A note on sequences determined by a recurrence relation

By C. E. Walsh, Dublin.

(*Received* 12*th May*, 1932. *Read* 4*th June*, 1932.)

§ 1. Copson and Ferrar have proved the following theorem[1]:

If $q_n = \rho_n + i\sigma_n$, $\underline{\lim}\ \rho_n = h > 0$, $\Sigma(1/p_n)$ *is a divergent series of positive terms, and* $(p_n + q_n)u_n - p_n u_{n-1} = y_n$ *always, where* $y_n \to 0$, *then* $u_n \to 0$.

Let $p_n/(p_n + q_n)$ and $1/(p_n + q_n)$ be denoted respectively by a_n and c_n; then

$$u_n = a_n u_{n-1} + c_n y_n.$$

Here we have

$$|a_n| \leqq \frac{p_n}{p_n + h} < 1; \qquad |c_n| \leqq \frac{1}{p_n + h} \leqq \frac{1}{h}(1 - |a_n|).$$

Further, since $\Sigma(1/p_n)$ is a divergent series of positive terms,

$$\prod_{r=1}^{n} a_r \to 0, \qquad \text{as } n \to \infty.$$

Thus the theorem of Copson and Ferrar is included in the following:

If $u_n = a_n u_{n-1} + c_n y_n$, *where* $|a_n| < 1$ *always,* $\prod_{r=1}^{n} a_r$ *tends to zero,* $|c_n| \leqq K(1 - |a_n|)$, *K being fixed, and* $y_n \to 0$, *then* $u_n \to 0$.

A proof of this can be obtained by putting $b_n = 0$ in Theorem I which occurs later in this paper. The latter concerns sequences (u_n) obtained from a recurrence relation of the form

$$u_n = a_n u_{n-1} + b_n u_{n-2} + c_n \theta_n, \tag{1}$$

where $\theta_n \to 0$.

§ 2. Supposing equations (1) to have been solved for u_n, we derive an equation of the form

$$u_n = A_n^n \theta_n + A_n^{n-1}\theta_{n-1} + \ldots + A_n^r \theta_r + \ldots + A_n^1 \theta_1 + B_n. \tag{2}$$

Clearly $A_n^n = c_n$. The following lemma enables us to set bounds to the other coefficients, A_n^{n-1}, etc., and to B_n.

[1] Copson and Ferrar, *Journal London Math. Soc.*, 4 (1929), 258-264. See also Izumi, *Tôhoku Math. J.*, 33 (1931), 181-186. Further references will be found in these papers.

Lemma. If β_n *be determined by the equations* $\beta_1 = l_1 \beta_0$, $\beta_r = l_r \beta_{r-1} + m_r \beta_{r-2}$, *then*

$$|\beta_r| \leqq |\beta_0 l_r| \prod_{n=1}^{r-1} \left\{ |l_n| + \left| \frac{m_{n+1}}{l_{n+1}} \right| \right\}.$$

The lemma is obvious when $r = 2$. Suppose that it is true for $2, 3, \ldots, r-1$. Then

$$|\beta_{r-2}| \leqq |\beta_0 l_{r-2}| \prod_{n=1}^{r-3} \left\{ |l_n| + \left| \frac{m_{n+1}}{l_{n+1}} \right| \right\}$$

$$\leqq |\beta_0| \prod_{n=1}^{r-2} \left\{ |l_n| + \left| \frac{m_{n+1}}{l_{n+1}} \right| \right\},$$

so that
$$|\beta_r| \leqq |l_r \beta_{r-1}| + |m_r . \beta_{r-2}|$$

$$\leqq |\beta_0 l_r| \prod_{n=1}^{r-1} \left\{ |l_n| + \left| \frac{m_{n+1}}{l_{n+1}} \right| \right\}.$$

Thus the result holds good for all values of r.

The coefficients A_n^r in (2) are determined, in virtue of (1), by the equations $A_r^r = c_r$, $A_{r+1}^r = a_{r+1} c_r$, $A_n^r = a_n A_{n-1}^r + b_n A_{n-2}^r$.
Writing $\beta_0 = A_r^r = c_r$, $\beta_s = A_{r+s}^r$, $l_s = a_{r+s}$, $m_s = b_{r+s}$ in the above lemma, it follows that

$$|A_n^r| \leqq |a_n c_r| \prod_{s=r+1}^{n-1} \left\{ |a_s| + \left| \frac{b_{s+1}}{a_{s+1}} \right| \right\},$$

that is

$$|A_n^r| \leqq |c_r| \prod_{s=r+1}^{n} |a_s| \prod_{s=r+1}^{n-1} \left\{ 1 + \frac{|b_{s+1}|}{|a_s a_{s+1}|} \right\}, \tag{3}$$

and similarly

$$|B_n| \leqq B \prod_{s=1}^{n} |a_s| \prod_{s=1}^{n-1} \left\{ 1 + \frac{|b_{s+1}|}{|a_s a_{s+1}|} \right\},$$

B being a constant.

§ 3. Now let u_n, given by equations (1), be expressed as in (2); then certain results connected with its convergence can be obtained. Our mode of proof uses a well known theorem, due to Toeplitz,[1] namely, that *if*

$$u_n = A_n^n \theta_n + A_n^{n-1} \theta_{n-1} + \ldots + A_n^1 \theta_1,$$

where $\theta_n \to 0$, *and* $A_n^r \to 0$ *as* $n \to \infty$, *for a fixed* r, *while* $\sum_{r=1}^{n} |A_n^r| < K$ (*a constant*) *then* $u_n \to 0$.

[1] K. Knopp, "Theory and Application of Infinite Series" (1928), 72.

Theorem I. *Let* (i) $\sum_{n=2}^{\infty} \frac{|b_n|}{|a_n a_{n-1}|}$ *converge;*

(ii) $|a_n| < 1$ *for all* n;

(iii) $\prod_{r=1}^{n} a_r \to 0$ *as* $n \to \infty$;

(iv) $|c_n| \leqq K(1 - |a_n|)$, *where* K *is fixed.*

Then if

$$u_n = a_n u_{n-1} + b_n u_{n-2} + c_n \theta_n$$

where

$$\lim_{n \to \infty} \theta_n = 0,$$

then

$$\lim_{n \to \infty} u_n = 0.$$

For, by (3), (i) and (iv), we have

$$|A_n^r| < L(1 - |a_r|) \prod_{s=r+1}^{n} |a_s|,$$

where L is a constant; for fixed r, $A_n^r \to 0$ as $n \to \infty$, in virtue of (iii). Also

$$\sum_{r=1}^{n} |A_n^r| < L.$$

Again, in virtue of (i) and (iii), B_n tends to zero.

Thus we may apply Toeplitz's Theorem to (2), and conclude that u_n tends to zero.

§ 4. Further similar results may be obtained immediately by the use of the lemma. In particular, the following are of some interest.

Theorem II. Let conditions (ii) *and* (iii) *of the Theorem still hold. Replace* (i) *and* (iv) *by*

$$|b_n| < k |a_n a_{n-1}|,$$

$$|c_n| < K(1 - |a_n|)(1 + k)^n,$$

where k, K *are constants. Then it follows that*

$$(1 + k)^{-n} u_n \to 0.$$

For the conditions of Toeplitz's theorem are satisfied, since

$$(1 + k)^{-n} |A_n^r| < K(1 - |a_r|) \prod_{s=r+1}^{n} |a_s|$$

and $$(1 + k)^{-n} B_n \to 0.$$

https://doi.org/10.1017/S0013091500013924 Published online by Cambridge University Press

Theorem III. If the conditions (i)-(iv) *of the Theorem be replaced by*

$$0 < k \leqq \prod_{r=1}^{n} |a_r| \leqq l,$$

$$|c_n| < K,$$

$$\sum_{n=1}^{\infty} \frac{|b_n|}{|a_n|} \textit{ convergent,}$$

k, l, K being constants, then

$$u_n/n \to 0.$$

In this case $|a_n| \geqq k/l$, so that the series

$$\Sigma \frac{|b_n|}{|a_n a_{n-1}|}$$

is convergent. It can easily be shewn that $|A_n^r|$, $|B_n|$ are bounded, so that the result follows at once from Toeplitz's Theorem. It is evident that this is still true if we only assume

$$|a_n| \leqq 1, \qquad |c_n| < K$$

and

$$\sum_{n=1}^{\infty} \frac{|b_n|}{|a_n|} \text{ convergent.}$$

Theorem IV. If the conditions (i)-(iv) *of the Theorem be replaced by*

$$|c_n| \leqq K l^n, \qquad |a_{n-1}| + \frac{|b_n|}{|a_n|} \leqq k < l,$$

where k, l, K are constants, then

$$u_n l^{-n} \to 0.$$

For here we have

$$|A_n^r l^{-n}| \leqq K (k/l)^{n-r}, \qquad |B_n| \leqq Bk^n.$$

§ 5. We may rewrite the inequality (3) in the form

$$|A_n^r| \leqq |a_n c_r| \prod_{s=r+1}^{n-1} \left[\frac{|b_{s+1}|}{|a_{s+1}|}\left(1 + \frac{|a_s a_{s+1}|}{|b_{s+1}|}\right)\right]$$

and deal similarly with the inequality for $|B_n|$. If we now impose suitable conditions on the infinite product

$$\prod_{n=1}^{\infty} \frac{b_n}{a_n},$$

theorems analogous to I, II, III above may be obtained. A formal statement of these is not, however, necessary here.

https://doi.org/10.1017/S0013091500013924 Published online by Cambridge University Press