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A partial resolution of Hedden's conjecture on satellite homomorphisms

BY RANDALL JOHANNINGSMEIER, HILLARY KIM AND ALLISON N. MILLER
Department of Mathematics and Statistics, Swarthmore College, 500 *College Avenue, Swarthmore, PA* 19081, *U.S.A.*
e-mails: randall.jmeier@gmail.com, hillarykim0626@gmail.com, amille11@swarthmore.edu

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Abstract

A pattern knot in a solid torus defines a self-map of the smooth knot concordance group. We prove that if the winding number of a pattern is even but not divisible by 8, then the corresponding map is not a homomorphism, thus partially establishing a conjecture of Hedden.

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1. *Introduction*

The satellite construction plays an important role in low-dimensional topology in general and knot concordance in particular. A knot in a solid torus, or *pattern*, induces a well-defined function on the set $\mathcal{C}$ of smooth concordance classes of knots, via the satellite operation illustrated in Figure 1. While these functions have been well-studied (see for example [**4–7, 13, 14, 18, 20**]), much remains open. In particular, $\mathcal{C}$ famously has the structure of an abelian group, with operation induced by connected sum [**9**]. While both the satellite operation and connected sum are geometrically defined operations, they do not interact well: $P(K_1\#K_2)$ is isotopic to $P(K_1)\#P(K_2)$ essentially only if P is isotopic to either a core or an unknot in the solid torus.

Our main result is the following, which roughly states that for patterns of certain (algebraic) winding number this behaviour must persist even modulo concordance.

THEOREM 1·1. *Let P be a pattern whose winding number is even but not divisible by* 8*. Then P does not induce a homomorphism of the smooth concordance group.*

This is progress towards establishing the following conjecture of Hedden.

CONJECTURE 1·2 ([**2, 19**]). *Let P be a pattern that induces a homomorphism of the smooth concordance group. Then P induces the zero map, the identity map, or reversal.*

It is straightforward to show that the zero map can only be induced by a winding number 0 pattern, the identity map by a winding number 1 pattern, and reversal by a winding

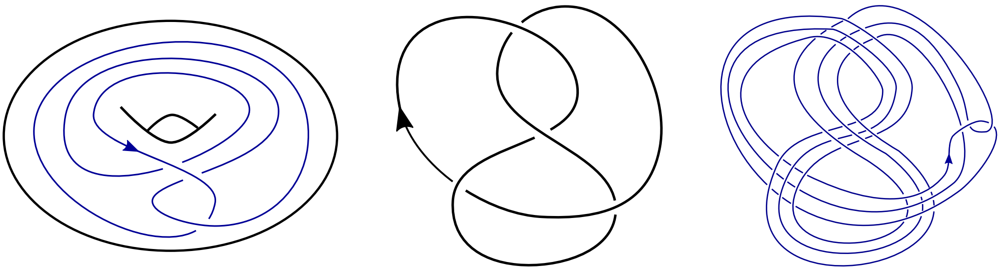

Fig. 1. A pattern P (left), a knot K (center), and the satellite knot $P(K)$ (right).

number -1 pattern. In particular, Hedden's conjecture would imply that patterns whose winding number is greater than one in absolute value cannot induce homomorphisms. Previous work in this area has obstructed specific examples like the Whitehead pattern [**10**] and $(n, 1)$-cables [**12**]. from inducing homomorphisms, as well as establishing new obstructions coming from Casson–Gordon signatures [**17**] and the d-invariants of Heegaard Floer homology [**16**]. However, Theorem 1·1 is the first result to obstruct all patterns of a given winding number from inducing homomorphisms. Our proof relies on the following obstruction, due to recent work of Lidman, Miller and Pinzón-Caicedo.

THEOREM 1·3 ([**16**]). *Let $P \subset S^1 \times D^2$ be a pattern of winding number w, where w is divisible by some prime power m. Let η_m denote a preferred lift of $\eta = \{pt\} \times \partial D^2$ to the m-fold branched cover $\Sigma_m(P(U))$, and let $t\colon \Sigma_m(P(U)) \to \Sigma_m(P(U))$ be a generator for the group of covering transformations.*

Suppose the following two conditions hold:

(i) η_m represents an odd order element of $H_1(\Sigma_m(P(U)))$;

(ii) for $k = 1, \ldots, m-1$, the rational linking numbers $lk_{\Sigma_m(P(U))}(\eta_m, t^k\eta_m)$ are not all equal to 0, while being either all non-positive or all non-negative.

Then there is a knot K such that $P(-K)\#P(K)$ is not smoothly slice, and hence P does not induce a homomorphism of the smooth concordance group.

The proof of Theorem 1·3 uses Heegaard Floer d-invariants to obstruct $\Sigma_m(P(-K)\#P(K))$ from bounding a *smooth* rational homology ball. Since these invariants provide no obstruction in the topoogical category, Theorem 1·3 and hence Theorem 1·1 only hold in the smooth category. Therefore, even though Hedden's conjecture is open in the topological category, our results do not apply there. In fact, it remains open whether $(m, 1)$-cabling induces a homomorphism of the topological concordance group.

To prove Theorem 1·1, we will split into cases according to whether the winding number w of P satisfies $w \equiv 2 \bmod 4$ (Corollary 3·1) or $w \equiv 4 \bmod 8$ (Corollary 3·1). When $w \equiv 2 \bmod 4$, we argue that $\mathrm{lk}_{\Sigma_2(P(U))}(\eta_2, t\eta_2) \neq 0$ in order to apply Theorem 1·3 with $m = 2$. When $w \equiv 4 \bmod 8$, we first observe that if $\mathrm{lk}_{\Sigma_2(P(U))}(\eta_2, t\eta_2) \neq 0$, then we can apply Theorem 1·3 with $m = 2$. We prove that if $\mathrm{lk}_{\Sigma_2(P(U))}(\eta_2, t\eta_2) = 0$, then $\mathrm{lk}_{\Sigma_4(P(U))}(\eta_4, t\eta_4) = 0$ (Proposition 3·5), before showing that $\mathrm{lk}_{\Sigma_4(P(U))}(\eta_4, t^2\eta_4) \neq 0$ in order to apply Theorem 1·3 with $m = 4$. In all of these cases our computation of linking numbers comes from relating an arbitrary winding number w pattern to the $(w, 1)$-cable pattern via crossing changes (Proposition 2·5),

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lifting the surgery curves realizing these crossing changes to the appropriate branched cover, and comparing linking numbers there.

Note that $\mathrm{lk}_{\Sigma_m(P(U))}(\eta_m, t^k\eta_m) = \mathrm{lk}_{\Sigma_m(P(U))}(\eta_m, t^{m-k}\eta_m)$ for all $1 \leq k \leq m-1$, so to verify condition (ii) of Theorem 1·3 we need consider at most $\lfloor m/2 \rfloor$ linking numbers. Also, we will only use Theorem 1·3 when m is a power of 2, in which case $|H_1(\Sigma_m(P(U)))|$ must be odd (see e.g. [**11**]) and so condition (i) of Theorem 1·3 will be automatically satisfied.

In another direction, note that for every odd integer w, there is an example of a pattern P with winding number w such that P is isotopic to $-P$, and hence such that $P(-K)$ is isotopic to $-P(K)$ for all knots K [**17**]. However, there are no known examples of a pattern Q with even winding number such that $Q(-K)$ is always concordant to $-Q(K)$, unless Q induces the 0-map. This leads to the following conjecture.

CONJECTURE 1·4. *Let P be a pattern with nonzero even winding number. Then there exists a knot K such that $P(-K)$ is not concordant to $-P(K)$.*

Since Theorem 1·3 obstructs a pattern P from having the property that $P(-K)$ is always concordant to $-P(K)$, our proof of Theorem 1·1 establishes Conjecture 1·4 for patterns whose winding numbers are not divisible by 8.

2. *Definitions and examples*

All manifolds are assumed to be smooth, compact and oriented.

The rational linking number of a pair of disjoint oriented simple closed curves in a rational homology sphere can be defined as follows, see [**21**, chapter 10, section 77] for further details.

Definition 2·1. Let γ_1 and γ_2 be disjoint oriented simple closed curves in a rational homology 3-sphere Y. Let $\ell \in \mathbb{N}$ be such that $\ell\gamma_2$ represents the trivial element of $H_1(Y;\mathbb{Z})$, and let F be a 2-chain in Y with boundary $\partial F = \ell\gamma_2$. Then the *(rational) linking number of γ_1 and γ_2* is

$$\mathrm{lk}_Y(\gamma_1, \gamma_2) = \frac{1}{\ell}(\gamma_1 \cdot F) \in \mathbb{Q}.$$

Note that here and throughout the paper, for a simple closed curve γ and a transverse 2-chain F, we use $\gamma \cdot F$ to denote the signed count of intersection points between γ and F. Although it is not particularly obvious from this definition, the linking number is symmetric and depends only on the homology class of one curve in the complement of the other.

There is a natural correspondence between patterns in the solid torus and certain ordered 2-component links in S^3. Given a pattern P in $S^1 \times D^2$, we let $\eta = \{\mathrm{pt}\} \times \partial D^2$. By considering the trivial embedding of $S^1 \times D^2$ in S^3, we obtain an ordered 2-component link $P(U) \cup \eta$ in S^3, such that the second component η is unknotted in S^3. In the other direction, given an ordered 2-component link $L_1 \cup L_2 \subseteq S^3$ with L_2 unknotted, we obtain a pattern by considering $L_1 \subseteq (S^3 \smallsetminus \nu(L_2)^\circ) \cong S^1 \times D^2$. We will frequently move between P and $P(U) \cup \eta$ without much discussion.

Example 2·2. The left-hand side of Figure 2 illustrates the 2 component link $P(U) \cup \eta$ describing the (6, 1) cable pattern, $C_{6,1}$. On the right we have performed an isotopy so that $P(U)$ is the standard unknot. Now, observe that $\Sigma_2(P(U)) = \Sigma_2(U) = S^3$, and the pre-image of η in $\Sigma_2(P(U))$ is the (6, 2) torus link, whose components have linking number 3.

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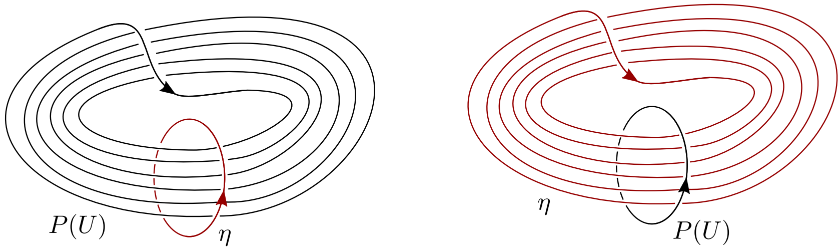

Fig. 2. The link $P(U) \cup \eta$ defining the $C_{6,1}$ cable pattern is symmetric.

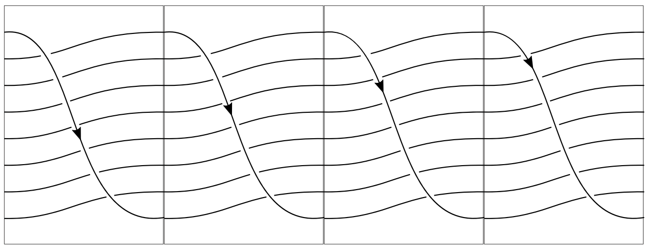

Fig. 3. The preimage of η in $\Sigma_4(C_{8,1}(U)) = S^3$, once the left and right-hand sides of the diagram are identified without twisting.

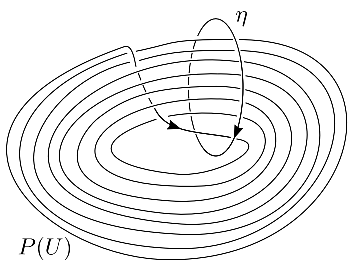

Fig. 4. A winding number 8 pattern that Theorem 1·3 does not obstruct from inducing a homomorphism.

This example generalises to give the following.

LEMMA 2·3. *Let $C_{n,1}(U) \cup \eta$ denote the 2-component link describing the (n, 1)-cable pattern. Then $lk_{\Sigma_m(C_{n,1}(U))}(\eta_m, t^j\eta_m) = n/m$ for all m dividing n and $1 \leq j \leq m-1$.*

Proof. Suppose that m divides n, and so $n = mk$ for some $k \in \mathbb{N}$. The link $C_{n,1}(U) \cup \eta$ is symmetric and the preimage of η in $\Sigma_m(C_{n,1}(U)) = S^3$ is the torus link $T(mk, m)$, as illustrated in Figure 3 for $n = 8$ and $m = 4$. Since $T(mk, m)$ is isotopic to $T(m, mk)$, the m-component link obtained from the m-component unlink by inserting k full twists between all components, we see that the linking number between any two distinct lifts of η is equal to $k = n/m$.

Example 2·4 (A winding number 8 pattern that Theorem 1·3 does not obstruct from inducing a homomorphism.) Consider the pattern depicted in Figure 4. The link $P(U) \cup \eta$ is symmetric, and since $P(U) = U$ and so $\Sigma_m(P(U)) = S^3$ for all m, we can use a diagrammatic

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approach akin to that of Example 2·2 and Lemma 2·3 to compute the linking numbers corresponding to the m-fold cyclic branched covers for all prime powers m dividing the winding number:

(1) $m=2$: $\mathrm{lk}_{S^3}(\eta_2, t\eta_2)=0$.
(2) $m=4$: $\mathrm{lk}_{S^3}(\eta_4, t\eta_4)=\mathrm{lk}_{S^3}(\eta_4, t^2\eta_4)=0$,
(3) $m=8$: $\mathrm{lk}_{S^3}(\eta_8, t\eta_8)=1$, $\mathrm{lk}_{S^3}(\eta_8, t^2\eta_8)=0$, and $\mathrm{lk}_{S^3}(\eta_8, t^3\eta_8)=\mathrm{lk}_{S^3}(\eta_8, t^4\eta_8)=-1$.

Since in each case the linking numbers are either identically zero or of mixed sign, Theorem 1·3 does not provide an obstruction.

Finally, we record the following for later use.

PROPOSITION 2·5. *Let P be a pattern with winding number $n>1$. Then there exists a sequence of crossing changes which transforms P into the cable pattern $C_{n,1}$.*

Proof. Note that a pattern with winding number n represents the same homotopy class as $C_{n,1}$ in the solid torus. As observed for example in [**15**, section 4], a homotopy of curves can be realized by a composition of isotopies and crossing changes, thereby giving the desired result.

In fact, an identical argument establishes Proposition 2·5 for any $n\in\mathbb{Z}$, so long as we interpret $C_{n,1}$ as a positively oriented core when $n=1$, an unknot in the solid torus when $n=0$, a negatively oriented core when $n=-1$, and as the reverse of $C_{|n|,1}$ when $n<-1$.

3. *Proof of main theorem*

The strategy for the proof of Theorem 1·1 is as follows. We will lift the curves realising the crossing changes converting the winding number n pattern P to the $(n, 1)$ cable to the m-fold cyclic branched cover $\Sigma_m(C_{n,1}(U))=S^3$. This will give us a surgery description for $(\Sigma_m(P(U)), \cup_{i=0}^{m-1} t^i\eta_m^P)$ in terms of $(\Sigma_m(C_{n,1}(U)), \cup_{i=0}^{m-1} t^i\eta_m^C)$, where by a mild abuse of notation we use t to refer to either of the covering transformation maps on $\Sigma_m(P(U))$ or on $\Sigma_m(C_{n,1}(U))$. The following theorem of Cha and Ko then allows us to relate the linking numbers of the components of $\bigcup_{i=0}^{m-1} t^i\eta_m^P$ in $\Sigma_m(P(U))$ to the linking numbers of the corresponding components of $\bigcup_{i=0}^{m-1} t^i\eta_m^C$ in $\Sigma_m(C_{n,1}(U))=S^3$, which we computed in Lemma 2·3.

THEOREM 3·1 ([**3**, theorem 3·1]). *Let $L=K_1\cup\ldots\cup K_\ell$ be an integrally framed link in S^3 such that the result of surgery on S^3 along L is a rational homology 3-sphere Y. Let $A=(A_{i,j})$ be the linking-framing matrix of L. Then for any two disjoint 1-cycles a and b in $S^3\smallsetminus L$, we have that*

$$lk_Y(a,b)=lk_{S^3}(a,b)-x^TA^{-1}y,$$

where $x=(x_i)$ and $y=(y_i)$ are column vectors with $x_i=lk_{S^3}(a,K_i)$ and $y_i=lk_{S^3}(b,K_i)$ for all $i=1,\ldots,\ell$.

We will also need the notion of a block circulant matrix, see for example [**1**].

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Definition 3·2. An $nm \times nm$ matrix A is called *$n \times n$ block circulant with $m \times m$ blocks* if there exist $m \times m$ matrices $B_1, \ldots, B_n$ such that

$$A = \begin{bmatrix} B_1 & B_2 & \ldots & B_{n-1} & B_n \\ B_n & B_1 & \ldots & B_{n-2} & B_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ B_3 & \ldots & B_n & B_1 & B_2 \\ B_2 & \ldots & B_{n-1} & B_n & B_1 \end{bmatrix}.$$

We record the following elementary proposition for future reference.

PROPOSITION 3·3. *Let A be an invertible $n \times n$ block circulant matrix with $m \times m$ blocks. Then A^{-1} is also an $n \times n$ block circulant matrix with $m \times m$ blocks.*

Proof. One can directly verify that a matrix C is $n \times n$ block circulant with $m \times m$ blocks if and only if $C = P_{n,m} C P_{n,m}^{-1}$, where $P_{n,m}$ is the $n \times n$ block circulant matrix with $m \times m$ blocks $B_2 = I_m$ and $B_1 = B_3 = \cdots = B_n = 0_m$, i.e.:

$$P_{n,m} := \begin{bmatrix} 0_m & I_m & 0_m & \ldots & 0_m \\ 0_m & 0_m & I_m & \ldots & 0_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & \ldots & \ldots & 0_m & I_m \\ I_m & 0_m & \ldots & 0_m & 0_m \end{bmatrix}.$$

Now observe that since A is invertible and $A = P_{n,m} A P_{n,m}^{-1}$, we have $A^{-1} = P_{n,m} A^{-1} P_{n,m}^{-1}$, i.e. A^{-1} is also $n \times n$ block circulant with $m \times m$ blocks.

3·1. *The case of $n \equiv 2$ mod 4*

The following result relates the 2-fold cover linking numbers of a winding number $2k$ pattern to those of $C_{2k,1}$.

LEMMA 3·4. *Let $P = P(U) \cup \eta$ be a pattern with even winding number $n = 2k$. Let η^1 and η^2 denote the lifts of η to $\Sigma_2(P(U))$. There exists $a \in \mathbb{Z}$ such that*

$$\mathrm{lk}_{\Sigma_2(P(U))}(\eta_1, \eta_2) = k + \frac{2a}{|H_1(\Sigma_2(P(U)))|}.$$

Proof. By Proposition 2·5, P can be transformed into the $(n, 1)$ cable $C_{n,1}$ by a sequence of m crossing changes for some $m \in \mathbb{N}$. We can realize the ith crossing change by doing ϵ_i-framed surgery (for $\epsilon_i = \pm 1$) along a small curve L_i linking $P(U)$ geometrically twice and algebraically zero times, while linking η zero times. We refer to the resulting link as $P_C \cup \eta_C \cup \bigcup_{i=1}^m L_i$. Note that blowing down the L_i curves transforms $P_C \cup \eta_C$ to $P \cup \eta$, while ignoring the L_i curves we see $P_C \cup \eta_C = C_{n,1}$.

Each framed curve L_i lifts to a 2-component framed link $L_i^1 \cup L_i^2$ in $\Sigma_2(C_{n,1}(U)) = S^3$, and the curve η_C lifts to the 2-component link $\eta_C^1 \cup \eta_C^2$. Doing surgery along $L = \bigcup_{i=1}^m (L_i^1 \cup L_i^2)$ according to the induced framing converts $(\Sigma_2(C_{n,1}(U)), \eta_C^1 \cup \eta_C^2)$ to $(\Sigma_2(P(U)), \eta^1 \cup \eta^2)$.

Let A be the linking-framing matrix for L with respect to the ordering $L_1^1, \ldots, L_m^1, L_1^2, \ldots L_m^2$. Observe that since $\mathrm{lk}(t\gamma_1, t\gamma_2) = \mathrm{lk}(\gamma_1, \gamma_2)$ for any two curves γ_1, γ_2,

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for any $1 \leq i, j \leq m$ we have

$$A_{i,j+m} = \mathrm{lk}(L_i^1, L_j^2) = \mathrm{lk}(L_i^2, L_j^1) = A_{i+m,j}$$
$$A_{i+m,j+m} = \mathrm{lk}(L_i^2, L_j^2) = \mathrm{lk}(L_i^1, L_j^1) = A_{i,j}.$$

That is, A is a 2×2 block circulant matrix with size $m \times m$ blocks.

Now define

$$x = (\mathrm{lk}(\eta_C^1, L_1^1), \ldots, \mathrm{lk}(\eta_C^1, L_m^1), \mathrm{lk}(\eta_C^1, L_1^2), \ldots, \mathrm{lk}(\eta_C^1, L_m^2))$$
$$y = (\mathrm{lk}(\eta_C^2, L_1^1), \ldots, \mathrm{lk}(\eta_C^2, L_m^1), \mathrm{lk}(\eta_C^2, L_1^2), \ldots, \mathrm{lk}(\eta_C^2, L_m^2)).$$

We have that

$$\mathrm{lk}(\eta_C^2, L_i^2) = \mathrm{lk}(t\eta_C^1, tL_i^1) = \mathrm{lk}(\eta_C^1, L_i^1)$$

and

$$\mathrm{lk}(\eta_C^1, L_i^2) = \mathrm{lk}(t\eta_C^2, tL_i^1) = \mathrm{lk}(\eta_C^2, L_i^1)$$

for all $i = 1, \ldots, m$.

Claim. $\mathrm{lk}(\eta_C^1, L_i^2) = -\mathrm{lk}(\eta_C^1, L_i^1)$.

Proof of claim. Let F_i be a surface in S^3 with boundary $\partial F_i = L_i$. Observe that $0 = \mathrm{lk}(\eta_C, L_i) = F_i \cdot \eta_C$. Let $\widetilde{F}_i$ denote the pre-image of F_i in $\Sigma_2(C_{n,1}(U)) = S^3$. Then $\widetilde{F}_i$ is a surface with boundary $\partial \widetilde{F}_i = L_i^1 \cup L_i^2$. Moreover,

$$\widetilde{F}_i \cdot (\eta_C^1 \cup \eta_C^2) = 2(F_i \cdot \eta_C) = 0.$$

Let G_i be a surface in $\Sigma_2(C_{n,1}(U)) = S^3$ with $\partial G_i = -L_i^1$. Then we can compute

$$\begin{aligned} \mathrm{lk}(\eta_C^1, L_i^2) + \mathrm{lk}(\eta_C^2, L_i^2) &= (\widetilde{F}_i \cup G_i) \cdot (\eta_C^1 \cup \eta_C^2) \\ &= G_i \cdot (\eta_C^1 \cup \eta_C^2) \\ &= (G_i \cdot \eta_C^1) + (G_i \cdot \eta_C^2) = -\mathrm{lk}(\eta_C^1, L_i^1) - \mathrm{lk}(\eta_C^2, L_i^1). \end{aligned}$$

We can now apply the previous observations that $\mathrm{lk}(\eta_C^2, L_i^1) = \mathrm{lk}(\eta_C^1, L_i^2)$ and $\mathrm{lk}(\eta_C^2, L_i^2) = \mathrm{lk}(\eta_C^1, L_i^1)$ to finish the proof of our claim.

We therefore have that $x = (v, -v)$ and $y = (-v, v)$ for some $v \in \mathbb{Z}^m$. By Theorem 3·1, we know that

$$\mathrm{lk}_{\Sigma_2(P(U))}(\eta_1, \eta_2) = \mathrm{lk}_{S^3}(\eta_C^1, \eta_C^2) - x^T A^{-1} y = k - x^T A^{-1} y,$$

where we use Lemma 2·3 to compute $\mathrm{lk}(\eta_C^1, \eta_C^2) = k$.

It now only remains to show that $x^T A^{-1} y = 2a/|H_1(\Sigma_2(P(U)))|$ for some $a \in \mathbb{Z}$. Since A is a block circulant matrix, Proposition 3·3 implies that A^{-1} is also block circulant, and hence can be written $A^{-1} = \begin{bmatrix} F & G \\ G & F \end{bmatrix}$ for some $F, G \in M_{m \times m}(\mathbb{Q})$. We obtain

$$x^T A^{-1} y = \begin{bmatrix} v^T & -v^T \end{bmatrix} \begin{bmatrix} F & G \\ G & F \end{bmatrix} \begin{bmatrix} -v \\ v \end{bmatrix} = 2v^T(G - F)v.$$

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The adjoint formula for a matrix inverse, applied to A, implies that the entries of F and G are all of the form $a/\det(A)$ for some $a \in \mathbb{Z}$, and so we obtain our desired result, since $|\det(A)| = |H_1(\Sigma_2(P(U)))|$.

The following proof now follows quickly.

Proof of Theorem 1·1. *for* $n \equiv 2 \bmod 4$. Let $P = P(U) \cup \eta$ be a pattern whose winding number is even but not divisible by 4. Then P does not induce a homomorphism of the concordance group.

Proof. Write $n = 2k$ for an odd integer k. By Lemma 3·4, we know that there is $a \in \mathbb{Z}$ such that

$$\mathrm{lk}_{\Sigma_2(P(U))}(\eta_1, \eta_2) = k + \frac{2a}{|H_1(\Sigma_2(P(U)))|} = \frac{k|H_1(\Sigma_2(P(U)))| + 2a}{|H_1(\Sigma_2(P(U)))|} \neq 0,$$

since k and $|H_1(\Sigma_2(P(U)))|$ are both odd. Theorem 1·3 with $m = 2$ then gives the desired result.

3·2. *The case of* $n \equiv 4$ *mod* 8

Our proof strategy for $n \equiv 4 \bmod 8$ will involve both the 2-fold and 4-fold branched cover linking numbers. The following result relates the two.

PROPOSITION 3·5. *Let* $P = P(U) \cup \eta$ *be a pattern with winding number that is divisible by* 4. *Let* η_4^1 *denote a preferred lift of* η *to* $\Sigma_4(P(U))$, *and let* $\eta_4^{j+1} = t^j \eta_4^1$ *for* $j = 1, 2, 3$. *Define* $\eta_2^1 = \pi(\eta_4^1)$ *and* $\eta_2^2 = \pi(\eta_4^2)$, *where* $\pi : \Sigma_4(P(U)) \to \Sigma_2(P(U))$ *is the branched covering projection map. Then*

$$\mathrm{lk}_{\Sigma_2(P(U))}(\eta_2^1, \eta_2^2) = 2\,\mathrm{lk}_{\Sigma_4(P(U))}(\eta_4^1, \eta_4^2).$$

For convenience, in this proof we will abbreviate $\mathrm{lk}_{\Sigma_2(P(U))}$ to lk_2 and $\mathrm{lk}_{\Sigma_4(P(U))}$ to lk_4.

Proof. Let $\ell = |H_1(\Sigma_4(P(U)))|$. Note that as observed by Fox [8] we have that

$$\begin{aligned} |H_1(\Sigma_4(P(U)))| &= |\Delta_{P(U)}(-1)\Delta_{P(U)}(i)\Delta_{P(U)}(-i)| \\ &= |H_1(\Sigma_2(P(U)))| \cdot |\Delta_{P(U)}(i)\Delta_{P(U)}(-i)|, \end{aligned}$$

where as usual $\Delta_{P(U)}(t)$ denotes the Alexander polynomial of $P(U)$. Therefore, every 1-cycle a in $\Sigma_2(P(U))$ has the property that ℓa bounds a 2-cycle.

Let F be a 2-cycle in $\Sigma_2(P(U))$ such that $\partial F = \ell \eta_2^2$, so $\mathrm{lk}_2(\eta_2^1, \eta_2^2) = (1/\ell)(\eta_2^1 \cdot F)$. Define $\widetilde{F} = \pi^{-1}(F) \subset \Sigma_4(P(U))$, and observe that $\partial \widetilde{F} = \ell(\eta_4^2 \cup \eta_4^4)$. Let G be a 2-cycle in $\Sigma_4(P(U))$ such that $\partial G = -\ell \eta_4^4$. Observe that for $j \in \{1, 3\}$ we have that

$$\mathrm{lk}_4(\eta_4^j, \eta_4^2) = \frac{1}{\ell}(\eta_4^j \cdot (\widetilde{F} \cup G)) = \frac{1}{\ell}(\eta_4^j \cdot \widetilde{F}) + \frac{1}{\ell}(\eta_4^j \cdot G) = \frac{1}{\ell}(\eta_4^j \cdot \widetilde{F}) - \mathrm{lk}_4(\eta_4^j, \eta_4^4).$$

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It follows that

$$\begin{aligned}4\,\mathrm{lk}_4(\eta_4^1,\eta_4^2) &= \mathrm{lk}_4(\eta_4^1,\eta_4^2)+\mathrm{lk}_4(\eta_4^1,\eta_4^4)+\mathrm{lk}_4(\eta_4^3,\eta_4^2)+\mathrm{lk}_4(\eta_4^3,\eta_4^4)\\ &= \frac{1}{\ell}((\eta_4^1\cup\eta_4^3)\cdot\widetilde{F})\\ &= \frac{1}{\ell}(\pi^{-1}(\eta_2^1)\cdot\pi^{-1}(F))\\ &= \frac{2}{\ell}(\eta_2^1\cdot F)\\ &= 2\,\mathrm{lk}_2(\eta_2^1,\eta_2^2),\end{aligned}$$

where the second-to-last equality follows from the fact that $\pi:\Sigma_4(P(U))\to\Sigma_2(P(U))$ is 2-to-1 everywhere besides the branch set $P(U)$.

We also need the analogue of Lemma 3·4 for patterns with winding number divisible by 4.

LEMMA 3·6. *Let $P=P(U)\cup\eta$ be a pattern of winding number $n=4k$. Let $\eta^1,\eta^2,\eta^3,\eta^4$ denote the lifts of η to $\Sigma_4(P(U))$, where for $i=1,2,3$, we obtain η^{i+1} from η^i by the action of the covering transformation. Then there exists $a\in\mathbb{Z}$ such that*

$$lk_{\Sigma_4(P(U))}(\eta^1,\eta^3)=k+\frac{2a}{|H_1(\Sigma_4(P(U)))|}.$$

Proof of Lemma 3·6. Our strategy imitates that of the proof of Lemma 3·4. Let $L_1\cup\ldots\cup L_m$ be a framed link of unknots, surgery along which realises the crossing changes of $P(U)$ that transform $P=P(U)\cup\eta$ to the $(n,1)$-cable pattern $C_{n,1}=P_C\cup\eta_C$. Note that

$$\mathrm{lk}(L_i,P(U))=\mathrm{lk}(L_i,\eta)=\mathrm{lk}(L_i,L_j)=0$$

for all $1\leq i\neq j\leq m$.

For each $i=1,\ldots,m$, pick a preferred lift L_i^1 of L_i to $\Sigma_4(C_{n,1}(U))=S^3$, and let $L_i^2=tL_i^1$, $L_i^3=t^2L_i^1$, and $L_i^4=t^3L_i^1$. Similarly, let η_C^1 be a preferred lift of η_C to $\Sigma_4(C_{n,1}(U))=S^3$, and let $\eta_C^2=t\eta_C^1$, $\eta_C^3=t^2\eta_C^1$, and $\eta_C^4=t^3\eta_C^1$. We have that $(\Sigma_4(P(U)),\bigcup_{i=1}^4\eta^i)$ is obtained from $(\Sigma_4(C_{n,1}(U)),\bigcup_{i=1}^4\eta_C^i)$ by performing appropriately framed surgery along

$$L=L_1^1\cup\cdots L_m^1\cup L_1^2\cup\cdots L_m^2\cup L_1^3\cup\ldots\cup L_m^3\cup L_1^4\cup\ldots\cup L_m^4.$$

Now let A be the linking-framing matrix of L and let x (respectively y) be the $4m$-component vector whose ith entry is the linking of η_C^1 (respectively η_C^3) with the ith component of L. Theorem 3·1 tells us that

$$\mathrm{lk}_{\Sigma_4(P(U))}(\eta^1,\eta^3)=\mathrm{lk}(\eta_C^1,\eta_C^3)-x^TA^{-1}y=k-x^TA^{-1}y,$$

where for the last equality we use Lemma 2·3.

Observe that for any $1\leq i,j\leq m$ and $1\leq a,b\leq 4$ we have

$$\mathrm{lk}(L_i^a,L_j^b)=\mathrm{lk}(t^{a-1}L_i^1,t^{b-1}L_j^1)=\mathrm{lk}(L_i^1,t^{b-a}L_j^1)=\mathrm{lk}(L_i^1,L_j^{b-a+1}),$$

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where all exponents are taken modulo 4. It follows that A is a 4×4 block circulant matrix with blocks of size $m\times m$, and hence by Proposition 3·3 has a block circulant inverse

$$A^{-1}=\begin{bmatrix} Q & R & S & R^T\\ R^T & Q & R & S\\ S & R^T & Q & R\\ R & S & R^T & Q\end{bmatrix},$$

for some $Q,R,S\in M_{m\times m}(\mathbb{Q})$ with $Q^T=Q$ and $S^T=S$. (We use here that A is a linking-framing matrix, hence symmetric, so A^{-1} is also symmetric.)

Claim. $\mathrm{lk}(\eta_C^1,L_i^4)=-\mathrm{lk}(\eta_C^1,L_i^1)-\mathrm{lk}(\eta_C^1,L_i^2)-\mathrm{lk}(\eta_C^1,L_i^3)$.

Proof of claim. Let F_i be a surface in S^3 with boundary $\partial F=L_i$, and observe that $0=\mathrm{lk}(\eta,L_i)=F\cdot\eta_C$. Taking the pre-image of F in $\Sigma_4(C_{n,1}(U))=S^3$, we obtain $\widetilde{F}$, a surface with boundary $L_i^1\cup L_i^2\cup L_i^3\cup L_i^4$ and with the property that

$$\widetilde{F}\cdot(\eta_C^1\cup\eta_C^2\cup\eta_C^3\cup\eta_C^4)=4(F\cdot\eta_C)=0.$$

Let G_1, G_2, and G_3 be surfaces in $\Sigma_4(C_{n,1}(U))=S^3$ with $\partial G_j=-L_i^j$, and observe that

$$\begin{aligned}\sum_{k=1}^{4}\mathrm{lk}(\eta_C^k,L_i^4)&=(\widetilde{F}\cup G_1\cup G_2\cup G_3)\cdot(\eta_C^1\cup\eta_C^2\cup\eta_C^3\cup\eta_C^4)\\ &=(G_1\cup G_2\cup G_3)\cdot(\eta_C^1\cup\eta_C^2\cup\eta_C^3\cup\eta_C^4)=\sum_{j=1}^{3}\sum_{k=1}^{4}-\mathrm{lk}(\eta_C^k,L_i^j).\end{aligned}$$

Rewriting, we obtain that

$$\begin{aligned}0=&(\mathrm{lk}(\eta_C^1,L_i^1)+\mathrm{lk}(\eta_C^1,L_i^2)+\mathrm{lk}(\eta_C^1,L_i^3)+\mathrm{lk}(\eta_C^1,L_i^4))\\ &+(\mathrm{lk}(\eta_C^2,L_i^2)+\mathrm{lk}(\eta_C^2,L_i^3)+\mathrm{lk}(\eta_C^2,L_i^4)+\mathrm{lk}(\eta_C^2,L_i^1))\\ &+(\mathrm{lk}(\eta_C^3,L_i^3)+\mathrm{lk}(\eta_C^3,L_i^4)+\mathrm{lk}(\eta_C^3,L_i^1)+\mathrm{lk}(\eta_C^3,L_i^2))\\ &+(\mathrm{lk}(\eta_C^4,L_i^4)+\mathrm{lk}(\eta_C^4,L_i^1)+\mathrm{lk}(\eta_C^4,L_i^2)+\mathrm{lk}(\eta_C^4,L_i^3)).\end{aligned}$$

Now observe that each 4-term parenthetical sum is equal, since $\mathrm{lk}(\eta_C^j,L_i^k)$ depends only on $1\le i\le m$ and the value of $k-j$ mod 4. So we obtain our desired claim.

We therefore have that $x=(u,v,w,-u-v-w)$, where $u_i=\mathrm{lk}(\eta_C^1,L_i^1)$, $v_i=\mathrm{lk}(\eta_C^1,L_i^2)$, and $w_i=\mathrm{lk}(\eta_C^1,L_i^3)$ for $i=1,\ldots,m$. Furthermore, since x records the linking of η_C^1 with the components of L and y records the linking of $\eta_C^3=t^2\eta_C^1$ with the components of L, we have that $y=(w,-u-v-w,u,v)$.

We can now compute

$$\begin{aligned}(*)&=x^TA^{-1}y\\ &=\begin{bmatrix}u^T & v^T & w^T & -u^T-v^T-w^T\end{bmatrix}\begin{bmatrix} Q & R & S & R^T\\ R^T & Q & R & S\\ S & R^T & Q & R\\ R & S & R^T & Q\end{bmatrix}\begin{bmatrix} w\\ -u-v-w\\ u\\ v\end{bmatrix}\\ &=u^TQw+v^TQ(-u-v-w)+w^TQu+(-u^T-v^T-w^T)Qv\\ &\quad+u^TR(-u-v-w)+v^TRu+w^TRv+(-u^T-v^T-w^T)Rw\end{aligned}$$

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$$
\begin{aligned}
&+u^T Su + v^T Sv + w^T Sw + (-u^T - v^T - w^T)S(-u-v-w)\\
&\quad + u^T R^T v + v^T R^T w + w^T R^T(-u-v-w) + (-u^T - v^T - w^T)R^T u\\
&= 2(u^T Qw - u^T Qv - v^T Qv - v^T Qw)\\
&\quad + 2(-u^T Ru - u^T Rv - 2u^T Rw - v^T Rw - w^T Rw + v^T Ru + w^T Rv)\\
&\quad + 2(u^T Su + u^T Sv + u^T Sw + v^T Sv + v^T Sw + w^T Sw),
\end{aligned}
$$

where to obtain the final equality we repeatedly use that $Q^T = Q$ and $S^T = S$. The entries of u, v, and w are integers and the entries of Q, R, and S are all of the form $a/\det(A)$ for $a \in \mathbb{Z}$, by the adjoint formula for a matrix inverse. Therefore, since $|\det(A)| = |H_1(\Sigma_4(P(U)))|$, we have our desired result.

Theorem 1·1 for $n \equiv 4 \mod 8$ follows quickly from this result together with Proposition 3·5.

Proof of Theorem 1·1 for $n \equiv 4 \mod 8$. Let P be a pattern with winding number n equivalent to 4 mod 8. Then P does not induce a homomorphism of the concordance group.

Proof. Write $n = 4k$ for some odd k. If $\mathrm{lk}_{\Sigma_2(P(U))}(\eta_2, t\eta_2) \neq 0$, then by applying Theorem 1·3 with $m = 2$ we are done. So assume that $\mathrm{lk}_{\Sigma_2(P(U))}(\eta_2, t\eta_2) = 0$. By Proposition 3·5, this implies that $\mathrm{lk}_{\Sigma_4(P(U))}(\eta_4, t\eta_4) = 0$ as well. However, by Lemma 3·6, we know that there is $a \in \mathbb{Z}$ such that

$$\mathrm{lk}_{\Sigma_4(P(U))}(\eta_4, t^2\eta_4) = k + \frac{2a}{|H_1(\Sigma_4(P(U)))|} \neq 0,$$

since $|H_1(\Sigma_4(P(U)))|$ is odd. Theorem 1·3 with $m = 4$ then gives the desired result.

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REFERENCES

[1] D. S. BERNSTEIN. *Matrix Mathematics* (Princeton University Press, Princeton, NJ, second edition, 2009). *Theory, facts and formulas*.

[2] Workshop Organisers BIRS. Final report. In *Synchronizing Smooth and Topological* 4-*Manifolds*, BIRS (Feb. 21-26 2016). https://www.birs.ca/workshops/2016/16w5145/report16w5145.pdf.

[3] J. CHOON CHA and K. HYOUNG KO. Signatures of links in rational homology spheres. *Topology* **41**(6) (2002), 1161–1182.

[4] T. COCHRAN and S. HARVEY. The geometry of the knot concordance space. *Algebr. Geom. Topol.* **18**(5) (2018), 2509–2540.

[5] T. D. COCHRAN, C. W. DAVIS and A. RAY. Injectivity of satellite operators in knot concordance. *J. Topol.* **7**(4) (2014), 948–964.

[6] T. D. COCHRAN, S. HARVEY and C. LEIDY. Primary decomposition and the fractal nature of knot concordance. *Math. Ann.* **351**(2) (2011), 443–508.

[7] C. W. DAVIS and A. RAY. Satellite operators as group actions on knot concordance. *Algebr. Geom. Topol.* **16**(2) (2016), 945–969.

https://doi.org/10.1017/S0305004125101485 Published online by Cambridge University Press

[8] R. H. FOX. A quick trip through knot theory. In *Topology of* 3*-Manifolds and Related Topics* (Proc. The Univ. of Georgia Institute, 1961). (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961), pp. 120–167.
[9] R. H. FOX and J. W. MILNOR. Singularities of 2-spheres in 4-space and cobordism of knots. *Osaka Math. J.* **3** (1966). 257–267.
[10] R. E. GOMPF. Smooth concordance of topologically slice knots. *Topology* **25**(3) (1986), 353–373.
[11] C. MCA. GORDON. Some aspects of classical knot theory. In *Knot Theory* (Proc. Sem., Plans-sur-Bex, 1977). Lecture Notes in Math. vol. 685 (Springer, Berlin, 1978), pp. 1–60.
[12] M. HEDDEN. On knot Floer homology and cabling. II. *Int. Math. Res. Not. IMRN* (12) (2009), 2248–2274.
[13] M. HEDDEN and J. PINZÓN-CAICEDO. Satellites of infinite rank in the smooth concordance group. *Invent. Math.* **225**(1) (2021), 131–157.
[14] A. SIMON LEVINE. Nonsurjective satellite operators and piecewise-linear concordance. *Forum Math. Sigma* 4:Paper No. e34, 47, 2016.
[15] J. P. LEVINE. An approach to homotopy classification of links. *Trans. Amer. Math. Soc.* **306**(1) (1988), 361–387.
[16] T. LIDMAN, A. N. MILLER and J. PINZÓN-CAICEDO. Linking number obstructions to satellite homomorphisms. *Quantum Topology* (2024), available at: https://ems.press/journals/qt/articles/14297997
[17] A. N. MILLER. Homomorphism obstructions for satellite maps. *Trans. Amer. Math. Soc., Series B* **10** (2023), 220–240.
[18] A. N. MILLER and L. PICCIRILLO. Knot traces and concordance. *J. Topol.* **11**(1) (2018), 201–220.
[19] Conference organisers MPIM. Problem list. In *Conference on* 4*-Manifolds and Knot Concordance*, MPIM (Oct. 17–21 2016). http://www.people.brandeis.edu/aruray/4manifoldsconference/problem_session.pdf.
[20] A. RAY. Satellite operators with distinct iterates in smooth concordance. *Proc. Amer. Math. Soc.* **143**(11) (2015), 5005–5020.
[21] H. SEIFERT and W. THRELFALL. *Seifert and Threlfall: a Textbook of Topology*, Pure Appl. Math. vol. 89 (Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980). Translated from the German edition of 1934 by Michael A. Goldman, with a preface by Joan S. Birman, With "Topology of 3-dimensional fibered spaces" by Seifert, Translated from the German by Wolfgang Heil.