DUAL INTEGRAL EQUATIONS WITH TRIGONOMETRIC KERNELS

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1. Consider the dual equations

$$\int_0^\infty \xi^{2\lambda-1}\psi(\xi)\cos(\xi x)d\xi = f(x), \quad 0 \leqq x \leqq 1, \tag{1}$$

$$\int_0^\infty \psi(\xi)\cos(\xi x)d\xi = 0, \qquad x > 1, \tag{2}$$

where $-\frac{1}{2} < \lambda < \frac{1}{2}$.

A solution of these equations has been obtained by Srivastava (**1**) in the form of a Neumann series. In this note a formal solution for the equations (1) and (2) is obtained by a generalisation of a method due to Tranter (**2**) who obtained a solution for the special case when $\lambda = 0$.

We shall use the representations ((**3**), pp. 48 and 170)

$$\frac{\sqrt{\pi}}{2} J_\lambda(r\xi) = \frac{1}{\Gamma(\frac{1}{2}+\lambda)}\left(\frac{\xi}{2r}\right)^\lambda \int_0^r \frac{\cos(\xi x)}{(r^2-x^2)^{\frac{1}{2}-\lambda}}dx, \tag{3}$$

$$= \frac{1}{\Gamma(\frac{1}{2}-\lambda)}\left(\frac{2r}{\xi}\right)^\lambda \int_r^\infty \frac{\sin(\xi x)}{(x^2-r^2)^{\frac{1}{2}+\lambda}}dx, \tag{4}$$

where $-\frac{1}{2} < \lambda < \frac{1}{2}$.

Equations in which $\sin(\xi x)$ replaces $\cos(\xi x)$ in (1) and (2) are also solved.

2. Integrating equation (2) with respect to x we find, as in (**2**), that

$$\int_0^\infty \xi^{-1}\psi(\xi)\sin(\xi x)d\xi = 0, \quad x > 1. \tag{5}$$

Multiplying equations (1) and (5) respectively by $\frac{(2r)^{-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}(r^2-x^2)^{\lambda-\frac{1}{2}}$ and $\frac{(2r)^\lambda}{\Gamma(\frac{1}{2}-\lambda)}(x^2-r^2)^{-\lambda-\frac{1}{2}}$ and integrating with respect to x between 0, r and r, ∞, we find, using the representations (3) and (4), that

$$\frac{\sqrt{\pi}}{2}\int_0^\infty \xi^{\lambda-1}\psi(\xi)J_\lambda(\xi r)d\xi = \frac{(2r)^{-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}\int_0^r \frac{f(x)}{(r^2-x^2)^{\frac{1}{2}-\lambda}}dx, \quad 0 < r < 1,$$

$$= 0, \quad r > 1. \tag{6}$$

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Applying the Hankel inversion theorem to equation (6) gives

$$\sqrt{\pi}\,\xi^{\lambda-2}\psi(\xi)=\frac{2^{1-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}\int_0^1 r^{1-\lambda}J_\lambda(\xi r)dr\int_0^r \frac{f(x)}{(r^2-x^2)^{\frac{1}{2}-\lambda}}\,dx, \tag{7}$$

as a solution of equations (1) and (2). When $\lambda=0$ this reduces to Tranter's solution.

3. To solve the similar pair of equations

$$\int_0^\infty \xi^{2\lambda-1}\psi(\xi)\sin(\xi x)d\xi=f(x),\quad 0\leqq x\leqq 1, \tag{8}$$

$$\int_0^\infty \psi(\xi)\sin(\xi x)d\xi=0,\quad x>1, \tag{9}$$

where $-\frac{1}{2}<\lambda<\frac{1}{2}$, we first differentiate (8) with respect to x and find

$$\int_0^\infty \xi^{2\lambda}\psi(\xi)\cos(\xi x)d\xi=f'(x),\quad 0\leqq x\leqq 1. \tag{10}$$

Operating on equations (10) and (9) in the same way as we have on equations (1) and (5) we get

$$\begin{aligned}\frac{\sqrt{\pi}}{2}\int_0^\infty \xi^{\lambda}\psi(\xi)J_\lambda(\xi r)d\xi &= \frac{(2r)^{-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}\int_0^r \frac{f'(x)}{(r^2-x^2)^{\frac{1}{2}-\lambda}}\,dx,\quad 0<r<1,\\ &=0,\quad r>1.\end{aligned} \tag{11}$$

Hence the solution of equations (8) and (9) is

$$\sqrt{\pi}\,\xi^{\lambda-1}\psi(\xi)=\frac{2^{1-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}\int_0^1 r^{1-\lambda}J_\lambda(\xi r)dr\int_0^r \frac{f'(x)}{(r^2-x^2)^{\frac{1}{2}-\lambda}}\,dx. \tag{12}$$

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