q and G_q; see [4, Section 4].

In particular, when $q = 1$, (2) is simplified to

$$F_1(y) = \mathrm{e}^{y^2/2}\int_{-y}^{\infty} \mathrm{e}^{-u^2/2}\,\mathrm{d}u = \sqrt{2\pi}\mathrm{e}^{y^2/2}\Phi(y), \qquad G_1(y) = \sqrt{2\pi}\mathrm{e}^{y^2/2}\Phi(-y), \qquad y \in \mathbb{R}, \tag{4}$$

where Φ denotes the standard normal distribution function, i.e.

$$\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \mathrm{e}^{-z^2/2}\,\mathrm{d}z, \qquad y \in \mathbb{R}.$$

Consequently, we also have $(G_1 + F_1)(y) = \sqrt{2\pi}\exp(y^2/2)$ for all $y \in \mathbb{R}$.

2.1. One-sided exit problem

For fixed integer $n \geq 0$, consider the single stopping problem:

$$U(t, x) := \sup_{t\leq\tau<1} \mathbb{E}_{t,x}[X_\tau^{2n+1}], \qquad 0 \leq t < 1,\ x \in \mathbb{R}. \tag{5}$$

Define the upcrossing time of the process $\{X_s/\sqrt{1-s}\}_{t\leq s\leq 1}$ as

$$\tau^+(B) := \inf\{s \geq t \colon X_s \geq B\sqrt{1-s}\}, \qquad B \in \mathbb{R}. \tag{6}$$

Following the arguments as in [3], we have, for any $B \in \mathbb{R}$,

$$\mathbb{E}_{t,x}[X_{\tau^+(B)}^{2n+1}] = (1-t)^{n+1/2}\frac{B^{2n+1}}{F_{2n+1}(B)}F_{2n+1}\left(\frac{x}{\sqrt{1-t}}\right), \qquad x < B\sqrt{1-t}, \tag{7}$$

which can be derived by solving (3) for $q = 2n+1$ and $y = x/\sqrt{1-t}$ with its boundary conditions; see [3, p. 172].

Ekström and Wanntorp [3] showed that (5) is solved by the stopping time (6) by choosing B that maximizes (7) or, equivalently, the function $B \mapsto B^{2n+1}/F_{2n+1}(B)$. Taking its derivative, we have

$$\frac{\partial}{\partial B}\frac{B^{2n+1}}{F_{2n+1}(B)} = \frac{B^{2n}}{F_{2n+1}(B)}\left[(2n+1) - \frac{BF'_{2n+1}(B)}{F_{2n+1}(B)}\right], \qquad B \in \mathbb{R}.$$

The sign of the above equation is determined by that of the function

$$B \mapsto (2n+1) - \frac{BF'_{2n+1}(B)}{F_{2n+1}(B)},$$

which is plotted in Figure 1. As is shown in [3], it is monotonically decreasing and there exists a unique zero $B^* > 0$ such that

$$B^*F'_{2n+1}(B^*) = (2n+1)F_{2n+1}(B^*) \tag{8}$$

and

$$\frac{\partial}{\partial B}\frac{B^{2n+1}}{F_{2n+1}(B)} > 0 \iff B < B^*, \qquad B \in \mathbb{R}. \tag{9}$$

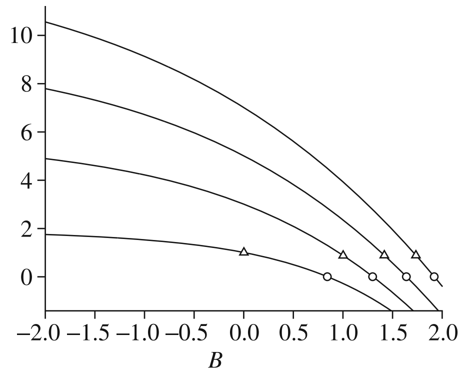

FIGURE 1: Plots of the function $B \mapsto (2n+1) - BF'_{2n+1}(B)/F_{2n+1}(B)$ for $n = 0, 1, 2, 3$. Triangles indicate the points at $\sqrt{n}$ and circles indicate the points at B^*.

Define the candidate value function $U^*(t,x) := \mathbb{E}_{t,x}[X^{2n+1}_{\tau^+(B^*)}]$ for $0 \le t < 1$ and $x \in \mathbb{R}$. The verification of optimality requires the following lower bound on B^*; as it is not included in [3], we shall give its proof. Note that this is also confirmed in the numerical plots of Figure 1.

Lemma 1. *We have* $B^* \ge \sqrt{n}$.

Proof. See Appendix A.

By Lemma 1, for $x > B^*\sqrt{1-t}$ (where $U^*(t,x) = x^{2n+1}$),

$$\mathcal{L}U^*(t,x) = (2n+1)\left[n - \frac{x^2}{1-t}\right]x^{2n-1} \le 0.$$

This together with the smooth fit at $B^*\sqrt{1-t}$ (which can be confirmed by simple algebra) verifies the optimality using martingale arguments via Itô's formula.

Theorem 1. ([3, Theorems 2.1 and 3.1].) (i) *An optimal stopping time for (5) is given by* $\tau^+(B^*)$ *and the value function* $U(t,x)$ *is given by*

$$U(t,x) = U^*(t,x) = \begin{cases} (1-t)^{n+1/2}(B^*)^{2n+1}\dfrac{F_{2n+1}(x/\sqrt{1-t})}{F_{2n+1}(B^*)} & \text{if } x < B^*\sqrt{1-t}, \\ x^{2n+1} & \text{if } x \ge B^*\sqrt{1-t}. \end{cases} \tag{10}$$

(ii) *In particular, when* $n = 0$, $B^* \simeq 0.84$ *is the unique solution to*

$$\sqrt{2\pi}(1-B^2)\mathrm{e}^{B^2/2}\Phi(B) = B. \tag{11}$$

The value function $U(t,x)$ *is given by, if* $x < B^*\sqrt{1-t}$,

$$\begin{aligned} U(t,x) &= U^*(t,x) \\ &= \frac{\sqrt{1-t}B^*\mathrm{e}^{x^2/(2(1-t))-(B^*)^2/2}\Phi(x/\sqrt{1-t})}{\Phi(B^*)} \\ &= \sqrt{2\pi(1-t)}(1-(B^*)^2)\mathrm{e}^{x^2/(2(1-t))}\Phi\left(\frac{x}{\sqrt{1-t}}\right) \end{aligned} \tag{12}$$

and it is equal to x otherwise.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

2.2. Two-sided exit problem

Consider now, for fixed integer $q > 0$, the problem of maximizing the absolute value

$$\overline{U}(t,x) := \sup_{t \le \tau < 1} \mathbb{E}_{t,x}[|X_\tau|^q], \qquad 0 \le t < 1,\ x \in \mathbb{R}. \tag{13}$$

It has been shown by [3] that the optimal stopping time is of the form

$$\tau(D) := \inf\{s \ge t : |X_s| \ge D\sqrt{1-s}\}, \qquad D \ge 0. \tag{14}$$

For $-D\sqrt{1-t} < x < D\sqrt{1-t}$, by [3], again solving (3) with desired boundary conditions

$$\mathbb{E}_{t,x}[(1-\tau(D))^{q/2}] = (1-t)^{q/2} \frac{(F_q+G_q)(x/\sqrt{1-t})}{(F_q+G_q)(D)} \tag{15}$$

and, hence,

$$\mathbb{E}_{t,x}[|X_{\tau(D)}|^q] = D^q \mathbb{E}_{t,x}[(1-\tau(D))^{q/2}] = (1-t)^{q/2} D^q \frac{(F_q+G_q)(x/\sqrt{1-t})}{(F_q+G_q)(D)}. \tag{16}$$

Here, note that $(F_q + G_q)$ is an even function.

The maximization of this expectation is equivalent to maximizing the function $D \mapsto D^q/(F_q+G_q)(D)$, whose derivative can be expressed as

$$\frac{\partial}{\partial D} \frac{D^q}{(F_q+G_q)(D)} = \frac{D^{q-1}}{(F_q+G_q)(D)} \left[q - \frac{D(F_q+G_q)'(D)}{(F_q+G_q)(D)} \right], \qquad D > 0.$$

Similarly to the arguments above for B^*, there exists a maximizer $D^* > 0$, which is a unique root of

$$0 = q - \frac{D(F_q+G_q)'(D)}{(F_q+G_q)(D)} \tag{17}$$

and

$$\frac{\partial}{\partial D} \frac{D^q}{(F_q+G_q)(D)} > 0 \iff D < D^*, \qquad D > 0. \tag{18}$$

In Figure 2 we show the function defined on the right-hand side of (17). Similarly to Lemma 1, we prove the following lower bound for D^*.

Lemma 2. *We have* $D^* \ge \sqrt{(q-1)/2 \vee 0}$.

Proof. See Appendix A.

Define the candidate value function $\overline{U}^*(t,x) := \mathbb{E}_{t,x}[|X_{\tau(D^*)}|^q]$ for $0 \le t < 1$ and $x \in \mathbb{R}$. Again, from Lemma 2, for $|x| > D^*\sqrt{1-t}$,

$$\mathcal{L}\overline{U}^*(t,x) = q\left[\frac{q-1}{2} - \frac{|x|^2}{1-t}\right]|x|^{q-2} \le 0.$$

This together with the smooth fit at $D^*\sqrt{1-t}$ and $-D^*\sqrt{1-t}$ verifies the optimality.

Theorem 2. ([3, Theorem 3.2].) *An optimal stopping time for (13) is given by* $\tau(D^*)$ *and the value function* $\overline{U}(t,x)$ *is given by*

$$\overline{U}(t,x) = \overline{U}^*(t,x) = \begin{cases} (1-t)^{q/2}(D^*)^q \dfrac{(F_q+G_q)(x/\sqrt{1-t})}{(F_q+G_q)(D^*)} & \text{if } |x| < D^*\sqrt{1-t}, \\ |x|^q & \text{if } |x| \ge D^*\sqrt{1-t}. \end{cases} \tag{19}$$

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

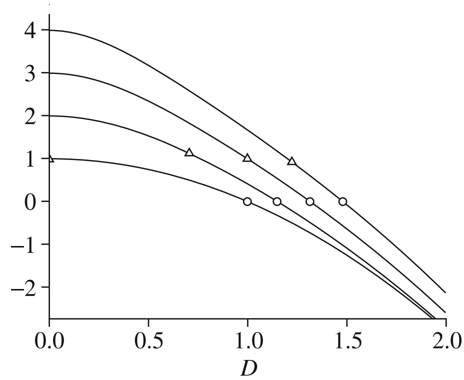

FIGURE 2: Plots of the function $D \mapsto q - D(F_q + G_q)'(D)/(F_q + G_q)(D)$ for $q = 1, 2, 3, 4$. The triangles indicate the points at $\sqrt{(q-1)/2}$ and circles indicate the points at D^*.

3. Problem 1

We first solve the optimal double stopping problem of the form:

$$V(t, x) := \sup_{t \le \tau_1 \le \tau_2 < 1} \mathbb{E}_{t,x}[X_{\tau_2} - X_{\tau_1}], \qquad 0 \le t < 1,\ x \in \mathbb{R}.$$

First, by the strong Markov property, we can write this as a two-stage problem

$$V(t, x) = \sup_{t \le \tau < 1} \mathbb{E}_{t,x}[f(\tau, X_\tau)], \tag{20}$$

where $f(t, x) := U(t, x) - x, 0 \le t < 1,\ x \in \mathbb{R}$, with $U(t, x)$ defined in (12) as the value function of a single stopping problem.

It is expected that the first optimal stopping time is of the form

$$\tau^-(C) := \inf\{s \ge t \colon X_s \le C\sqrt{1-s}\} \quad \text{for some } C \in \mathbb{R}. \tag{21}$$

The corresponding second stopping time can be written as $\inf\{s \ge \tau^-(C) \colon X_s \ge B^*\sqrt{1-s}\}$.

For $C \in \mathbb{R}$ and $x > C\sqrt{1-t}$, by (4), (7), and symmetry,

$$\begin{aligned}
\mathbb{E}_{t,x}[C\sqrt{1-\tau^-(C)}] &= \mathbb{E}_{t,x}[X_{\tau^-(C)}] \\
&= -\mathbb{E}_{t,-x}[X_{\tau^+(-C)}] \\
&= \frac{C}{\Phi(-C)} \mathrm{e}^{-C^2/2}\sqrt{1-t}\,\Phi\left(\frac{-x}{\sqrt{1-t}}\right)\mathrm{e}^{x^2/(2(1-t))}.
\end{aligned}$$

Now we focus on the function, for $C \le B^*$,

$$V_C(t, x) := \mathbb{E}_{t,x}[f(\tau^-(C), X_{\tau^-(C)})], \qquad 0 \le t < 1,\ x \in \mathbb{R}.$$

This can be written as $f(t, x)$ for $x \le C\sqrt{1-t}$ whereas for $x > C\sqrt{1-t}$, by (12),

$$\begin{aligned}
V_C(t, x) &= \mathbb{E}_{t,x}[\sqrt{2\pi(1-\tau^-(C))}(1-(B^*)^2)\mathrm{e}^{C^2/2}\Phi(C) - X_{\tau^-(C)}] \\
&= \left(\sqrt{2\pi}(1-(B^*)^2)\mathrm{e}^{C^2/2}\frac{\Phi(C)}{C} - 1\right)\mathbb{E}_{t,x}[C\sqrt{1-\tau^-(C)}] \\
&= v(C)\sqrt{1-t}\sqrt{2\pi}\,\Phi\left(\frac{-x}{\sqrt{1-t}}\right)\mathrm{e}^{x^2/(2(1-t))},
\end{aligned}$$

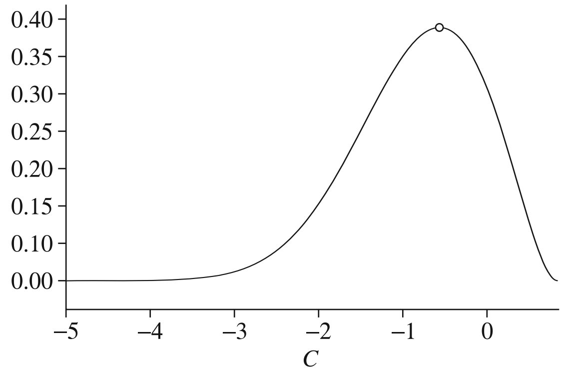

FIGURE 3: The function v on $[-5, B^*]$. The graph indicates that it has a unique maximum $C^* \simeq -0.564$ (the point indicated by the circle); note that $|C^*|$ is smaller than $B^* \simeq 0.84$.

where we define

$$v(C) := \frac{1}{\Phi(-C)}\left[(1-(B^*)^2)\Phi(C) - C\frac{e^{-C^2/2}}{\sqrt{2\pi}}\right], \qquad C \le B^*.$$

The idea now is to identify a C that maximizes $V_C(\cdot,\cdot)$ (or equivalently $v(\cdot)$) and then use a verification lemma to show the optimality of the corresponding strategy. Hence, we consider optimizing the function $v(C)$ on $(-\infty, B^*]$. In Figure 3 we show a plot of this function (using the definition of B^* above). We remark that only the maximality of $v(C)$ and V_C over $(-\infty, B^*]$ is needed; Lemma 3 below is used for the proof of Lemma 4, where only the maximality over $(-\infty, B^*]$ is necessary. It is clearly suboptimal to choose $C > B^*$ as the corresponding strategies would lead to an expected payoff of 0 at or above the boundary $x = C\sqrt{1-t} > B^*\sqrt{1-t}$ with the first and second stoppings happening at the same time.

Lemma 3. *There exists a unique* $C^* < 0$ *that maximizes* $v(\cdot)$ *over* $(-\infty, B^*]$ *such that* $u(C^*) = 0$, *where we define*

$$u(C) := 1-(B^*)^2 - (1-C^2)\Phi(-C) - \frac{C}{\sqrt{2\pi}}e^{-C^2/2}, \qquad C \le B^*.$$

Proof. For all $C \le B^*$, using $\Phi(C)+\Phi(-C)=1$,

$$v'(C) = \frac{e^{-C^2/2}}{\sqrt{2\pi}(\Phi(-C))^2}u(C) \tag{22}$$

and

$$u'(C) = 2C\Phi(-C) - \frac{C^2-1}{\sqrt{2\pi}}e^{-C^2/2} - \frac{1}{\sqrt{2\pi}}e^{-C^2/2} + \frac{C^2}{\sqrt{2\pi}}e^{-C^2/2} = 2C\Phi(-C).$$

On $(-\infty, 0)$, $u'(C)$ is uniformly negative. Moreover, $u(-\infty) = \infty$ and $u(0) = \frac{1}{2} - (B^*)^2 < 0$. Thanks to the continuity of u, this implies that there is a unique solution to the equation $u(C) = 0$ on $(-\infty, 0)$, which we call C^*.

It remains to show that C^* indeed maximizes the function v over $(-\infty, B^*]$. From (22), we see that $v'(C)$ and $u(C)$ are of the same sign. Hence, on $(-\infty, 0)$, v is strictly increasing on $(-\infty, C^*)$ and is strictly decreasing on $(C^*, 0)$ showing that C^* is the unique maximizer on $(-\infty, 0]$.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

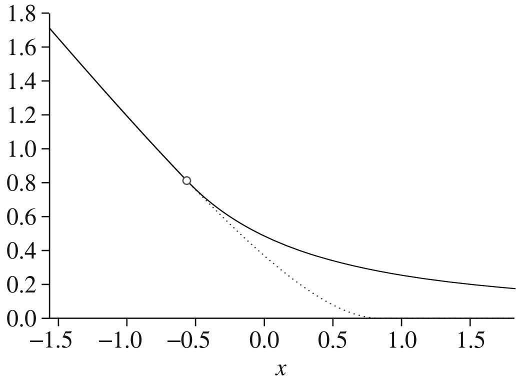

FIGURE 4: Plots of $V^*(0, \cdot)$ (*solid*) and $f(0, \cdot)$ (*dotted*). The circle indicates the point at C^*.

We now extend this result to the domain $(0, B^*]$. On $(0, B^*]$, u' is uniformly positive and, hence, u is monotonically increasing. Hence, v will be strictly increasing on (z, ∞) as soon as $v'(z) = 0$ (or $u(z) = 0$) for some $z > 0$. This means that v on $(0, B^*]$ is dominated by the maximum value of $v(0)$ and $v(B^*)$. Finally, observe that $v(B^*) = 0$ (by how B^* is chosen as in (11)), which is smaller than $v(C^*)$. This completes the proof.

Now we define our candidate value function, for $0 \le t < 1$ and $x \in \mathbb{R}$,

$$V^*(t, x) := V_{C^*}(t, x) = \begin{cases} \sqrt{1-t}\sqrt{2\pi}\,\Phi\left(\dfrac{-x}{\sqrt{1-t}}\right)\mathrm{e}^{x^2/(2(1-t))}v(C^*), & x > C^*\sqrt{1-t}, \\ f(t, x), & x \le C^*\sqrt{1-t}. \end{cases} \tag{23}$$

In Figure 4 we plot the functions V^* and f for fixed $t = 0$ as a function of x; this suggests Lemmas 4 and 5, which we shall prove analytically below.

Lemma 4. *We have* $V^*(t, x) \ge f(t, x)$ *for any* $0 \le t < 1$ *and* $x \in \mathbb{R}$.

Proof. To derive the inequality, we remark that we only need to consider $x < B^*\sqrt{1-t}$. Indeed, for $x \ge B^*\sqrt{1-t}$ it holds that $V^*(t, x) \ge 0 = f(t, x)$.

Consider $C^*\sqrt{1-t} < x < B^*\sqrt{1-t}$. Due to continuous fit and the maximality of C^* on $(-\infty, B^*]$ as in Lemma 3, we derive that

$$V^*(t, x) = V_{C^*}(t, x) \ge V_{x/\sqrt{1-t}}(t, x) = f(t, x).$$

Finally, for $x \le C^*\sqrt{1-t}$, we have $V^*(t, x) = f(t, x)$.

Before we verify the optimality, we shall prove the smoothness so as to use Itô's formula. In view of (23), $V^*(t, x)$ is twice-differentiable in x at any (t, x) such that $x \ne C^*\sqrt{1-t}$. Hence, the smoothness on $x = C^*\sqrt{1-t}$ is our only concern.

Lemma 5. *We have smooth fit:*

$$\lim_{x \downarrow C^*\sqrt{1-t}} \frac{\partial}{\partial x} V^*(t, x) = \lim_{x \uparrow C^*\sqrt{1-t}} \frac{\partial}{\partial x} f(t, x), \qquad 0 \le t < 1. \tag{24}$$

Proof. See Appendix A.

Note that V^* is continuously differentiable in t for any (t, x) such that $x \neq C^*\sqrt{1-t}$; the differentiability on $x = C^*\sqrt{1-t}$ in t can be shown by slightly modifying the proof of Lemma 5.

Lemma 6. (i) *For* (t, x) *such that* $x > C^*\sqrt{1-t}$, *we have* $\mathcal{L}V^*(t, x) = 0$.

(ii) *For* (t, x) *such that* $x < C^*\sqrt{1-t}$, *we have* $\mathcal{L}V^*(t, x) \leq 0$.

Proof of Lemma 6(i). It is clear by the construction of the expected value as a solution to the ODE (3).

Proof of Lemma 6(ii). As $V^*(t, x) = U(t, x) - x$ and because $\mathcal{L}U(t, x) = 0$ for $x < C^*\sqrt{1-t} < B^*\sqrt{1-t}$ in view of (12),

$$\mathcal{L}V^*(t, x) = \mathcal{L}U(t, x) + \frac{x}{1-t} = \frac{x}{1-t},$$

which is negative as $x < C^*\sqrt{1-t} < 0$.

We now have the main result of this section.

Theorem 3. *The function* V^* *as defined in (23) is the value function. Namely,* $V(t, x) = V^*(t, x)$ *for every* $0 \leq t < 1$ *and* $x \in \mathbb{R}$. *Optimal stopping times are*

$$\tau_1^* := \tau^-(C^*), \qquad \tau_2^* := \inf\{s \geq \tau^-(C^*) \colon X_s \geq B^*\sqrt{1-s}\}.$$

Proof. Thanks to the smooth fit as in Lemma 5, Itô's formula applies and for all (s, X_s) such that $X_s \neq C^*\sqrt{1-s}$,

$$\mathrm{d}V^*(s, X_s) = \mathcal{L}V^*(s, X_s)\,\mathrm{d}s + \frac{\partial}{\partial x}V^*(s, X_s)\,\mathrm{d}W_s \leq \frac{\partial}{\partial x}V^*(s, X_s)\,\mathrm{d}W_s, \tag{25}$$

where the inequality holds by Lemma 6.

In (20), because stopping at or above $B^*\sqrt{1-t}$ attains a zero payoff, which is clearly suboptimal, we can focus on stopping times ν such that $X_\nu < B^*\sqrt{1-\nu}$ (and hence $f(\nu, X_\nu) > 0$) a.s. For any such $[t, 1)$-valued stopping time ν with $\tau(M)$ as defined in (14) for $M > B^*$, we have

$$\mathbb{E}_{t,x}[f(\nu \wedge \tau(M), X_{\nu\wedge\tau(M)})] \leq \mathbb{E}_{t,x}[V^*(\nu \wedge \tau(M), X_{\nu\wedge\tau(M)})] \leq V^*(t, x),$$

where the first and second inequalities hold by Lemma 4 and (25), respectively.

In order to take $M \to \infty$, we decompose the left-hand side as

$$\mathbb{E}_{t,x}[f(\nu \wedge \tau(M), X_{\nu\wedge\tau(M)})] = \mathbb{E}_{t,x}[f(\nu, X_\nu)\,\mathbf{1}_{\{\nu<\tau(M)\}}] + \mathbb{E}_{t,x}[f(\tau(M), X_{\tau(M)})\,\mathbf{1}_{\{\nu\geq\tau(M)\}}].$$

The first expectation on the right-hand side converges via monotone convergence to $\mathbb{E}_{t,x}[f(\nu, X_\nu)]$ because f is nonnegative. On the other hand, $f(\tau(M), X_{\tau(M)})\,\mathbf{1}_{\{\nu\geq\tau(M)\}}$ is uniformly integrable for $\{\tau(M), M > B^*\}$. Indeed, $f(s, y) = 0$ for $y \geq B^*\sqrt{1-s}$. In addition, for $y < B^*\sqrt{1-s}$, from the first equation of (12), it follows that $U(s, y) \leq \sqrt{1-s}B^*$ and, hence, we have a bound

$$|f(s, y)| = U(s, y) - y \leq \sqrt{1-s}B^* + |y|, \qquad y < B^*\sqrt{1-s}.$$

Therefore,

$$|f(\tau(M), X_{\tau(M)})|\,\mathbf{1}_{\{\nu\geq\tau(M)\}} \leq \sqrt{1-\tau(M)}B^* + |X_{\tau(M)}|,$$

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

which is uniformly integrable in view of (16) (which is maximized by setting $D = D^*$). Now as $M \to \infty$, because $\tau(M) \to 1$ and $X_{\tau(M)} \to 0$ a.s., we have

$$\mathbb{E}_{t,x}[f(\tau(M), X_{\tau(M)})\, \mathbf{1}_{\{\nu \geq \tau(M)\}}] \to 0.$$

In summary, we have $\mathbb{E}_{t,x}[f(\nu, X_\nu)] \leq V^*(t,x)$.

This together with the fact V^* is attained by an admissible stopping time $\tau^-(C^*) \in [t, 1)$ concludes the proof.

4. Problem 2

We now consider the problem, for given integer $n \geq 0$,

$$\begin{aligned} J(t,x) := \sup_{t \leq \tau_1 \leq \tau_2 < 1} & \mathbb{E}_{t,x}[(X_{\tau_2}^{2n+1} - X_{\tau_1}^{2n+1})\, \mathbf{1}_{\{X_{\tau_1} \leq 0\}} \\ & + (X_{\tau_1}^{2n+1} - X_{\tau_2}^{2n+1})\, \mathbf{1}_{\{X_{\tau_1} > 0\}}], \qquad 0 \leq t < 1,\ x \in \mathbb{R}. \end{aligned}$$

By the strong Markov property, we can write it as

$$J(t,x) = \sup_{t \leq \tau < 1} \mathbb{E}_{t,x}[g(\tau, X_\tau)],$$

where $g(t,x) := (U(t,x) - x^{2n+1})\mathbf{1}_{\{x \leq 0\}} + (U(t,-x) + x^{2n+1})\mathbf{1}_{\{x>0\}}$, $0 \leq t < 1,\ x \in \mathbb{R}$.

By the symmetry of g with respect to x, we expect, for some $D \geq 0$, that the first stopping time has a form $\tau(D)$ defined as in (14). The second stopping time can be written as

$$\begin{aligned} &\inf\{s \geq \tau(D) \colon X_s \geq B^*\sqrt{1-s}\} \quad \text{if } X_{\tau(D)} \leq -D\sqrt{1-\tau(D)}, \\ &\inf\{s \geq \tau(D) \colon X_s \leq -B^*\sqrt{1-s}\} \quad \text{if } X_{\tau(D)} \geq D\sqrt{1-\tau(D)}. \end{aligned}$$

Define the corresponding payoff by

$$J_D(t,x) := \mathbb{E}_{t,x}[g(\tau(D), X_{\tau(D)})], \qquad 0 \leq t < 1,\ x \in \mathbb{R}.$$

We first write it as a function of F_{2n+1} and G_{2n+1} as defined in (2).

Lemma 7. *Given $D \geq 0$, we have for all $0 \leq t < 1$ and $-D\sqrt{1-t} \leq x \leq D\sqrt{1-t}$,*

$$J_D(t,x) = (1-t)^{n+1/2}(F_{2n+1} + G_{2n+1})\left(\frac{x}{\sqrt{1-t}}\right) j(D), \tag{26}$$

where

$$j(D) := \frac{1}{(F_{2n+1} + G_{2n+1})(D)} \left[D^{2n+1} + (B^*)^{2n+1} \frac{G_{2n+1}(D)}{F_{2n+1}(B^*)} \right].$$

Proof. Under the initial condition $-D\sqrt{1-t} \leq x \leq D\sqrt{1-t}$, we have $\mathbb{P}_{t,x}$-a.s.,

$$\begin{aligned} &g(\tau(D), X_{\tau(D)}) \\ &\quad = (U(\tau(D), X_{\tau(D)}) - X_{\tau(D)}^{2n+1})\, \mathbf{1}_{\{X_{\tau(D)} \leq 0\}} + (U(\tau(D), -X_{\tau(D)}) + X_{\tau(D)}^{2n+1})\, \mathbf{1}_{\{X_{\tau(D)} > 0\}} \\ &\quad = (U(\tau(D), -D\sqrt{1-\tau(D)}) + D^{2n+1}(1-\tau(D))^{n+1/2})\, \mathbf{1}_{\{X_{\tau(D)} \leq 0\}} \\ &\qquad + (U(\tau(D), -D\sqrt{1-\tau(D)}) + D^{2n+1}(1-\tau(D))^{n+1/2})\, \mathbf{1}_{\{X_{\tau(D)} > 0\}} \\ &\quad = U(\tau(D), -D\sqrt{1-\tau(D)}) + D^{2n+1}(1-\tau(D))^{n+1/2}. \end{aligned}$$

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

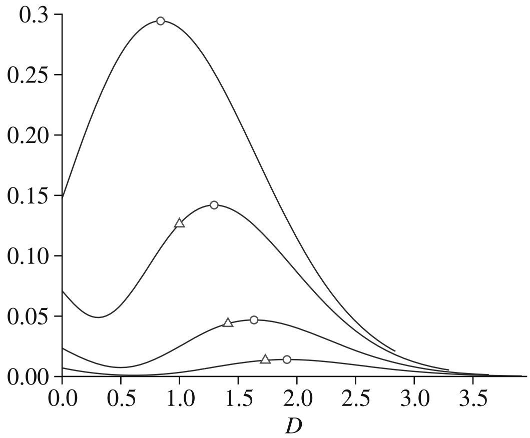

FIGURE 5: Plots of j for $n = 0, 1, 2, 3$. Circles indicate the points at B^* and triangles indicate the points at $\sqrt{n}$.

Hence, by (10), we can write

$$\begin{aligned} J_D(t, x) &= \mathbb{E}_{t,x}[U(\tau(D), -D\sqrt{1 - \tau(D)}) + D^{2n+1}(1 - \tau(D))^{n+1/2}] \\ &= \mathbb{E}_{t,x}[(1 - \tau(D))^{n+1/2}]\bigg[D^{2n+1} + (B^*)^{2n+1}\frac{F_{2n+1}(-D)}{F_{2n+1}(B^*)}\bigg]. \end{aligned}$$

The proof is now complete by (15).

In view of (26), we want to maximize the function j. It turns out that it is maximized by B^* as in (8). See Figure 5 for a numerical plot of this function.

Lemma 8. *For any $n \geq 0$, B^* maximizes j.*

Proof. For any $D \geq 0$,

$$\begin{aligned} j(D) &= j(B^*) + \frac{D^{2n+1}}{(F_{2n+1} + G_{2n+1})(D)} - \frac{(B^*)^{2n+1}}{(F_{2n+1} + G_{2n+1})(D)}\frac{F_{2n+1}(D)}{F_{2n+1}(B^*)} \\ &= j(B^*) + \frac{D^{2n+1}}{(F_{2n+1} + G_{2n+1})(D)}\bigg[1 - \frac{(B^*)^{2n+1}}{D^{2n+1}}\frac{F_{2n+1}(D)}{F_{2n+1}(B^*)}\bigg]. \end{aligned}$$

Because B^* is the maximizer of $B \mapsto B^{2n+1}/F_{2n+1}(B)$ (see (9)) and F_{2n+1} is nonnegative, we have

$$1 - \frac{(B^*)^{2n+1}}{D^{2n+1}}\frac{F_{2n+1}(D)}{F_{2n+1}(B^*)} \leq 0,$$

which shows that $j(D) \leq j(B^*)$, as desired.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

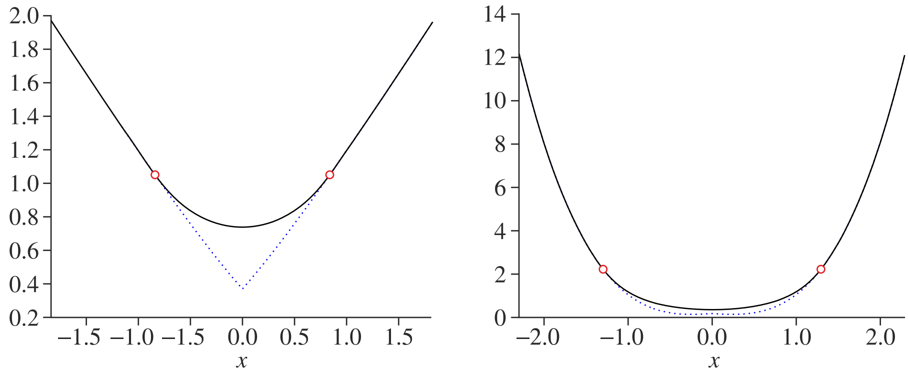

FIGURE 6: Plots of $J^*(0, \cdot)$ (*solid*) and $g(0, \cdot)$ (*dotted*) for $n = 0$ (*left*) and $n = 1$ (*right*). Circles indicate the points at B^* and $-B^*$.

Remark 1. We have $j(B^*) = (B^*)^{2n+1}/F_{2n+1}(B^*)$.

Now setting $D = B^*$, we have our candidate value function

$$J^*(t, x) := J_{B^*}(t, x)$$
$$= \begin{cases} (1-t)^{n+1/2}(F_{2n+1} + G_{2n+1})\left(\dfrac{x}{\sqrt{1-t}}\right) j(B^*), & |x| < B^*\sqrt{1-t}, \\ g(t, x), & |x| \geq B^*\sqrt{1-t}. \end{cases} \tag{27}$$

In Figure 6 we plot J^* and g for $t = 0$ as a function of x; this suggests Lemmas 9 and 10, which we shall prove analytically below.

Lemma 9. *We have* $J^*(t, x) \geq g(t, x)$ *for any* $0 \leq t < 1$ *and* $x \in \mathbb{R}$.

Proof. Suppose that $0 < x < B^*\sqrt{1-t}$. Due to continuous fit and the maximality of B^* on $[0, \infty)$, we derive that

$$J^*(t, x) = J_{B^*}(t, x) \geq J_{x/\sqrt{1-t}}(t, x) = g(t, x).$$

Suppose that $0 \geq x > -B^*\sqrt{1-t}$. By the symmetry of J^* and g with respect to x,

$$J^*(t, x) = J^*(t, |x|) \geq g(t, |x|) = g(t, x).$$

Finally, for $|x| \geq B^*\sqrt{1-t}$, we have $J^*(t, x) = g(t, x)$.

In view of (27), $J^*(t, x)$ is twice-differentiable in x at any (t, x) such that $|x| \neq B^*\sqrt{1-t}$. As we shall show next, on $|x| = B^*\sqrt{1-t}$, differentiability holds (the differentiability with respect to t holds similarly).

Lemma 10. *We have smooth fit. For all* $0 \leq t < 1$,

$$\lim_{x \uparrow B^*\sqrt{1-t}} \frac{\partial}{\partial x} J^*(t, x) = \lim_{x \downarrow B^*\sqrt{1-t}} \frac{\partial}{\partial x} g(t, x),$$
$$\lim_{x \downarrow -B^*\sqrt{1-t}} \frac{\partial}{\partial x} J^*(t, x) = \lim_{x \uparrow -B^*\sqrt{1-t}} \frac{\partial}{\partial x} g(t, x).$$

Proof. See Appendix A.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

Lemma 11. (i) *For* (t, x) *such that* $|x| < B^*\sqrt{1-t}$, *we have* $\mathcal{L}J^*(t, x) = 0$.

(ii) *For* (t, x) *such that* $|x| > B^*\sqrt{1-t}$, *we have* $\mathcal{L}J^*(t, x) \le 0$.

Proof of Lemma 11(i). This holds immediately in view of (27) by the fact that $F_{2n+1}+G_{2n+1}$ solves the ODE (3).

Proof of Lemma 11(ii). For $x > B^*\sqrt{1-t}$, as $J^*(t, x) = U(t, -x) + x^{2n+1}$ and because $\mathcal{L}U(t, -x) = 0$ for $-x < 0 < B^*\sqrt{1-t}$ in view of (10),

$$\mathcal{L}J^*(t, x) = \mathcal{L}U(t, -x) + (2n+1)\left[n - \frac{x^2}{1-t}\right]x^{2n-1} = -(2n+1)\left[\frac{x^2}{1-t} - n\right]|x|^{2n-1}.$$

For $x < -B^*\sqrt{1-t}$, as $J^*(t, x) = U(t, x) - x^{2n+1}$ and because $\mathcal{L}U(t, x) = 0$ for $x < 0 < B^*\sqrt{1-t}$,

$$\mathcal{L}J^*(t, x) = \mathcal{L}U(t, x) - (2n+1)\left[n - \frac{x^2}{1-t}\right]x^{2n-1} = -(2n+1)\left[\frac{x^2}{1-t} - n\right]|x|^{2n-1}.$$

Hence, the proof is complete by Lemma 1.

By Lemmas 10 and 11, the verification of optimality is immediate. We omit the proof of the following theorem because it is essentially the same as that of Theorem 3.

Theorem 4. *The function* J^* *is the value function. Namely,* $J(t, x) = J^*(t, x)$ *for every* $0 \le t < 1$ *and* $x \in \mathbb{R}$. *Optimal stopping times are*

$$\tau_1^* := \tau(B^*),$$
$$\tau_2^* := \begin{cases} \inf\{s \ge \tau(B^*) \colon X_s \ge B^*\sqrt{1-s}\} & \text{if } X_{\tau(B^*)} \le -B^*\sqrt{1-\tau(B^*)}, \\ \inf\{s \ge \tau(B^*) \colon X_s \le -B^*\sqrt{1-s}\} & \text{if } X_{\tau(B^*)} \ge B^*\sqrt{1-\tau(B^*)}. \end{cases}$$

5. Problem 3

Our last problem is to solve, for fixed $q > 0$,

$$W(t, x) := \sup_{t \le \tau_1 \le \tau_2 < 1} \mathbb{E}_{t,x}[|X_{\tau_2}|^q - |X_{\tau_1}|^q], \qquad 0 \le t < 1,\ x \in \mathbb{R}.$$

By the strong Markov property, it can be written as

$$W(t, x) = \sup_{t \le \tau < 1} \mathbb{E}_{t,x}[h(\tau, X_\tau)], \qquad 0 \le t < 1,\ x \in \mathbb{R}, \tag{28}$$

where

$$h(t, x) := \begin{cases} \overline{U}(t, x) - |x|^q & \text{if } |x| < D^*\sqrt{1-t}, \\ 0 & \text{if } |x| \ge D^*\sqrt{1-t}. \end{cases}$$

Here, $\overline{U}(t, x)$ is the value function of (13) and is written as (19) with the same D^* that satisfies (18).

It is easily conjectured that the optimal stopping time for the problem (28) is given by

$$\sigma(A) := \inf\{s \ge t \colon |X_s| \le A\sqrt{1-s}\}$$

for some $A \in [0, D^*]$. Let us define its corresponding expected payoff by

$$W_A(t, x) := \mathbb{E}_{t,x}[h(\sigma(A), X_{\sigma(A)})], \qquad 0 \le t < 1,\ x \in \mathbb{R}.$$

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

Lemma 12. *For all* (t, x) *such that* $|x| \geq A\sqrt{1-t}$,

$$W_A(t, x) = (1-t)^{q/2} G_q\left(\frac{|x|}{\sqrt{1-t}}\right) w(A), \tag{29}$$

where we define

$$w(A) := \frac{1}{G_q(A)}\left[(D^*)^q \frac{(F_q+G_q)(A)}{(F_q+G_q)(D^*)} - A^q\right], \qquad 0 \leq A \leq D^*.$$

Proof. Suppose first that $x \geq A\sqrt{1-t}$ (then we must have $X_{\sigma(A)} = A\sqrt{1-\sigma(A)}$ a.s.). By (19),

$$W_A(t, x) = \mathbb{E}_{t,x}[(1-\sigma(A))^{q/2}]\left[(D^*)^q \frac{(F_q+G_q)(A)}{(F_q+G_q)(D^*)} - A^q\right] \tag{30}$$

and, hence, the problem reduces down to computing the expectation on the right-hand side. As we have discussed in Section 2, we can write

$$\mathbb{E}_{t,x}[(1-\sigma(A))^{q/2}] = (1-t)^{q/2}\zeta\left(\frac{x}{\sqrt{1-t}}\right),$$

where the function ζ satisfies the ODE (3) with boundary conditions $\zeta(A) = 1$ and $\lim_{y\to\infty}\zeta(y) = 0$. A general solution of (3) is given by $\zeta(y) = \alpha F_q(y) + \beta G_q(y)$ with the values of α and β to be determined. Because $\lim_{y\to\infty} F_q(y) = \infty$ and $\lim_{y\to\infty} G_q(y) = 0$, we must have $\alpha = 0$. Solving $\zeta(A) = 1$, we have $\beta = G_q(A)^{-1}$. Substituting this into the right-hand side of (30) gives (29).

Finally, by symmetry, we have $W_A(t, x) = W_A(t, -x)$, $x \geq 0$. Hence, the result can be extended to $x \leq -A\sqrt{1-t}$ as well.

In view of (29), we shall maximize the function w. As shown in Figure 7, w admits a unique maximizer. We shall show this analytically in Lemma 13 below. Note that

$$w(0) = \frac{2(D^*)^q}{(F_q+G_q)(D^*)} > 0, \qquad w(D^*) = 0, \tag{31}$$

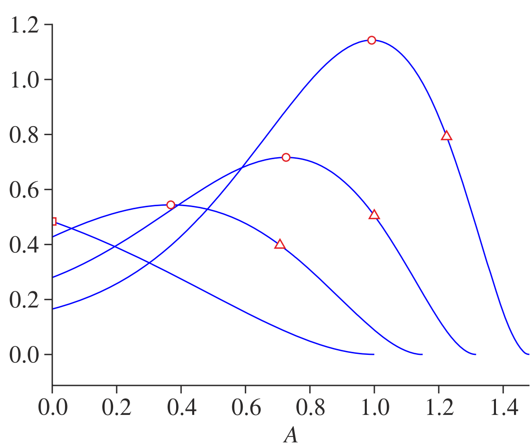

FIGURE 7: Plots of $w(A)$ on $[0, D^*]$ for $q = 1, 2, 3, 4$. For $q = 2, 3, 4$, the circles (respectively triangles) indicate the points at A^* (respectively $\sqrt{(q-1)/2}$). For $q = 1$, the square indicates the point at $A^* = \sqrt{(q-1)/2} = 0$.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

and, for all $A > 0$,

$$w'(A) = \left[(D^*)^q \frac{(F_q + G_q)'(A)}{(F_q + G_q)(D^*)} - qA^{q-1}\right] \frac{1}{G_q(A)} - \left[(D^*)^q \frac{(F_q + G_q)(A)}{(F_q + G_q)(D^*)} - A^q\right] \frac{G_q'(A)}{(G_q(A))^2}. \tag{32}$$

Lemma 13. (i) *There exists a unique maximizer of w over $[0, D^*]$, which we call A^*, such that*

$$A^* \le \sqrt{\frac{(q-1)}{2} \vee 0} \tag{33}$$

and

$$w'(A) \le 0 \iff A \ge A^*, \qquad 0 \le A \le D^*.$$

(ii) *Moreover, $A^* = 0$ if and only if $q \le 1$.*

Proof of Lemma 13(i). For all $A > 0$, let us define

$$\begin{aligned} L(A) &:= \frac{(F_q + G_q)'(A)G_q(A) - (F_q + G_q)(A)G_q'(A)}{(G_q(A))^2} \\ &= \frac{F_q'(A)G_q(A) - F_q(A)G_q'(A)}{(G_q(A))^2} \\ &> 0. \end{aligned}$$

Then, for any $A > 0$,

$$\begin{aligned} \frac{w'(A)}{L(A)} &= \frac{1}{L(A)}\left[-\frac{qA^{q-1}}{G_q(A)} + \frac{A^q G_q'(A)}{(G_q(A))^2}\right] + \frac{(D^*)^q}{(F_q + G_q)(D^*)} \\ &= \frac{-qA^{q-1}G_q(A) + A^q G_q'(A)}{F_q'(A)G_q(A) - F_q(A)G_q'(A)} + \frac{(D^*)^q}{(F_q + G_q)(D^*)}. \end{aligned} \tag{34}$$

Because F_q and G_q satisfy (3),

$$F_q''(A) = AF_q'(A) + qF_q(A), \qquad G_q''(A) = AG_q'(A) + qG_q(A).$$

It follows that

$$\begin{aligned} F_q''(A)G_q(A) - F_q(A)G_q''(A) &= [AF_q'(A) + qF_q(A)]G_q(A) - F_q(A)[AG_q'(A) + qG_q(A)] \\ &= A[F_q'(A)G_q(A) - F_q(A)G_q'(A)]. \end{aligned}$$

Now differentiating (34) with respect to A,

$$\begin{aligned} &[F_q'(A)G_q(A) - F_q(A)G_q'(A)]^2 \frac{\partial}{\partial A} \frac{w'(A)}{L(A)} \\ &\quad = [-q(q-1)A^{q-2}G_q(A) + A^q G_q''(A)][F_q'(A)G_q(A) - F_q(A)G_q'(A)] \\ &\qquad - [-qA^{q-1}G_q(A) + A^q G_q'(A)][F_q''(A)G_q(A) - F_q(A)G_q''(A)] \end{aligned}$$

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

$$= [-q(q-1)A^{q-2}G_q(A) + A^{q+1}G_q'(A) + qA^qG_q(A)]$$
$$\times [F_q'(A)G_q(A) - F_q(A)G_q'(A)]$$
$$- [-qA^qG_q(A) + A^{q+1}G_q'(A)][F_q'(A)G_q(A) - F_q(A)G_q'(A)]$$
$$= 2qA^{q-2}G_q(A)[F_q'(A)G_q(A) - F_q(A)G_q'(A)]\left[A^2 - \frac{(q-1)}{2}\right].$$

That is, $w'(A)/L(A)$ is differentiable and

$$\left(\frac{F_q'(A)G_q(A) - F_q(A)G_q'(A)}{2qA^{q-2}G_q(A)}\right)\left(\frac{\partial}{\partial A}\right)\left(\frac{w'(A)}{L(A)}\right) = A^2 - \frac{q-1}{2}.$$

Note that $F_q'(A)G_q(A) - F_q(A)G_q'(A)$ and $G_q(A)$ are both positive for $A > 0$. Hence, we see that

$$\left(\frac{\partial}{\partial A}\right)\left(\frac{w'(A)}{L(A)}\right) > 0 \iff A > \sqrt{\frac{(q-1)}{2} \vee 0}, \qquad A > 0.$$

Recall D^* or the unique root of (17). The equivalence above together with $w'(D^*) = 0$ (and $w'(D^*)/L(D^*) = 0$) and recalling that $D^* \geq \sqrt{(q-1)/2 \vee 0}$ (as in Lemma 2) and $L(\cdot) > 0$ shows that there exists a unique $A^* \in [0, \sqrt{(q-1)/2 \vee 0}]$ such that, for all $0 < A < D^*$,

$$w'(A) \leq 0 \iff A \geq A^* \tag{35}$$

as desired.

Proof of Lemma 13(ii). First suppose that $q \leq 1$. Then $A^* \leq \sqrt{(q-1)/2 \vee 0} = 0$ and, hence, we must have $A^* = 0$. Now suppose that $q > 1$. Then, by taking a limit in (32) and noting that $(F_q + G_q)'(0) = 0$,

$$w'(0+) = -(D^*)^q \frac{(F_q + G_q)(0)}{(F_q + G_q)(D^*)} \frac{G_q'(0)}{(G_q(0))^2} > 0.$$

This together with (35) shows that $A^* > 0$.

Now we define

$$W^*(t,x) := W_{A^*}(t,x) = \begin{cases} (1-t)^{q/2} G_q\left(\dfrac{|x|}{\sqrt{1-t}}\right) w(A^*), & |x| > A^*\sqrt{1-t}, \\ h(t,x), & |x| \leq A^*\sqrt{1-t} \end{cases} \tag{36}$$

as our candidate value function, and verify the optimality. In Figure 8 we plot W^* and h for $t = 0$ as a function of x; this suggests Lemmas 14 and 15, which we shall prove analytically below.

Lemma 14. *We have* $W^*(t,x) \geq h(t,x)$ *for* $0 \leq t < 1$ *and* $x \in \mathbb{R}$.

Proof. Due to the symmetry of both W^* and h with respect to x, it is sufficient to show for $x \geq 0$.

When $x \geq D^*\sqrt{1-t}$, then $W^*(t,x) \geq 0 = h(t,x)$. When $0 \leq x \leq A^*\sqrt{1-t}$, then $W^*(t,x) = h(t,x)$ by definition. Finally, if $A^*\sqrt{1-t} < x < D^*\sqrt{1-t}$, due to continuous fit and the maximality of A^* on $[0,\infty)$ we derive that $W^*(t,x) = W_{A^*}(t,x) \geq W_{x/\sqrt{1-t}}(t,x) = h(t,x)$, as desired.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

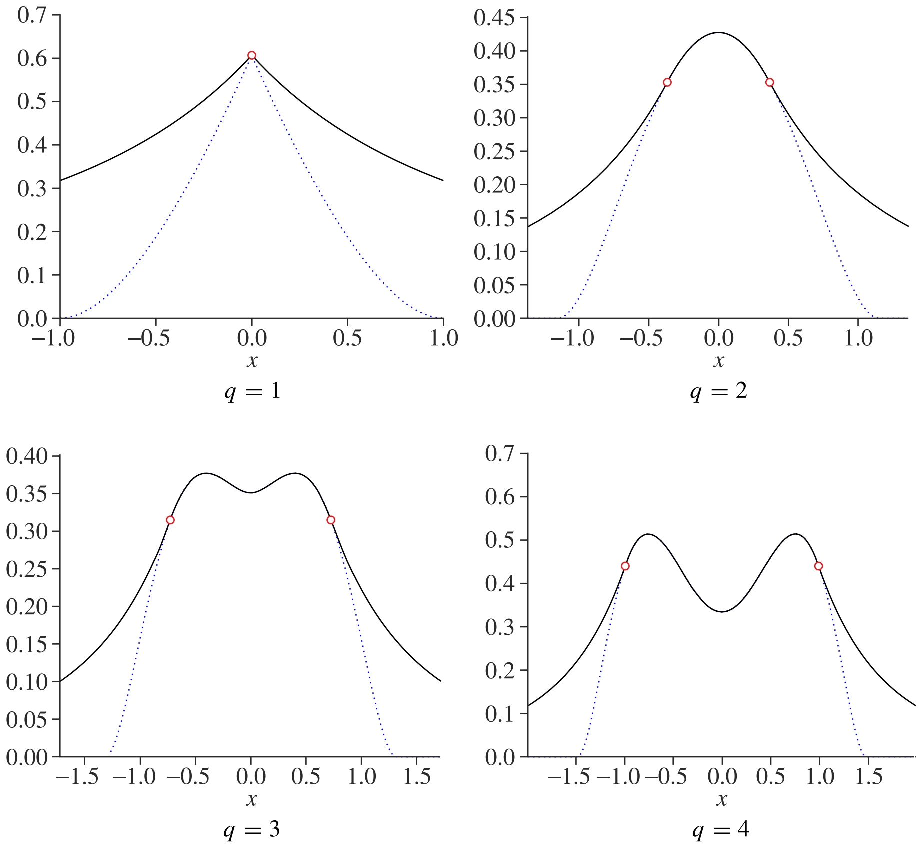

FIGURE 8: Plots of $W^*(0, \cdot)$ (*solid*) and $h(0, \cdot)$ (*dotted*) for $q = 1, 2, 3, 4$. Circles indicate the points at A^* and $-A^*$.

Similarly to Problems 1 and 2, $W^*(t, x)$ is smooth enough to apply Itô's formula at any (t, x) such that $|x| \neq A^*\sqrt{1-t}$; see also (40) below regarding the smoothness at 0 when $A^* \neq 0$. Regarding the smoothness on $|x| = A^*\sqrt{1-t}$, we have the following.

Lemma 15. *Fix* $0 \le t < 1$. (i) *If* $A^* > 0$, *then smooth fit*

$$\lim_{x \downarrow A^*\sqrt{1-t}} \frac{\partial}{\partial x} W^*(t, x) = \lim_{x \uparrow A^*\sqrt{1-t}} \frac{\partial}{\partial x} h(t, x), \tag{37}$$

$$\lim_{x \uparrow -A^*\sqrt{1-t}} \frac{\partial}{\partial x} W^*(t, x) = \lim_{x \downarrow -A^*\sqrt{1-t}} \frac{\partial}{\partial x} h(t, x) \tag{38}$$

holds.

(ii) *If* $A^* = 0$, *we have*

$$\lim_{x \downarrow 0} \frac{\partial}{\partial x} W^*(t, x) < \lim_{x \uparrow 0} \frac{\partial}{\partial x} W^*(t, x). \tag{39}$$

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

Proof of Lemma 15(i). We have

$$\lim_{x\downarrow A^*\sqrt{1-t}} \frac{\partial}{\partial x} W^*(t,x) = (1-t)^{(q-1)/2} \frac{G_q'(A^*)}{G_q(A^*)}\left[(D^*)^q \frac{(F_q+G_q)(A^*)}{(F_q+G_q)(D^*)} - (A^*)^q\right].$$

On the other hand,

$$\lim_{x\uparrow A^*\sqrt{1-t}} \frac{\partial}{\partial x} h(t,x) = (1-t)^{(q-1)/2}\left[(D^*)^q \frac{(F_q+G_q)'(A^*)}{(F_q+G_q)(D^*)} - q(A^*)^{q-1}\right].$$

When $A^* > 0$, Lemma 13 implies that $w'(A^*) = 0$. In view of (32), the two equations above are the same. Hence, (37) holds. The proof of (38) holds by symmetry.

Proof of Lemma 15(ii). Now suppose that $A^* = 0$. In this case, by (31),

$$W^*(t,x) = 2(1-t)^{q/2}(D^*)^q \frac{G_q(|x|/\sqrt{1-t})}{(F_q+G_q)(D^*)}$$

and, hence, taking derivatives and then limits, we obtain

$$\lim_{x\downarrow 0} \frac{\partial}{\partial x} W^*(t,x) = 2(1-t)^{(q-1)/2}(D^*)^q \frac{G_q'(0)}{(F_q+G_q)(D^*)},$$
$$\lim_{x\uparrow 0} \frac{\partial}{\partial x} W^*(t,x) = -2(1-t)^{(q-1)/2}(D^*)^q \frac{G_q'(0)}{(F_q+G_q)(D^*)}.$$

Because F_q and G_q are nonnegative and G_q' is negative, we have (39) as desired.

Lemma 16. (i) *For* (t,x) *such that* $|x| > A^*\sqrt{1-t}$*, we have* $\mathcal{L}W^*(t,x) = 0$.

(ii) *If* $A^* > 0$ *for* (t,x) *such that* $0 < |x| < A^*\sqrt{1-t}$*, we have* $\mathcal{L}W^*(t,x) \le 0$.

Proof of Lemma 16(i). It is clear by the fact that G_q solves (3) in view of (36).

Proof of Lemma 16(ii). By Lemma 13(ii), $A^* > 0$ guarantees that $q > 1$. Because $|x| < A^*\sqrt{1-t} < D^*\sqrt{1-t}$, we must have $\mathcal{L}\overline{U}(t,x) = 0$ in view of (19).

For $0 < x < A^*\sqrt{1-t}$ as $W^*(t,x) = \overline{U}(t,x) - x^q$, we have

$$\mathcal{L}W^*(t,x) = -\left[\frac{(q-1)}{2} - \frac{x^2}{(1-t)}\right]qx^{q-2},$$

whereas for $-A^*\sqrt{1-t} < x < 0$, we have $\mathcal{L}W^*(t,x) = -[(q-1)/2 - x^2/(1-t)]q|x|^{q-2}$. Now (33) completes the proof.

We now have the following optimality results by Lemmas 14, 15, and 16. An important difference with the verification of Problems 1 and 2 is the potential nondifferentiability at $x = 0$, but this does not cause any issue.

Recall from (39) that for the $A^* = 0$ case, the smooth fit at A^* fails. However, this can be resolved easily by using the following version of Itô's formula (see, e.g. [10]), for all $t \le u < 1$,

$$\begin{aligned} W^*(u, X_u) &= W^*(t,x) + \int_t^u \mathcal{L}W^*(s,X_s)\,\mathbf{1}_{\{X_s\neq 0\}}\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_t^u \left(\lim_{z\downarrow 0}\frac{\partial}{\partial z}W^*(s,z) - \lim_{z\uparrow 0}\frac{\partial}{\partial z}W^*(s,z)\right)\mathrm{d}l_s^0 + \int_t^u \frac{\partial}{\partial x}W^*(s,X_s)\,\mathbf{1}_{\{X_s\neq 0\}}\,\mathrm{d}W_s, \end{aligned}$$

where $\{l_s^0\}_{t\le s<1}$ denotes the local time of X at 0. The supermartingale property of the process $\{W^*(u, X_u)\}_{t\le u\le 1}$ still holds by (39) and Lemma 16. For the $A^* > 0$ case, Lemma 13(ii) guarantees that $q > 1$. In this case

$$\lim_{z\downarrow 0}\frac{\partial}{\partial z}W^*(s,z) = \lim_{z\downarrow 0}\frac{\partial}{\partial z}h(s,z) = \lim_{z\uparrow 0}\frac{\partial}{\partial z}h(s,z) = \lim_{z\uparrow 0}\frac{\partial}{\partial z}W^*(s,z) \tag{40}$$

by using the equality $(F_q + G_q)'(0) = 0$ and $x^{q-1} \to 0$ as $x \to 0$. The rest of the proof is omitted as it is similar to that of Theorem 3.

Theorem 5. *The function W^* is the value function. Namely, $W(t,x) = W^*(t,x)$ for every $0 \le t < 1$ and $x \in \mathbb{R}$. Optimal stopping times are*

$$\tau_1^* := \sigma(A^*), \qquad \tau_2^* := \inf\{s \ge \sigma(A^*) \colon |X_s| \ge D^*\sqrt{1-s}\}.$$

Appendix A. Proofs

Proof of Lemma 1. The $n = 0$ case is trivial (as $B^* > 0$) and, hence, we shall focus on the $n \ge 1$ case.

Assume for contradiction that $B^* < \sqrt{n}$. Then we can take (t,x) such that $x/\sqrt{1-t} \in (B^*, \sqrt{n})$. By Itô's formula

$$\mathrm{d}X_s^{2n+1} = (2n+1)\bigg[n - \frac{X_s^2}{1-s}\bigg]X_s^{2n-1}\,\mathrm{d}s + (2n+1)X_s^{2n}\,\mathrm{d}W_s, \qquad t \le s < 1. \tag{41}$$

Define the first downcrossing time of X as

$$T^-(\delta) := \inf\{u \ge t \colon X_u \le \delta\}, \qquad \delta \ge 0. \tag{42}$$

Fix $0 < \varepsilon < x$. For $s \in [t, \tau^+(\sqrt{n}) \wedge T^-(\varepsilon)]$, we have $\varepsilon/\sqrt{1-s} \le X_s/\sqrt{1-s} \le \sqrt{n}$ (and hence $\varepsilon \le X_s \le \sqrt{n}\sqrt{1-s}$). Therefore, taking the expectation of the integral of (41), we have

$$\begin{aligned}\mathbb{E}_{t,x}[X^{2n+1}_{\tau^+(\sqrt{n})\wedge T^-(\varepsilon)}] &= x^{2n+1} + \mathbb{E}_{t,x}\bigg[\int_t^{\tau^+(\sqrt{n})\wedge T^-(\varepsilon)}(2n+1)\bigg[n - \frac{X_s^2}{1-s}\bigg]X_s^{2n-1}\,\mathrm{d}s\bigg]\\ &\ge x^{2n+1}.\end{aligned}$$

From dominated convergence, upon $\varepsilon \downarrow 0$, we have

$$\mathbb{E}_{t,x}\Big[X^{2n+1}_{\tau^+(\sqrt{n})\wedge T^-(0)}\Big] \ge x^{2n+1}.$$

Moreover, because

$$X^{2n+1}_{\tau^+(\sqrt{n})} \ge X^{2n+1}_{\tau^+(\sqrt{n})\wedge T^-(0)} \quad \text{a.s.,}$$

we also have $\mathbb{E}_{t,x}[X^{2n+1}_{\tau^+(\sqrt{n})}] \ge x^{2n+1}$.

On the other hand, by (7) and (9), and because $\sqrt{n} > x/\sqrt{1-t} > B^*$, we must have

$$x^{2n+1} = \mathbb{E}_{t,x}[X^{2n+1}_{\tau^+(x/\sqrt{1-t})}] > \mathbb{E}_{t,x}[X^{2n+1}_{\tau^+(\sqrt{n})}].$$

This is a contradiction and, hence, $B^* \ge \sqrt{n}$.

https://doi.org/10.1239/aap/1449859807 Published online by Cambridge University Press

Proof of Lemma 2. Because the $q \le 1$ case is trivial, we focus on the $q > 1$ case. Assume for contradiction that $D^* < \sqrt{(q-1)/2}$. Then we can take (t, x) such that $x/\sqrt{1-t} \in (D^*, \sqrt{(q-1)/2})$.

Using arguments similar to the ones in the proof of Lemma 1, we have

$$\mathbb{E}_{t,x}[|X_{\tau(\sqrt{(q-1)/2})\wedge T^-(0)}|^q] = \mathbb{E}_{t,x}[X^q_{\tau(\sqrt{(q-1)/2})\wedge T^-(0)}] \ge x^q,$$

where $T^-(0)$ is defined as in (42). Moreover, because $|X_{\tau(\sqrt{(q-1)/2})}|^q \ge X^q_{\tau(\sqrt{(q-1)/2})\wedge T^-(0)}$ a.s., we also have $\mathbb{E}_{t,x}[|X_{\tau(\sqrt{(q-1)/2})}|^q] \ge x^q$.

On the other hand, by (16) and (18), and because $\sqrt{(q-1)/2} > x/\sqrt{1-t} > D^*$, we must have

$$x^q = \mathbb{E}_{t,x}[|X_{\tau(x/\sqrt{1-t})}|^q] > \mathbb{E}_{t,x}[|X_{\tau(\sqrt{(q-1)/2})}|^q].$$

This is a contradiction, as desired.

Proof of Lemma 5. Differentiating (23), we have

$$\frac{\partial}{\partial x}V^*(t,x) = v(C^*)\sqrt{2\pi}\bigg(\frac{x}{\sqrt{1-t}}\mathrm{e}^{x^2/(2(1-t))}\Phi\bigg(\frac{-x}{\sqrt{1-t}}\bigg) - \frac{1}{\sqrt{2\pi}}\bigg)$$

and, hence,

$$\begin{aligned}\lim_{x\downarrow C^*\sqrt{1-t}}\frac{\partial}{\partial x}V^*(t,x) &= v(C^*)\sqrt{2\pi}\bigg(C^*\mathrm{e}^{(C^*)^2/2}\Phi(-C^*) - \frac{1}{\sqrt{2\pi}}\bigg)\\ &= C^*\sqrt{2\pi}(1-(B^*)^2)\mathrm{e}^{(C^*)^2/2}\Phi(C^*) - (C^*)^2\\ &\quad - \frac{\Phi(C^*)}{\Phi(-C^*)}(1-(B^*)^2) + C^*\frac{1}{\sqrt{2\pi}}\frac{1}{\Phi(-C^*)}\mathrm{e}^{-(C^*)^2/2}.\end{aligned}$$

On the other hand,

$$\lim_{x\uparrow C^*\sqrt{1-t}}\frac{\partial}{\partial x}f(t,x) = \sqrt{2\pi}(1-(B^*)^2)C^*\mathrm{e}^{(C^*)^2/2}\Phi(C^*) - (B^*)^2.$$

Taking the difference between the two, we have

$$\begin{aligned}&\lim_{x\downarrow C^*\sqrt{1-t}}\frac{\partial}{\partial x}V^*(t,x) - \lim_{x\uparrow C^*\sqrt{1-t}}\frac{\partial}{\partial x}f(t,x)\\ &\quad = 1-(C^*)^2 - \bigg(1+\frac{\Phi(C^*)}{\Phi(-C^*)}\bigg)(1-(B^*)^2) + C^*\frac{1}{\sqrt{2\pi}}\frac{1}{\Phi(-C^*)}\mathrm{e}^{-(C^*)^2/2}\\ &\quad = \frac{1}{\Phi(-C^*)}\bigg[\Phi(-C^*)(1-(C^*)^2) - (\Phi(-C^*)+\Phi(C^*))(1-(B^*)^2)\\ &\qquad\qquad + C^*\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(C^*)^2/2}\bigg]\\ &\quad = -\frac{u(C^*)}{\Phi(-C^*)},\end{aligned}$$

which equals 0 thanks to our choice of C^* as in Lemma 3.

Proof of Lemma 10. We show the differentiability at $x = B^*\sqrt{1-t}$ (that of $x = -B^*\sqrt{1-t}$ holds by symmetry). Differentiating (27) with respect to x, we obtain

$$\frac{\partial}{\partial x} J^*(t,x) = (1-t)^n (F_{2n+1} + G_{2n+1})'\left(\frac{x}{\sqrt{1-t}}\right) j(B^*),$$

and by Remark 1,

$$\begin{aligned}\lim_{x \uparrow B^*\sqrt{1-t}} \frac{\partial}{\partial x} J^*(t,x) &= (1-t)^n (B^*)^{2n+1} \frac{(F_{2n+1} + G_{2n+1})'(B^*)}{F_{2n+1}(B^*)} \\ &= (1-t)^n (B^*)^{2n+1} \frac{F'_{2n+1}(B^*) - F'_{2n+1}(-B^*)}{F_{2n+1}(B^*)}. \end{aligned} \tag{43}$$

On the other hand,

$$\frac{\partial}{\partial x} g(t,x) = -(1-t)^n (B^*)^{2n+1} \frac{F'_{2n+1}(-x/\sqrt{1-t})}{F_{2n+1}(B^*)} + (2n+1)x^{2n}.$$

Hence,

$$\lim_{x \downarrow B^*\sqrt{1-t}} \frac{\partial}{\partial x} g(t,x) = -(1-t)^n (B^*)^{2n+1} \left[\frac{F'_{2n+1}(-B^*)}{F_{2n+1}(B^*)} - \frac{2n+1}{B^*}\right],$$

which equals (43) by (8), as desired.

Acknowledgements

The authors thank the anonymous referee for his/her insightful comments. E. J. Baurdoux was visiting CIMAT, Guanajuato when part of this work was carried out and he is grateful for their hospitality and support. B. A. Surya would like to thank the Department of Statistics of the London School of Economics and the MathFinance Institute of Goethe University of Frankfurt for their hospitality during his stay in 2013, and also the Department of Industrial Engineering and Operations Research of Columbia University during his visit in 2014, where part of this work was carried out. K. Yamazaki is in part supported by MEXT KAKENHI (grant no. 26800092), the Inamori foundation research grant, and the Kansai University Subsidy for Supporting Young Scholars 2014.

References

[1] Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York.
[2] Avellaneda, M. and Lipkin, M. D. (2003). A market-induced mechanism for stock pinning. *Quant. Finance* **3,** 417–425.
[3] Ekström, E. and Wanntorp, H. (2009). Optimal stopping of a Brownian bridge. *J. Appl. Prob.* **46,** 170–180.
[4] Ekström, E., Lindberg, C. and Tysk, J. (2011). Optimal liquidation of a pairs trade. In *Advanced Mathematical Methods for Finance*, Springer, Heidelberg, pp. 247–255.
[5] Gallego, G. and van Ryzin, G. (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Manag. Sci.* **40,** 999–1020.
[6] Ivashko, A. A. (2014). Gain maximization problem in the urn scheme. In *Transactions of Karelian Research Centre of Russian Academy of Science, No 4. Mathematical Modeling and Information Technologies*, pp. 62–66.
[7] Leung, T. and Li, X. (2015). Optimal mean reversion trading with transaction cost and stop-loss exit. *Internat. J. Theoret. Appl. Finance* **18,** 1550020.
[8] Lin, T. C. W. (2013). The new investor. *UCLA Law Rev.* **60,** 678–735.
[9] Mazalov, V. V. and Tamaki, M. (2007). Duration problem on trajectories. *Stochastics* **79,** 211–218.
[10] Peskir, G. (2005). A change-of-variable formula with local time on curves. *J. Theoret. Prob.* **18,** 499–535.

[11] Shepp, L. A. (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.* **40,** 993–1010.
[12] Sofronov, G., Keith, J. M. and Kroese, D. P. (2006). An optimal sequential procedure for a buying-selling problem with independent observations. *J. Appl. Prob.* **43,** 454–462.
[13] Song, Q. S., Yin, G. and Zhang, Q. (2009). Stochastic optimization methods for buying-low-and-selling-high strategies. *Stoch. Anal. Appl.* **27,** 523–542.
[14] Tamaki, M. (2001). Optimal stopping on trajectories and the ballot problem. *J. Appl. Prob.* **38,** 946–959.