The application of γ-matrices to Taylor series

By P. VERMES

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1. *Introduction.* In a recent paper[1] some general properties of γ-matrices were proved and Dienes' theorem on regular γ-matrices[2] extended to semiregular γ-matrices and the binomial series.[3] In section 2 of this paper the previous results will be extended to certain classes of Taylor series. Section 3 gives some new results on Borel's exponential summation, and section 4 introduces matrices efficient for Taylor series on the circle of convergence and others efficient for Dirichlet series on the line of convergence. A knowledge of the definitions and results of the paper mentioned above is assumed.

2. *On the γ-sum of the Taylor series.*

[2.I] *If the semiregular γ-matrix G sums the Taylor series $\Sigma a_k z^k$ of the function $f(z)$ at $z = z_0$ to the value S, then it also sums the Taylor series of the function $F(z) \equiv z^p f(z)$ $(p = 1, 2, \ldots)$ at $z = z_0$ to the sum $z_0{}^p S$.*

Proof: By hypothesis $\lim_{n \to \infty} \sum_{k=0}^{\infty} g_{n,k} a_k z_0^k = S$, and, G being semiregular,

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} g_{n,k+p} a_k z_0^k = S,$$

which multiplied by z_0^p can be rewritten

$$\lim_{n \to \infty} \sum_{k=p}^{\infty} g_{n,k}\, a_{k-p} z_0^k = z_0^p S. \tag{2.1}$$

But the Taylor series of $F(z)$ is $\sum_{k=0}^{\infty} a_k z^{k+p} = \sum_{k=p}^{\infty} a_{k-p} z^k$, and so (2.1) proves the theorem.

Corollary. Under the conditions of the theorem, if $P(z)$ is a polynomial and $F(z) \equiv P(z) f(z)$, then G sums the Taylor series of $F(z)$ at z_0 to $P(z_0)S$.

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[1] P. Vermes, "On γ-matrices and their application to the binomial series," these *Proceedings* **8** (1947), 1-13. This paper will be referred to as γ-*M*.

[2] P. Dienes, *The Taylor Series* (Oxford), 1931, 418. This book will be referred to as *T.S.*

[3] γ-*M*, section 5.

[2.II] *If a semiregular* γ*-matrix* G *sums the Taylor series of a meromorphic function* $f(z)$ *at a regular point* $z = z_0$, *then the sum is the "right" value* $f(z_0)$.

Proof: Let $f(z) = \Sigma a_k z^k$ for $|z| < R$. By hypothesis if $\rho > |z_0|$, the only singularities of $f(z)$ in the circle $|z| \leqq \rho$ are poles a_i of order m_i, so that if $F(z) \equiv P(z) f(z)$, where $P(z) \equiv \Pi(z - a_i)^{m_i}$ (the product being taken over all poles in the circle $|z| \leqq \rho$), $F(z)$ is analytic in and on this circle. Hence the Taylor series of $F(z)$ is convergent at z_0, and its G-sum therefore exists and is $P(z_0)f(z_0)$. Applying the Corollary of [2.I] we have $P(z_0)f(z_0) = P(z_0)S$, whence $S = f(z_0)$.

Corollary. The theorem readily extends to general Taylor series for values of z_0 *in the circle of meromorphy.*

[2.III] *If the semiregular* γ*-matrix* G *sums the series* $\Sigma a_k z^k$ *in the domain* D *to* $s(z)$, *then the* γ*-matrix* $H \equiv \sum_{i=0}^{\infty} \lambda_i G^{(i)} / \sum_{i=0}^{\infty} \lambda_i$ *sums the series to the same value, provided that condition* (*b*) *of theorem* [1.III] *of* γ*-M is satisfied by the* λ_i, *and that*

(i) $|g_{n,k}| \geqq |g_{n,k+1}|$ *for every* n *and* k,

(ii) $|\sigma_n^{(i)}(z)| \equiv |\sum_{k=0}^{\infty} g_{n,k+i} a_k z^k| \leqq N(z)$ *for every* i, n, *and a fixed* z *in* D.

Moreover H *is semiregular with respect to this series.*[1]

NOTE: It will be seen in [3.I] that (i) and under certain conditions (ii) hold for the Borel-matrix.

Proof: Since $g_{n,k}^{(i)} = g_{n,k+i}$ we see by [1.I] of γ-*M* that all conditions of [1.III] of γ-*M* are satisfied. Hence H *exists and is a* γ*-matrix.*

Since by hypothesis, for a fixed n, the series $\sigma_n^0(z) \equiv \Sigma g_{n,k} a_k z^k$ converges for every z in D, it converges absolutely in D, *i.e.*

$$\Sigma \, | g_{n,k} a_k z^k | = S_n(|z|) \text{ is finite in } D.$$

Also
$$\sum_{k=0}^{\infty} \sum_{i=0}^{p} | \lambda_i g_{n,k}^{(i)} a_k z^k | = \sum_{i=0}^{p} |\lambda_i| \sum_{k=0}^{\infty} | g_{n,k}^{(i)} a_k z^k |$$

$$\leqq \sum_{i=0}^{p} |\lambda_i| \sum_{k=0}^{\infty} | g_{n,k} a_k z^k | \leqq S_n(|z|) \sum_{i=0}^{\infty} |\lambda_i| = S_n(|z|) L.$$

Since the last member is independent of p, the double series

$$\rho_n(z) \equiv \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \lambda_i g_{n,k}^{(i)} a_k z^k \tag{2.2}$$

[1] Here $G^{(i)}$ denotes the i-th *diminutive* of G, *e.g.* (3.1) of γ - *M*.

converges absolutely for every z in D, and thus we can reverse the order of summation, *i.e.*

$$(2.3) \qquad \rho_n(z) = \sum_{i=0}^{\infty} \lambda_i \sum_{k=0}^{\infty} g_{n,k}^{(i)} a_k z^k \equiv \sum_{i=0}^{\infty} \lambda_i \sigma_n^{(i)}(z).$$

Comparing the series on the right-hand side of (2.3) with the series $\Sigma \mid \lambda_i \mid N = NL$, we see that it converges uniformly for every n, so that

$$(2.4) \quad \lim_{n\to\infty} \rho_n(z) = \sum_{i=0}^{\infty} \lambda_i \lim_{n\to\infty} \sigma_n^{(i)}(z) = \sum_{i=0}^{\infty} \lambda_i s(z) = l\, s(z),$$

and it follows by (2.2) and (2.4) that

$$(2.5) \quad \lim_{n\to\infty} \sum_{k=0}^{\infty} h_{n,k} a_k z^k = \lim_{n\to\infty} \sum_{k=0}^{\infty} \left\{ \frac{1}{l} \sum_{i=0}^{\infty} \lambda_i g_{n,k}^{(i)} \right\} a_k z^k = s(z).$$

i.e. H *sums the series to* $s(z)$.

Also $H^{(1)} \equiv \Sigma \lambda_i G^{(i+1)} / \Sigma \lambda_i$ satisfies the conditions of this theorem. Hence (2.5) applies to $H^{(1)}$, showing that H *is semiregular with respect to* $\Sigma a_k z^k$. This concludes the proof.

Corollary. If $\Sigma \lambda_i = 0$, *the matrix* $\Sigma \lambda_i G^{(i)}$ (*which is not a* γ-*matrix*) *sums the series to zero.*

This follows from (2.4).

[2.IV] *We suppose that* $f(z) \equiv \Sigma a_k z^k$ *in a circle* Γ *round the origin, that the semiregular* γ-*matrix* G *sums the series to* $s(z)$ *in the domain* D, *and that conditions* (i) *and* (ii) *of* [2.III] *are satisfied. If the function* $F(z)$ *is regular in a circle* C *with centre at the origin, then* G *sums the Taylor series of* $F(z)f(z)$ *about the origin to the sum* $F(z)s(z)$ *in the domain* CD.

Proof: By hypothesis $F(z) \equiv \Sigma b_i z^i$ in C, whence

$$F(z)f(z) \equiv \sum_{k=0}^{\infty} z^k \sum_{i=0}^{k} a_{k-i} b_i \text{ in } C\Gamma;$$

and if we write $a_i = 0$ for $i = -1, -2, -3, \ldots$,

$$(2.6) \quad F(z)f(z) \equiv \sum_{k=0}^{\infty} z^k \sum_{i=0}^{\infty} a_{k-i} b_i \quad \text{in} \quad C\Gamma.$$

By hypothesis

$$(2.7) \qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} g_{n,k} a_k z^k = s(z) \qquad \text{in } D, \text{ and hence in } CD.$$

Since the series $\Sigma b_i z^i$ converges absolutely in CD, we can apply [2.III] or its corollary with $\lambda_i \equiv b_i z^i$ and $l \equiv F(z)$ to (2.7); and we have as in (2.2) and (2.4)

$$\lim_{n\to\infty} \rho_n(z) \equiv \lim_{n\to\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} b_i z^i g_{n,k+i} a_k z^k = F(z)s(z) \qquad \text{in } CD.$$

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Writing $k-i$ for k, we have

$$\lim_{n\to\infty}\sum_{k=i}^{\infty} g_{n,k}z^k \sum_{i=0}^{\infty} b_i a_{k-i} = F(z)s(z) \qquad \text{in } CD,$$

and again putting $a_i = 0$ for $i = -1, -2, \ldots$, we have

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} g_{n,k}z^k \sum_{i=0}^{\infty} a_{k-i}b_i = F(z)s(z) \qquad \text{in } CD. \tag{2.8}$$

The left-hand side is the G-sum of the series (2.6). Thus (2.8) proves the theorem.

3. *Borel's exponential summation.*

This is a summation method by the γ-matrix

$$g_{n,k} \equiv \frac{1}{k!}\int_0^n e^{-t}t^k dt = 1 - e^{-n}\left(1 + n + \frac{n^2}{2!} + \ldots \frac{n^k}{k!}\right), k, n=0, 1, 2, \ldots \tag{3.1}$$

The following well-known properties[1] will be used in this section:

$$g_{n,k} \geqq g_{n,k+1} \geqq 0 \text{ for every } n \text{ and } k,$$
$$g_{n,k} \to 0 \text{ as } k \to \infty \text{ for every fixed } n.$$

When G sums the series Σc_k, the order of summation and integration can be interchanged,[2] *i.e.*

$$(B) \text{ sum of } \sum_{k=0}^{\infty} c_k = \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{c_k}{k!}\int_0^n e^{-t}t^k dt \tag{3.2}$$
$$= \lim_{n\to\infty}\int_0^n e^{-t}\left(\sum_{k=0}^{\infty}\frac{c_k t^k}{k!}\right)dt = \int_0^{\infty} e^{-t}\left(\sum_{k=0}^{\infty}\frac{c_k t^k}{k!}\right)dt.$$

G is semiregular, and Hardy gave an example of a series summable by this method but with respect to which the summation is not regular.[3]

Using the notation[4] $u_0(t) \equiv \sum_{k=0}^{\infty}\frac{c_k t^k}{k!}$,

we see from (3.2) that when G sums the series Σc_k, $u_0(t)$ is an integral function of t, and so $u_0(t)$ can be integrated repeatedly, giving

$$u_j(t) \equiv \sum_{k=0}^{\infty}\frac{c_k t^{k+j}}{(k+j)!}, \qquad j = 0, 1, 2, \ldots, \tag{3.3}$$

where $u_j(t)$ is an integral function of t. We also see that

$$u_{j+1}(t) = \int_0^t u_j(t)dt, \qquad j = 0, 1, 2, \ldots. \tag{3.4}$$

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[1] *T.S.* 401.
[2] *T.S.* 401.
[3] *T.S.* 419-420.
[4] *T.S.* 403-404.

Since G is semiregular, if $n \to \infty$

(3.5) $\sigma_n^0 \equiv \sum_{k=0}^{\infty} g_{n,k}c_k \to s$ implies $\sigma_n^j \equiv \sum_{k=0}^{\infty} g_{n,k+j}c_k \to s.$

Finally, we know that $\Sigma a_k z^k$ is summable by G, if z is an inner point of the "polygon of summability,"[1] to the "right" sum. G is inefficient outside the polygon.

[3.I] *If Σc_k is summable (B), then $|\sigma_n^j|$ is bounded for every n and j whenever $|u_0(t)|$ is bounded for $0 \leqq t \leqq \infty$.*

Proof: By (3.1), (3.2), (3.3), (3.4) and (3.5)

$$\sigma_n^j = \int_0^n e^{-t}u_j(t)dt = \int_0^n e^{-t}u_{j-1}(t)dt + \Big[e^{-t}u_j(t)\Big]_n^0.$$

(3.6) From (3.4) $u_j(0) = 0$ for $j = 1, 2, 3, \ldots$, and so by repeated integration by parts

(3.7) $$\sigma_n^j = \sigma_n^0 - e^{-n}[u_j(n) + u_{j-1}(n) + \ldots + u_1(n)].$$

By (3.5) $|\sigma_n^0| \leqq K_0$ for every n. Also applying Taylor's theorem to $u_j(n)$ and considering (3.4) and (3.6), we have $u_j(n) = u_0(\Omega_j)n^j/j!$ where $0 \leqq \Omega_j \leqq n$. Hence

$$|\sigma_n^j| \leqq K_0 + e^{-n}\Big[n\,|u_0(\Omega_1)| + \frac{n^2}{2!}|u_0(\Omega_2)| + \ldots + \frac{n^j}{j!}|u_0(\Omega_j)|\Big],$$

and, since by hypothesis $|u_0(t)| \leqq K$ for $0 \leqq t \leqq \infty$,

$$|\sigma_n^j| \leqq K_0 + e^{-n}\Big[n + \frac{n^2}{2!} + \ldots + \frac{n^j}{j!}\Big]K \leqq K_0 + K,$$

which proves the theorem.

Examples. The divergent series $\Sigma(-2)^k$ is summable (B), and $u_0(t) = e^{-2t}$ is bounded in $(0, \infty)$. The convergent series $\Sigma(\frac{1}{2})^k$ is summable (B), and $u_0(t) = e^{\frac{1}{2}t}$ is not bounded. But $u_j(n)$ and σ_n^j are positive; hence by (3.7) σ_n^j is bounded, so that the condition of this theorem is not necessary.

[3.II] *Borel's γ-matrix is regular with respect to all Taylor series in the polygon of summability.*

Proof: When $z_0 = 0$ the proof is trivial. If $z_0 \neq 0$ is in the polygon of summability, then

[1] *T.S.* 305.

(3.8) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} g_{n,k} a_k z_0^k = a_0 + \sum\limits_{k=1}^{\infty} g_{n,k} a_k z_0^k$ exists and is equal to $f(z_0)$.

The function $F(z) = \{f(z) - a_0\}/z$ has singularities at the same finite points as $f(z)$ and at no other points. Hence it has the same polygon of summability. Thus G sums the series for $F(z)$, $a_1 + a_2 z + a_3 z^2 + \ldots$, at z_0, *i.e.*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k-1} a_k z_0^{k-1} = F(z_0) = \frac{1}{z_0}\left\{f(z_0) - a_0\right\},$$

whence

(3.9) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} g_{n,k-1} a_k z_0^k = f(z_0) - a_0.$

Comparing (3.8) and (3.9) we have for z_0 in the polygon of summability

(3.10) $$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k} a_k z_0^k \rightrightarrows \lim_{n\to\infty} \sum_{k=1}^{\infty} g_{n,k-1} a_k z_0^k,$$

which proves the theorem, since the semiregularity of G would be represented by (3.10) with the arrows reversed.

4. *γ-matrices efficient on the boundaries of convergence-domains.*

Given a sequence $\rho_0, \rho_1, \rho_2, \ldots$ satisfying the conditions

$0 < \rho_n < 1$ for every n, $\qquad \rho_n \to 1$ as $n \to \infty$,

we construct the matrix R: $\qquad r_{n,k} = \rho_n^{k+1} \quad (k, n = 0, 1, 2, \ldots)$.

Then we have:

[4.I] *R is a regular γ-matrix, which sums every Taylor series at those points z_0 of its circle of convergence for which the function represented by the series tends to a limiting value when $z \to z_0$ along the radius.*

Proof: By definition

$\sum\limits_{k=0}^{\infty} | r_{n,k} - r_{n,k+1} | = \sum\limits_{k=0}^{\infty} (r_{n,k} - r_{n,k+1}) = \rho_n < 1$ for every n, and $\lim\limits_{n\to\infty} r_{n,k} = 1$ for every fixed k. Thus *R is a γ-matrix.*

Also $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} r_{n,k+1} c_k \leftrightarrows \lim\limits_{n\to\infty} \{\rho_n \sum\limits_{k=0}^{\infty} r_{n,k} c_k\} \rightleftarrows \lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} r_{n,k} c_k.$

Hence *R is regular*. If z_0 is on the circle of convergence of $\Sigma a_k z^k$, representing $f(z)$,

$\sigma_n \equiv \sum\limits_{k=0}^{\infty} r_{n,k} a_k z_0^k = \rho_n \sum\limits_{k=0}^{\infty} a_k (\rho_n z_0)^k$ converges to $\rho_n f(\rho_n z_0)$.

Hence $\lim\limits_{n\to\infty} \sigma_n = \lim\limits_{\rho_n\to 1} f(\rho_n z_0)$ whenever the limit on the right-hand side exists. This proves the last statement.

Examples. (a) If $\rho_n = \theta^{1/(n+1)}$, $0 < \theta < 1$, then $r_{n,k} = \theta^{(k+1)/(n+1)}$

(b) If $\rho_n = (n+\beta)^{-p/(n+1)}$, $p > 0, \beta > 0$,
then $r_{n,k} = (n+\beta)^{-p(k+1)/(n+1)}$.

The matrix R can be constructed independently of the series to which it applies. A somewhat similar construction can be used for a particular class of Dirichlet series, given in the usual notation as

(4.1) $\Sigma a_k \exp\{-\lambda(k)s\}$, representing the function $f(s)$ where $s = \sigma + it$, where $\lambda(k) \to \infty$ with k, $0 < \lambda(k) < \lambda(k+1)$.
Given the class of series characterized by $\{\lambda(k)\}$, we construct the matrix L as follows:
We define a sequence $0 < \mu(1) < \mu(2) < \ldots.$ where $\mu(n) \to \infty$, and make

(4.2) $$l_{n,k} = \exp\{-p\lambda(k)/\mu(n)\}, \qquad p > 0, \qquad n, k = 1, 2, 3, \ldots.$$

[4.II] *If the series* (4.1) *has a finite abscissa of convergence* s_0, *then the* γ*-matrix* L *given by* (4.2) *sums the series at all points* $s_0 = \sigma_0 + it$ *of its line of convergence at which* $f(s_0 + 0)$ *exists, and the sum is* $f(s_0 + 0)$.

Proof: By (4.2) $l_{n,k} > l_{n,k+1} > 0$ for every n, $k \geqq 1$,
and $l_{n,k} \to 0$ for a fixed n as $k \to \infty$,
$l_{n,k} \to 1$ for a fixed k as $n \to \infty$.

Hence $$\sum_{k=1}^{\infty} | l_{n,k} - l_{n,k+1} | = \sum_{k=1}^{\infty} (l_{n,k} - l_{n,k+1}) = l_{n,1} \leqq 1.$$

Thus L is a γ-matrix. Also for $s_0 = \sigma_0 + it$

$$S_n \equiv \sum_{k=1}^{\infty} l_{n,k} a_k \exp\{-\lambda(k)s_0\} = \sum_{k=1}^{\infty} a_k \exp[-\lambda(k)\{s_0 + p/\mu(n)\}]$$
$$= f\{s_0 + p/\mu(n)\},$$

and therefore $S_n \to f(s_0 + 0)$ as $n \to \infty$ whenever the limit exists. This concludes the proof.

Example. For the class of special Dirichlet series $\Sigma a_k/k^s = \Sigma a_k \exp(-s \log k)$ the matrix L is given by

$$l_{n,k} = \exp\{-p \log k/\mu(n)\} = k^{-p/\mu(n)}.$$

BIRKBECK COLLEGE,
UNIVERSITY OF LONDON.