SYMMETRISABLE OPERATORS*

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1. Introduction

About 14 years ago A. C. Zaanen [7] published a series of papers on compact symmetrisable linear operators in Hilbert space. Four years later I was encouraged by Dr. F. Smithies to study the spectral properties of general symmetrisable operators in Hilbert space and the resulting research formed the basis of part of a dissertation I submitted to the University of Cambridge in 1952 [4]. For various personal reasons I have not previously been able to, publish these results more widely, although I believe some of them, at least, to be of general interest.

I propose to publish my work in three parts. The first part will be introductory in the sense that it will give a complete theory of symmetrisable operators in unitary spaces. This will be followed by a brief discussion of the way in which these results might generalise to Hilbert Space. A summary of the main results of the general theory will complete this part. Part II will give a rigorous discussion of symmetrisable operators in Hilbert space with an eye to maximum generality. Part III will discuss operators symmetrisable by bounded operators.

2. Notation and Definitions

We use standard Hilbert space notation as was used for instance by v. Neumann [3]. As far as possible Greek letters will be used for scalars and lower case letters for elements of the space, upper case letters for operators, German capitals for sets, i.e. spaces, vector subspaces etc., but the vector subspace spanned by the vectors (g_s) will be denoted by $[g_s]$ if closed, $\{g_s\}$ if not necessarily so. The null operator will be denoted by 0 and the identity operator by I. An operator H is called Hermitian if its domain is dense in $\mathfrak{H}$, and $H \subset H^*$. A Hermitian operator is self-adjoint if $H = H^*$. If H is a non-negative self-adjoint operator, $H^{\frac{1}{2}}$ or $\sqrt{H}$ will denote the positive square root, i.e. $(H^{\frac{1}{2}})^2 = H$, $(H^{\frac{1}{2}}x, x) \geqq 0$ (cf. Nagy [2]). Two vectors x, y are called H-orthogonal if $(Hx, y) = (x, Hy) = 0$. $P_{\mathfrak{M}}$ will be used to denote

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the orthogonal projector onto the smallest closed vector subspace containing $\mathfrak{M}$. The symbol $\mathfrak{M}^{\perp}$ will be used to denote the orthogonal complement of $\mathfrak{M}$ in the whole space. If $\mathfrak{X}$ and $\mathfrak{Y}$ are vector subspaces of $\mathfrak{M}$ we shall call $\mathfrak{Y}$ a vector complement of $\mathfrak{X}$ in $\mathfrak{M}$ if every element f of $\mathfrak{M}$ has a unique representation $f = x + y$ where $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$ and we then write $\mathfrak{M} = \mathfrak{X} + \mathfrak{Y}$. (The definition implies that $\mathfrak{X}$ and $\mathfrak{Y}$ only have the zero vector in common.) Some special items of notation will be introduced in the text, notably in Remarks 3.2 and 3.3.

We shall discuss linear operators A called symmetrisable because for some non-negative Hermitian operator H the operator HA is Hermitian. Strictly speaking this means that A is symmetrisable to the left. The theory of operators symmetrisable to the right is strictly analogous by virtue of lemma 3.7 and the properties of A^*. For brevity we shall largely restrict discussion to left symmetrisability.

In order to make the discussion non-trivial it is necessary to restrict the nullspace of H because the operator $0A$ is Hermitian for all operators A with dense domain. The weakest feasible assumptions for H are therefore that its domain $\mathfrak{D}_H$ contain $\mathfrak{R}_A$ the range of A and

(i) for all $x \in \mathfrak{D}_H$

$$(2.1) \qquad (Hx, x) \geqq 0$$

(ii) the closure $\overline{\mathfrak{N}_A}$ of the nullspace of A contains the nullspace $\mathfrak{N}_H$ of H, i.e.

$$(2.2) \qquad \overline{\mathfrak{N}_A} \supset \mathfrak{N}_H .$$

In particular if A is closed

$$(2.3) \qquad Hx = 0 \quad \text{implies} \quad Ax = 0.$$

DEFINITION: Let A be an operator and H be a Hermitian operator such that $\mathfrak{D}_H \supset \mathfrak{R}_A$ and (2.1) and (2.2) are satisfied then A is symmetrisable to the left (or the right) if HA (or AH) are Hermitian.

It is implied by the definition that the domain of A is dense in $\mathfrak{H}$.

The following general result will be used on several occasions and it is therefore convenient to state it here as

LEMMA 2.1. Let A be a linear mapping of a vector space $\mathfrak{X}$ into a vector space $\mathfrak{Y}$. Let $\mathfrak{I}$ be an index set, (x_i) $i \in \mathfrak{I}$ be a set of vectors in $\mathfrak{X}$ and (y_i) the set of vectors in $\mathfrak{Y}$ defined by $y_i = Ax_i$ for all $i \in \mathfrak{I}$.

(i) If the set (y_i) is linearly independent so is the set (x_i).

(ii) If all vectors of the set (x_i) belong to the vector subspace $\mathfrak{M}$, and $\mathfrak{M}$ and $\mathfrak{N}_A$, the nullspace of A, only share the zerovector then if the set (x_i) is linearly independent so is the set (y_i).

(iii) If $\mathfrak{M}$ is a finite dimensional vector subspace of $\mathfrak{X}$ and $\mathfrak{M}$ and $\mathfrak{N}_A$ only share the zero vector then $A(\mathfrak{M}) = \mathfrak{M}'$ has the same dimension as $\mathfrak{M}$. We give simple proofs of these familiar facts:

(i) Suppose the set (x_i) is linearly dependent; then for some finite index set $\mathfrak{J} \subset \mathfrak{I} : \sum_{j \in \mathfrak{J}} \alpha_j x_j = 0,\ \alpha_j \neq 0$. It follows that $\sum_{j \in \mathfrak{J}} \alpha_j y_j = A \sum \alpha_j x_j = 0$ and the set (y_i) is linearly dependent also; hence the result.

(ii) Suppose the set (x_i) is linearly independent but for some finite index set $\mathfrak{J} \subset \mathfrak{I}$ and $\alpha_j \neq 0 : \sum_{j \in \mathfrak{J}} \alpha_j y_j = 0$; then $A \sum \alpha_j x_j = 0$ so that $\sum \alpha_j x_j \in \mathfrak{N}_A$ and $\sum \alpha_j x_j \neq 0$. This is impossible if $\mathfrak{M} \cap \mathfrak{N}_A = [0]$.

(iii) If the set (y_i), $i \in \mathfrak{I}$ is a basis in $\mathfrak{M}'$ then there exists a set (x_i), $i \in \mathfrak{I}$, in $\mathfrak{M}$ such that $y_i = Ax_i$. By (i) the set (x_i) is linearly independent so that $\dim(\mathfrak{M}) \geqq \dim(\mathfrak{M}')$. If (g_j) $j \in \mathfrak{J}$ is a basis in $\mathfrak{M}$ then since $\mathfrak{M} \cap \mathfrak{N}_A = [0]$ the set (Ag_j) $j \in \mathfrak{J}$ is a linearly independent set in M' so that $\dim(\mathfrak{M}) \leqq \dim(\mathfrak{M}')$ and so we must have $\dim(\mathfrak{M}) = \dim(\mathfrak{M}')$.

3. Symmetrisable Operators in Unitary Spaces $\mathfrak{U}_n$

The properties established in this section are properties which, with one exception, generalise to Hilbert space with only minor obvious modifications. The proofs of these properties are all on standard lines and very simple. However, since they carry over to the general case many of them are given to be used again later. The account of this subject given in Zaanen [8] is in a different spirit, though many results will be found there.

Throughout this section (2.2) and (2.3) are of course equivalent.

The fundamental spectral properties are given in theorems 3.1, 3.2 and 3.3. They rely on the following

LEMMA 3.1. If A is symmetrisable by H so is A^p for $p = 2, 3, \cdots$. Also for all real $\lambda : [H(A - \lambda I)^q]^* = H(A - \lambda I)^q$ for $q = 1, 2, \cdots$.

PROOF. For any x, y in $\mathfrak{U}_n$

$$(HA^p x, y) = (A^{p-1}x, HAy) = (HA^{p-1}x, Ay).$$

This process can be continued until eventually one obtains

$$(HA^p x, y) = (x, HA^p y).$$

Since the nullspace of A^p contains the nullspace of A, condition (2.3) is also satisfied.

For any real λ

$$\begin{aligned}(H(A - \lambda I)x, y) &= (HAx, y) - \lambda(Hx, y)\\ &= (x, HAy) - (x, H\lambda y)\\ &= (x, H(A - \lambda I)y).\end{aligned}$$

By replacing A by $(A - \lambda I)$ in the first part of the proof the lemma is established.

We can now prove

THEOREM 3.1. Any symmetrisable operator A has the following properties:

(i) all eigenvalues are real, i.e. $Ax = \lambda x$ for $x \neq 0$ implies $\lambda = \bar{\lambda}$.

(ii) all non-zero eigenvalues are of index 1, i.e. if $\lambda \neq 0$ and $(A - \lambda I)^p y = 0$ for $p > 1$ then

$$(A - \lambda I)y = 0.$$

(iii) 0 is an eigenvalue of index 2 at most, i.e. $A^q y = 0$ for some $q \geqq 2$ implies $HAy = 0$ which implies $A^2 y = 0$ which implies $HAy = 0$.

(iv) Eigenvectors belonging to different eigenvalues are H-orthogonal. The principal vectors are H-orthogonal to the eigenvectors with nonzero eigenvalues.

REMARK 3.1. The fact that an eigenvalue is of index 1 does not prevent it from being repeated.

We call y a principal vector with principal value λ if $(A - \lambda I)^p y = 0$ for some p greater than 1 but $(A - \lambda I)\, y \neq 0$.

PROOF. To prove (i) we need only consider nonzero eigenvalues. Then if x is an eigenvector corresponding to λ

$$(HAx, x) = \lambda(Hx, x) = \frac{\lambda}{\bar{\lambda}}\,(Hx, Ax) = \frac{\lambda}{\bar{\lambda}}\,(HAx, x).$$

Hence $\lambda = \bar{\lambda}$, since $(Hx, x) \neq 0$ by (2.3).

Now let (λ_i) be the set of eigenvalues and (x_i) a set of corresponding eigenvectors. Suppose part (ii) of the theorem not true, then for some $\lambda \neq 0$ and y and $p \geqq 2$

$$(A - \lambda I)^p y = 0$$

but

$$(A - \lambda I)^{p-1} y = x \neq 0.$$

Hence x is an eigenvector with eigenvalue λ and λ is therefore real. Without loss of generality we can take x to be x_i with eigenvalue λ_i. Then clearly, using Lemma 3.1

$$\begin{aligned}(H(A - \lambda_i I)^{p-1} y, x_i) &= (Hx_i, x_i) \\ &= (H(A - \lambda_i I)^{p-2} y, (A - \lambda_i I)x_i) \\ &= 0.\end{aligned}$$

Thus $\sqrt{H}x_i = 0$ and hence $Hx_i = 0$ and therefore by (2.3) $Ax_i = 0$. Since

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$\lambda_i \neq 0$ this implies $x_i = 0$, i.e. $(A - \lambda I)^{p-1} y = 0$. This argument can now be repeated recursively for $p-1, p-2, \cdots, 2$ to prove assertion (ii).

The assertion (iii) follows from the fact that for $p \geqq 3$

$$A^p y = 0 \Rightarrow (HA^2 A^{p-2} y, A^{p-2} y) = 0 \Rightarrow \sqrt{H} A^{p-1} y = 0 \Rightarrow HA^{p-1} y = 0$$
$$\Rightarrow (HA^2 A^{p-3} y, A^{p-3} y) = 0 \Rightarrow HA^{p-2} y = 0 \Rightarrow A^{p-1} y = 0.$$

This argument can be continued until $p = 3$. For $p = 2$ the first four steps of the argument are valid with $A^0 = I$; they lead to

$$A^2 y = 0 \Rightarrow \sqrt{H} A y = 0 \Rightarrow HAy = 0.$$

The proof of (iii) is completed by appeal to condition (2.3).

To prove (iv) we first suppose x_1, x_2 to be eigenvectors with eigenvalues λ_1 and λ_2 respectively. Then

$$(HAx_1, x_2) = (x_1, HAx_2)$$

and hence

$$\lambda_1 (Hx_1, x_2) = \lambda_2 (x_1, Hx_2) = \lambda_2 (Hx_1, x_2)$$

and hence if $\lambda_1 \neq \lambda_2$ then $(Hx_1, x_2) = 0$ as required.

We have shown that A has at most 1 principal value, namely 0. Then should a principal vector y exist we have for any eigenvector x with nonzero eigenvalue λ

$$(HAx, y) = \lambda (Hx, y) = (x, HAy) = 0$$

by (iii). Since $\lambda \neq 0$ it follows that $(Hx, y) = 0$. Q.E.D.

It is important to realise that the eigenvalue zero may not be of index 1. This can be seen by considering the example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

REMARK 3.2. We partition the nullspace of A into two vector subspaces $\mathfrak{Z} = \mathfrak{N}_A \cap \mathfrak{R}_A$ and $\mathfrak{W}$ so that $\mathfrak{N}_A = \mathfrak{Z} + \mathfrak{W}$. Let sequences (z_i) $(i = 1, 2, \cdots r)$ and (w_j) $(j = 1, 2, \cdots, q)$ be bases in $\mathfrak{Z}$ and $\mathfrak{W}$ respectively. Further let $\mathfrak{Y}$ be a complement of $\mathfrak{N}_A$ in $\mathfrak{N}_{A^2}$. Now clearly $y \in \mathfrak{Y}$ implies $Ay \in \mathfrak{Z}$ so that $A(\mathfrak{Y}) \subset \mathfrak{Z}$. Also since $\mathfrak{Z} \subset \mathfrak{R}_A$ we have for every $z \in \mathfrak{Z}$ at least one $x \in \mathfrak{N}_{A^2}$ such that $Ax = z$. However by the unique decomposition $x = y + y'$ where $y \in \mathfrak{Y}$ and $y' \in \mathfrak{N}_A$ it follows that $Ay = z$ so that $A(\mathfrak{Y}) \supset \mathfrak{Z}$ which proves $A(\mathfrak{Y}) = \mathfrak{Z}$. By lemma 2.1 it follows that $\mathfrak{Z}$ and $\mathfrak{Y}$ have the same dimension and for every basis (z_i) in $\mathfrak{Z}$ there is a basis (y_i) in $\mathfrak{Y}$ such that $Ay_i = z_i$.

Assertion (iii) of theorem 3.1 implies that $\mathfrak{Z} \subset \mathfrak{N}_H$. The following lemma proves that for all symmetrisable operators we have symmetrising operators with $\mathfrak{N}_H = \mathfrak{Z}$.

LEMMA 3.2. (i) If A is symmetrisable by H then there exists a symmetring operator H_1 whose nullspace $\mathfrak{N}_{H_1}$ is $\mathfrak{N}_A \cap \mathfrak{N}_A$.

(ii) $\mathfrak{N}_{A*} = \mathfrak{R}_A^\perp, \mathfrak{N}_A = \mathfrak{R}_{A*}^\perp$ where $\mathfrak{R}_A^\perp, \mathfrak{R}_{A*}^\perp$ are the orthogonal complements of $\mathfrak{R}_A$ and $\mathfrak{R}_{A*}$ respectively.

PROOF. Let $P_{\mathfrak{R}_A^\perp}$ be the projector onto the orthogonal complement of $\mathfrak{R}_A$. We prove that $H_1 = H + P_{\mathfrak{R}_A^\perp}$ satisfies the requirements of the lemma. We prove the statements in reverse order.

(ii) is well known but referred to again later and a brief demonstration is therefore included.

For all f and g we have

$$(f, A^*g) = (Af, g).$$

Hence if g is a nullvector of A^*, i.e. $A^*g = 0$ then $(Af, g) = 0$ for all f, i.e. g is orthogonal to $\mathfrak{R}_A$ so that $\mathfrak{N}_{A*} \subset \mathfrak{R}_A^\perp$. Again if g is orthogonal to $\mathfrak{R}_A$ the *RHS* of the above equation vanishes so for all f we conclude $(f, A^*g) = 0$ which implies $A^*g = 0$ or $\mathfrak{R}_A^\perp \subset \mathfrak{N}_{A*}$ so that $\mathfrak{N}_{A*} = \mathfrak{R}_A^\perp$. The second statement follows analogously.

(i) Now for every $w \in \mathfrak{W}$ and all $f \in \mathfrak{U}_n$

$$0 = (f, HAw) = (HAf, w) = (f, A^*Hw)$$

so that $Hw \in \mathfrak{N}_{A*}$ and thus by (ii) $Hw \in R_A^\perp$.

The operator $H_1 = H + P_{\mathfrak{R}_A^\perp}$ is clearly Hermitian and since for all $f \in \mathfrak{U}_n$

$$(H_1 f, f) = (Hf, f) + (P_{\mathfrak{R}_A^\perp} f, f) \geqq (Hf, f)$$

it is non-negative and $\mathfrak{N}_{H_1} \subset \mathfrak{N}_H$. Also

$$(H_1 Af, f) = (HAf, f) = (f, HAf) = (f, H_1 Af)$$

so that H_1 is a symmetrising operator. It therefore follows that $\mathfrak{N}_{H_1} \supset \mathfrak{Z}$. Now suppose w is any nonzero vector of $\mathfrak{W}$ then $H_1 w = 0$ would imply

$$0 = (H_1 w, w) = (Hw, w) + (P_A w, w)$$

and this would mean

$$(Hw, w) = -\,||P_{\mathfrak{R}_A^\perp} w||^2 = 0$$

since H is non-negative; this would mean $w \in \mathfrak{R}_A$ which is impossible if $w \in \mathfrak{W}$. Hence $H_1 f = 0$ implies $f \in \mathfrak{Z}$ or $\mathfrak{Z} \supset \mathfrak{N}_{H_1}$ which leads to $\mathfrak{N}_{H_1} = \mathfrak{Z}$ as required.

Our next aim is to use eigenvectors and principal vectors to define a basis in $\mathfrak{U}_n$. We start by proving the simple

LEMMA 3.3. (i) H-orthogonal vectors not in the nullspace of H are linearly independent.

(ii) If $\mathfrak{M}$, $\mathfrak{N}$ are two vector subspaces such that $\mathfrak{M} \supset \mathfrak{N}$ and $\mathfrak{M} \cap \mathfrak{N}_H = \{0\}$ (i.e. the zero vector is the only vector common to $\mathfrak{M}$ and $\mathfrak{N}_H$) then the H-orthogonal vector complement of $\mathfrak{N}$ in $\mathfrak{M}$ is unique.

Let $(x_1, x_2, \cdots, x_m)$ be a set of H-orthogonal vectors such that $Hx_i \neq 0$ for $i = 1, 2, \cdots, m$. Then suppose the set linearly dependent, i.e. $\sum \alpha_i x_i = 0$ for some sequence (α_i) such that not all α_i vanish. Thus

$$(Hx_j, \sum_{i=1}^{m} \alpha_i x_i) = 0 \quad \text{for} \quad j = 1, 2, \cdots, m.$$

But this means $\bar{\alpha}_j(Hx_j, x_j) = 0$ for each j and hence $\alpha_j = 0$ for $j = 1, 2, \cdots$, m contrary to hypothesis.

To prove (ii) we have to show that every $f \in \mathfrak{M}$ has a unique resolution $f = g + h$ where $g \in \mathfrak{N}$ and h is H-orthogonal to $\mathfrak{N}$. Suppose there are two

$$f = g + h = g_1 + h_1.$$

Then

$$g - g_1 = h_1 - h$$

and

$$\begin{aligned}(H(h_1 - h), h_1 - h) &= (H(g - g_1), h_1 - h) \\ &= (H(g - g_1), h_1) - (H(g - g_1), h) \\ &= 0\end{aligned}$$

since $g - g_1 \in \mathfrak{N}$. Since $h_1 - h \in \mathfrak{M}$ it now follows by hypothesis that $h = h_1$ and hence $g = g_1$. Q.E.D.

It is now convenient to introduce some further notation which will be adhered to for the rest of this paper.

REMARK 3.3. Let A be a symmetrisable operator. We associate with it four subspaces. Let $\mathfrak{X}$ denote the vector subspace determined by the eigenvectors with nonzero eigenvalues. Let, as before, $\mathfrak{Z}$ denote the intersection of the range $\mathfrak{R}_A$ and the nullspace $\mathfrak{N}_A$ of A and $\mathfrak{W}$ its orthogonal vector complement in $\mathfrak{N}_A$. For $\mathfrak{Y}$ we choose the H-orthogonal vector complement of $\mathfrak{W}$ in $A^{-1}(\mathfrak{Z}) \cap \mathfrak{Z}^{\perp}$, i.e. the set of vectors y such that $Ay \in \mathfrak{Z}$, and $(Hy, w) = 0$, $(y, z) = 0$ all $w \in \mathfrak{W}$, $z \in \mathfrak{Z}$. We assume here, just as we do in the sequel without explicit reference to it that $\mathfrak{N}_H = \mathfrak{Z}$.

The Schmidt orthogonalisation procedure can clearly be generalised to H-orthogonalise any bases in vector subspaces of principal vectors and eigenvectors with equal eigenvalues.* Then let $1 \leqq i \leqq p$, $1 \leqq j \leqq q$, $1 \leqq k \leqq r$ and

(x_j) be an H-orthogonal basis in $\mathfrak{X}$ (i.e. a complete H-orthogonal set of eigenvectors with eigenvalues λ_i other than 0.)

* The procedure would break down in $\mathfrak{Z}$ but any basis in $\mathfrak{Z}$ is automatically H-orthogonal.

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(w_j) be an H-orthogonal basis in $\mathfrak{W}$ (i.e. a complete H-orthogonal set of eigenvectors with eigenvalue 0 and orthogonal to $\mathfrak{R}_A \cap \mathfrak{R}_A$.)

(y_k) be an H-orthogonal basis in $\mathfrak{Y}$ (i.e. a complete H-orthogonal set of principal vectors H-orthogonal to $\mathfrak{W}$ and orthogonal to $\mathfrak{Z} = \mathfrak{R}_A \cap \mathfrak{R}_A$.)

(z_k) be the basis in $\mathfrak{Z}$ defined by $z_k = Ay_k$. (The z_k are eigenvectors with zero eigenvalue but in the range of A.) Then the set

$$(g_s) = (x_1, \cdots, x_p, w_1, \cdots, w_q, y_1, \cdots, y_r, z_1, \cdots, z_r)$$

where $s = 1, 2, \cdots, p+q+2r$ and $g_s = x_s (1 \leqq s \leqq p)$, $g_s = w_{s-p}(p+1 \leqq s \leqq p+q)$, $g_s = y_{s-p-q}(p+q+1 \leqq s \leqq p+q+r)$, $g_s = z_{s-p-q-r}$ $(p+q+r+1 \leqq s \leqq p+q+2r)$ is called a complete H-orthogonal system of eigenvectors and principal vectors. The vectors (g_s) are linearly independent for the following reasons: (1) the set $((x_i), (w_j), (y_k))$ satisfies the conditions of lemma 3.3. (2) For the set (z_k) we may argue as follows: (z_k) not independent of the other (g_s) if there exist sequences of complex numbers (α_k), (β_t) not all zero and such that

$$z = \sum_1^r \alpha_k z_k = \sum_1^{p+q+r} \beta_t g_t;$$

but then since $A^2 z = 0$

$$0 = \sum_1^p \lambda_t^2 \beta_t x_t$$

so that $\beta_1 = \beta_2 = \cdots = \beta_p = 0$.

Also since $Az = 0$ it follows that

$$\sum_{p+q+1}^{p+q+r} \beta_t z_{t-p-q} = 0.$$

Since $z_k = Ay_k$ this would imply $\sum_{k=1}^r \beta_{p+q+k} y_k$ belongs to $\mathfrak{N}_A$ which is impossible unless all $\beta_t = 0$ since $\mathfrak{N}_A$ and $\mathfrak{Y}$ are vector complements in $\mathfrak{N}_{A^2}$. The same argument shows that $\sum \alpha_k z_k = A \sum \alpha_k y_k = 0$ implies $\alpha_k = 0$ for all k. Hence (z_k) is a linearly independent set, linearly independent of (x_i) and (y_k) and by definition also linearly independent of (w_j). We conclude therefore that the vector subspace $\mathfrak{G}$, say, spanned by the (g_s) has dimension $p+q+2r$.

It is a fact well known for finite dimensional vector spaces that $p+q+2r = n$ (n the dimension of $\mathfrak{U}_n$). Since it is our aim to use geometrical concepts wherever possible we shall give an independent proof of

LEMMA 3.4. The set (g_s) forms a basis in $\mathfrak{U}_n$.

We prove this by showing that if (g_s) is not a basis in $\mathfrak{U}_n$ then $\mathfrak{U}_n$ is infinite dimensional.

Suppose (g_s) not a basis so that $p+q+2r < n$. Then there exists a vector $f \neq 0$ orthogonal to $\mathfrak{N}_H$ and H-orthogonal to $\mathfrak{G}$ the subspace spanned by the (g_s). Further for some vector h_1 which can be taken as orthogonal to $\mathfrak{N}_H$ and H-orthogonal to the vectors g_s and f and for certain numbers $\beta_1, \beta_{1i}, \gamma_{1j}, \delta_{1k}, \eta_{1k}$.

$$Af = \beta_1 f + h_1 + \sum_1^p \beta_{1i}x_i + \sum_1^q \gamma_{1j}w_j + \sum_1^r \delta_{1k}y_k + \sum_1^r \eta_{1k}z_k.$$

But since $(HAf, x_i) = \lambda_i(Hf, x_i) = 0$, $(HAf, w_j) = 0$ and $(HAf, y_k) = 0$ and $(Hx_i, x_i) \neq 0$, $(Hw_j, w_j) \neq 0$, $(Hy_k, y_k) \neq 0$ we must have $\beta_{1i} = 0$, $\gamma_{1j} = 0, \delta_{1k} = 0$ for $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$, $k = 1, 2, \cdots, r$. Therefore we either have

$$Af = h_1 + \sum_1^r y_{1k}z_k = h_1 + z^{(1)}$$

and

$$A^2 f = Ah_1$$

or $\beta_1 \neq 0$ in which case we put

$$f_1 = f + \sum_{k=1}^r \frac{\eta_{1k}}{\beta_1} z_k \tag{3.1}$$

and then

$$Af_1 = \beta_1 f_1 + h_1.$$

In either case if h_1 were the zero vector f_1 or f would be an eigenvector or principal vector contrary to assumption that (g_s) a complete system of eigenvectors and principal vectors. We must conclude that $h_1 \neq 0$. We put $f_1 = f$ if $\beta_1 = 0$ and otherwise use the definition (3.1). Again there will exist an h_2 H-orthogonal to the set (g_s) and f_1 and h_1 and orthogonal to $\mathfrak{N}_H$ such that

$$Ah_1 = \beta_2 h_1 + h_2 + \alpha_{21} f_1 + \sum \eta_{2k} z_k$$

where $\alpha_{21} = (Hh_1, h_1)/(Hf, f) > 0$. We now put $f_2 = h_1 + \sum_1^r(\eta_{2k}/\beta_2) z_k$ if $\beta_2 \neq 0$ and $f_2 = h_1$ if $\beta_2 = 0$ so that for some $z^{(2)} \in \mathfrak{Z} = \mathfrak{N}_H$

$$Af_2 = \beta_2 f_2 + h_2 + \alpha_{21} f_1 + z^{(2)}$$

where of course $z^{(2)} = 0$ unless $\beta_2 = 0$.

Now we have that if $h_2 = 0$

$$A(f_1 + \alpha f_2) = (\beta_1 + \alpha\alpha_{21})f_1 + (1 + \alpha\beta_2)f_2 + z^{(1)} + \alpha z^{(2)}$$

so that if we choose for α one of the roots of

$$\alpha^2\alpha_{21} + \alpha(\beta_1 - \beta_2) - 1 = 0$$

we obtain writing $\lambda = \beta_1 + \alpha\alpha_{21}$

$$A(f_1 + \alpha f_2) = \lambda(f_1 + \alpha f_2) + z^{(1)} + \alpha z^{(2)}.$$

Clearly if $\lambda = 0$ then $f_1 + \alpha f_2$ is a principal vector or an eigenvector and if $\lambda \neq 0$, $f_1 + \alpha f_2 + \lambda^{-1}(z^{(1)} + \alpha z^{(2)})$ is an eigenvector contrary to hypothesis. We conclude that $h_2 \neq 0$. Again for some h_3 which is H-orthogonal to $[g_s, f_1, f_2, h_2]$ and orthogonal to $\mathfrak{N}_H$

$$Ah_2 = \beta_3 h_2 + h_3 + \alpha_{32} f_2 + \sum \eta_{3k} z_k. \tag{3.2}$$

We note that $(HAh_2, f_1) = (Hh_2, \beta_1 f_1 + h_1) = 0$ by the definition of h_2 and hence f_1 does not appear on the RHS of (3.2); also $\alpha_{32} = (Hh_2, h_2)/(Hf_2, f_2) > 0$. Again when β_3 not zero we remove the term in $\mathfrak{N}_H$ by putting $f_3 = h_2 + \sum_1^r \eta_{3k}/\beta_3 z_k$; otherwise $f_3 = h_2$. It is now clear that we are recursively defining an H-orthogonal set of vectors $f_1, f_2, \cdots, f_t, \cdots$ in a vector complement of $\mathfrak{N}_H$ for which

$$Af_t = \beta_t f_t + \alpha_{t,t-1} f_{t-1} + f_{t+1} + \tilde{z}^{(t)} \tag{3.3}$$

where $\alpha_{10} = 0$, $\alpha_{t,t-1} > 0$ for $t > 1$ and $\tilde{z}^{(t)}$ is some vector in $\mathfrak{N}_H$ and possibly the zero vector. If $f_{t+1} = 0$ for some t then for the vector $\hat{f} = \sum_{r=1}^t \alpha_r f_r$ with

$$\begin{aligned}
\alpha_1 &= 1 \\
\alpha_2 &= (\lambda - \beta_1)/\alpha_{21} \\
\alpha_r &= [(\lambda - \beta_{r-1})\alpha_{r-1} - \alpha_{r-2}]/\alpha_{r,r-1} \qquad r = 3, 4, \cdots, t,
\end{aligned}$$

where λ is a solution of the polynomial equation

$$(\beta_t - \lambda)\alpha_t + \alpha_{t-1} = 0$$

it is easy to verify that

$$A\hat{f} = \lambda\hat{f} + \hat{z}$$

where $\hat{z}$ is an element of $\mathfrak{N}_H$. It follows that either $\lambda \neq 0$ and $\hat{f} + \lambda^{-1}\hat{z}$ is an eigenvector, or $\hat{f}$ is an eigenvector or principal vector depending on whether $\hat{z}$ is the zero vector or not. Neither of these possibilities is consistent with the hypothesis regarding the set (g_s). Hence $f_{t+1} \neq 0$ however large t which is only possible if $\mathfrak{U}_n$ is infinite dimensional.

Remark. The above construction can, of course, be used to determine the eigenvalues and eigenvectors of A starting not with the set (g_s) but with an arbitrary vector. (Cf. Silberstein [5]).

The properties of A^* are closely related to the properties of A as can be seen from the following lemmas

Lemma 3.5

(i) If λ is an eigenvalue of A it is an eigenvalue of A^*.

(ii) The range of A^* contains the vector subspace

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$$[Hx_1, Hx_2, \cdots, Hx_p, Hy_1, \cdots, Hy_r] \quad \text{i.e.} \quad H(\mathfrak{X} + \mathfrak{Y}).$$

(iii) A complete set of eigenvectors and principal vectors of A^* is given by (Hx_i, Hw_j, Hy_k) and (u_k) respectively where $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$, $k = 1, 2, \cdots, r$ and $u_k = A^{*-1}Hy_k$, A^{*-1} being a "minimal inverse" defined in the proof (Equations (3.6) and (3.7)).

(iv) The eigenvalues corresponding to Hx_i, Hw_j, Hy_k are λ_i, 0, 0 respectively. $A^{*2}u_k = A^*Hy_k = 0$, i.e. u_k is a principal vector of index 2 with principal value 0.

PROOF.

(i) follows from the fact that λ is real and the eigenvalues of A^* are, of course, $\bar{\lambda}$. (A simple "geometrical" proof of the relationship between eigenvalues of A and A^* is as follows:

For all $f \in \mathfrak{U}_n$ and any eigenvector g with eigenvalue λ

$$0 = ((A - \lambda I)g, f) = (g, (A^* - \bar{\lambda} I)f)$$

hence g is orthogonal to the range of $A^* - \bar{\lambda}I$. Hence this range has dimension $(n - 1)$ at most. Then if (f_i) $(i = 1, 2, \cdots, n)$ be a basis in $\mathfrak{U}_n$ the set of n vectors $(A^* - \bar{\lambda}I)f_i$ must be linearly dependent, i.e. for some non-null sequence (α_i): $\sum \alpha_i (A^* - \bar{\lambda}I)f_i = 0$. This however means $\sum \alpha_i f_i$ is an eigenvector with eigenvalue $\bar{\lambda}$.)

(ii) By statement (ii) of lemma 3.2 $\mathfrak{R}_{A^*}$ is the orthogonal complement of $\mathfrak{N}_A$. By Remark (3.3) $\mathfrak{X}$, $\mathfrak{Y}$ are H-orthogonal to $\mathfrak{N}_A$ which means Hx_i, Hy_k are orthogonal to $\mathfrak{N}_A$ for all i, k as required. To prove (iii) and (iv) we note first that for all $f \in \mathfrak{U}_n$ and with λ_s denoting the eigenvalue or principal value of the eigenvector or principal vector g_s

$$0 = (H(A - \lambda_s I)g_s, f) = g_s, (H(A - \lambda_s I) f) = ((A^* - \lambda_s I)Hg_s, f)$$

and hence Hg_s is an eigenvector unless it is a zero vector, which means $g_s \in \mathfrak{Z}$. Again the set (Hg_s) where $1 \leqq s \leqq p + q + r$ is linearly independent since $\sum_1^{p+q+r} \alpha_s Hg_s = 0$ implies $H \sum \alpha_s g_s = 0$ or $\sum_1^{p+q+r} \alpha_s g_s \in \mathfrak{Z}$ which by the linear independence of the g_s means $\alpha_s = 0$ for all s.

The set (Hy_k) is linearly independent and by (ii) every Hy_k is in the range of A^* so that for $k = 1, 2, \cdots, r$ there exists at least one u_k for each Hy_k such that

$$A^*u_k = Hy_k. \tag{3.4}$$

Since for all $f \in \mathfrak{U}_n (A^{*2}u_k, f) = (A^*Hy_k, f) = (y_k, HAf) = (HAy_k, f) = 0$, u_k is a principal vector with index 2. The u_k are so far only defined to within an arbitrary vector in the nullspace of A^*. However, A^* is single-valued and (Hy_k) is linearly independent so that if (u_k) is a set of vectors satisfying (3.4) then by Lemma 2.1 they form a linearly independent set.

We proceed to prove that the set $(Hg_s;\ u_k)$ $(s = 1, \cdots, p+q+r,$ $k = 1, \cdots, r)$ is linearly independent. Suppose

$$\sum_1^{p+q+r} \alpha_s Hg_s + \sum_1^r \beta_k u_k = 0$$

for some (α_s), (β_k) not all zero. Then

$$A^*(\sum \alpha_s Hg_s + \sum \beta_k u_k) = \sum \alpha_s A^* Hg_s + \sum \beta_k Hy_k = 0 \tag{3.5}$$

but this means

$$\sum_1^p \alpha_s \lambda_s Hx_s + \sum_1^r \beta_k Hy_k = 0 \tag{3.6}$$

which is impossible since the Hx_i and Hy_k are linearly independent. Hence the set (Hg_s, u_k) is a set of n linearly independent vectors and therefore a basis in $\mathfrak{U}_n$. Furthermore (3.5) and (3.6) then implies that $\mathfrak{N}_{A^*}$, the null-space of A^*, is spanned by the (Hw_j, Hy_k). It follows that $\mathfrak{N}_{A^*}$ has $p+r$ dimensions so that $H(\mathfrak{X} + \mathfrak{Y}) = \mathfrak{X}_{A^*}$. We define A^{*-1} as a mapping of $\mathfrak{N}_{A^*}$ into $\mathfrak{U}_n$ as follows. Consider the restriction of A^* to $H(\mathfrak{X}) + \mathfrak{U}$ where $\mathfrak{U}$ is the r-dimensional orthogonal complement of $\mathfrak{X} + \mathfrak{Y} + \mathfrak{W}$. Then A^* maps $H(\mathfrak{X})$ onto $H(\mathfrak{X})$ and $\mathfrak{U}$ onto $H(\mathfrak{Y})$ in a one-one manner because clearly $H(\mathfrak{X})$ reduces A^* since $A^*(Hx_i) = \lambda_i Hx_i$ and for any $u \in \mathfrak{U}$ there exists some $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ such that

$$A^* u = \alpha Hx + \beta Hy$$

and

$$(A^* u, x) = \alpha(Hx, x) + \alpha(Hy, x);$$

but by Remark 3.3 $(Hy, x) = 0$ and $(A^* u, x) = (u, Ax) = 0$ since $Ax \in \mathfrak{X}$ so that $\alpha = 0$, $A^* u \in H(\mathfrak{Y})$. Explicitly A^{*-1} is defined on its domain $\mathfrak{N}_{A^*}$ by

$$\left.\begin{aligned} A^{*-1}(Hx_i) &= \lambda_i^{-1} Hx_i \qquad (i = 1, 2, \cdots, p) \\ A^{*-1}(Hy_k) &= u_k \qquad\qquad (k = 1, 2, \cdots, r) \end{aligned}\right\} \tag{3.6}$$

where if u_k^+ is any solution of $A^* u_k^+ = Hy_k$ then

$$u_k = u_k^+ - \sum_{j=1}^q \frac{(u_k^+, w_j)}{(Hw_j, w_j)} Hw_j - \sum_m^r \frac{(u_k^+, y_m)}{(Hy_m, y_m)} Hy_m. \tag{3.7}$$

We introduce further notation

REMARK 3.4. The eigenvectors and principal vectors of A^* as described in lemma 3.5 will be denoted by

$$x_i^* = Hx_i, w_j^* = Hw_j, y_k^* = Hy_k, z_k^* = u_k \qquad (\text{thus } A^* z_k^* = y_k^*)$$

and collectively they will be referred to as (g_s^*), i.e. (g_s^*) is a complete system of eigenvectors and principal vectors of A^*.

LEMMA 3.6. If (g_s) is any complete H-orthogonal system of eigenvectors and principal vectors of A and (g_s^*) the set of vectors defined in Remark 3.4 then (g_s) and (g_s^*) are bi-orthogonal sets. By choosing a suitably normalised set (g_s) the (g_s) and (g_s^*) are complete bi-orthonormal systems in $\mathfrak{U}_n$.

By Remark 3.3 $x_i^* = Hx_i$ is orthogonal to all $g_s \neq x_i$ since (g_s) an H-orthogonal set; similarly w_j^* is orthogonal to all $g_s \neq w_j$ and y_k^* is orthogonal to all $g_s \neq y_k$. By the definition of u_k (equation (3.7)) we have since $z_k^* = u_k$

$$(z_k^*, w_j) = 0 \qquad (j = 1, 2, \cdots, q)$$
$$(z_k^*, y_m) = 0 \qquad (m = 1, 2, \cdots, r).$$

Also

$$(z_k^*, x_i) = \lambda_i^{-1}(z_k^*, Ax_i) = \lambda_i^{-1}(A^*z_k^*, x_i) = \lambda_i^{-1}(y_k^*, x_i) = 0.$$

Finally

$$(z_k^*, z_m) = (z_k^*, Ay_m) = (A^*z_k^*, y_m) = (y_k^*, y_m). \tag{3.8}$$

Hence by the orthogonality properties of y_k^* and y_k we have

$$(z_k^*, z_m) = 0 \quad \text{unless} \quad k = m.$$

The normalisation of the systems is best carried out by modifying the set (g_s). Since $(Hg_s, g_s) \neq 0$ for $1 \leqq s \leqq p + q + r$ we can replace g_s by

$$g_s' = \frac{g_s}{\sqrt{(Hg_s, g_s)}} \qquad s = 1, 2, \cdots, p + q + r$$

and then $(g_s'^*, g_s') = (Hg_s', g_s') = 1$ for $s = 1, \cdots, p + q + r$. On defining $z_k' = Ay_k'$, $z_k'^* = z_k^*/\sqrt{(Hy_k, y_k)}$ it follows from (3.8) that $(z_k'^*, z_k') = 1$ also. Hence the sets (g_s) and (g_s^*) can, without loss of generality, be taken to be bi-orthonormal.

We have given a complete account of the spectral properties of operators that are symmetrisable to the left and of their adjoints. Provided right symmetrisability is defined with a suitable restriction on the nullspace of H, the operators symmetrisable to the right are seen to have analogous properties to operators symmetrisable to the left by virtue of the following

LEMMA 3.7 (i) If A is symmetrisable to the left (right) by a positive Hermitian operator H then A^* is symmetrisable to the right (left) by the same operator.

(ii) If A is symmetrisable to the left with $\mathfrak{N}_H = \mathfrak{N}_A \cap \mathfrak{R}_A$ then A^*H is Hermitian and $\mathfrak{N}_H = \mathfrak{N}_{A^*}^\perp \cap \mathfrak{R}_{A^*}^\perp$ and conversely.

(iii) If A is such that AH is Hermitian and $\mathfrak{N}_H \subset \mathfrak{N}_A$ then we can take

$\mathfrak{N}_H = [0]$ and A is symmetrisable to the right and A^* is symmetrisable to the left.

Proof. In the first place we have that if for all f, g in

$$(3.9) \qquad (HAf, g) = (f, HAg) \quad \text{then} \quad (f, A^*Hg) = (A^*Hf, g)$$

or if

$$(3.10) \qquad (AHf, g) = (f, AHg) \quad \text{then} \quad (f, HA^*g) = (HA^*f, g)$$

and conversely. Hence it is merely a matter of discussing $\mathfrak{N}_H$.

(i) Since $\mathfrak{N}_H = [0]$ the result is immediate.

(ii) This is the standard case dealt with previously and the definition of $\mathfrak{N}_H$ in terms of $\mathfrak{N}_{A*}$ and $\mathfrak{R}_{A*}$ follows from lemma 3.2.

(iii) By hypothesis we have for all f, g

$$(AHf, g) = (f, AHg).$$

If we replace H by $H_1 = H + P\mathfrak{N}_H$ which is clearly positive then since $\mathfrak{N}_H \subset \mathfrak{N}_A$ we have $AH = AH_1$; thus AH_1 is Hermitian and (3.10) gives the required result.

To complete this section we investigate the following question: given that an operator A has only real eigenvalues, that all nonzero eigenvalues are of index 1 and that zero eigenvalues are of index 2 at most, is A always symmetrisable by an H satisfying (2.1) and (2.3)? The answer is in the affirmative and relies on the following

Lemma 3.8. Let K be the transformation which transforms every $f = \sum \gamma_s h_s$ into $f^* = \sum \gamma_s h_s^*$. Then if (h_s), (h_s^*) are any complete bi-orthonormal systems, K is a Hermitian positive definite linear transformation. If some of the h_s^* in the definition of K are replaced by the zero vector, K will be Hermitian non-negative.

The reader will very easily verify the statements in this lemma. We show that the K described above symmetrises A for particular sets (h_s), (h_s^*). Let (g_s) (g_s^*) denote a complete bi-orthonormal system of eigenvectors and principal vectors of A and A^* as described for instance in lemma 3.6. Then for any $f \in \mathfrak{U}_n$

$$\begin{aligned} Af &= A \sum g_i(f, g_i^*) \\ &= \sum_{i=1}^{p} \lambda_i x_i(f, x_i^*) + \sum_{k=1}^{r} z_k(f, y_k^*). \end{aligned}$$

Hence for K defined as in lemma 3.8 with $h_s = g_s$ and $h_s^* = g_s^*$ for $s = 1, 2 \cdots, n$ and any f, g in u_n

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$$(KAf, g) = \sum_{i=1}^{p} \lambda_i (x_i^*, g)(f, x_i^*) + \sum_{k=1}^{r} (z_k^*, g)(f, y_k^*)$$

$$(f, KAg) = \sum_{i=1}^{p} \lambda_i (f, x_i^*)\overline{(g, x_i^*)} + \sum_{k=1}^{r} (f, z_k^*)\overline{(g, y_k^*)}.$$

Hence if the subspace $\mathfrak{Z} = [z_j]$ is empty then K symmetrises A. If on the other hand $r \neq 0$ then the symmetrisation is achieved by the non-negative operator H which maps g_s on g_s^* except for the elements of the set (z_k) which are mapped on the zero element.

Finally we consider the positive Hermitian operator $K_1 (= K^{-1})$ which maps $\sum \gamma_s g_s^*$ on $\sum \gamma_s g_s$. We find

$$\begin{aligned}(AK_1 f, g) &= (AK_1 \sum (f, g_s) g_s^*, g) = (A \sum (f, g_s) g_s, g) \\ &= \sum_1^p (f, x_i)\lambda_i (x_i, g) + \sum_1^r (f, y_k)(z_k, g)\end{aligned}$$

$$f, AK_1 g) = \sum_1^p (f, x_i)\lambda_i \overline{(g, x_i)} + \sum_1^r (f, z_k)\overline{(g, y_k)}.$$

Again we see that symmetrisation is achieved if we modify K_1 by requiring that y_k^* be mapped on 0 for all k, but for all other g_s^* the map is still g_s. We shall call the non-negative Hermitian operator so defined H_1. Also using lemma (3.7) H_1 symmetrises A^* to the left. In the above argument we have only used the spectral properties of A (and A^*), not its symmetrisability. Hence we have proved

THEOREM 3.2. Let A be a linear operator in $\mathfrak{U}_n$ whose non-zero eigenvalues (λ_i), if any, are real and of index 1, and for which 0 is an eigenvalue of index 2 at most. Let a complete set of principal vectors be (y_k) $(k = 1, \cdots, r)$ and a complete set of eigenvectors be the sets (x_i) (with $\lambda_i \neq 0$, $i = 1, \cdots, p$), (w_j) (with $\lambda_j = 0$, $j = 1, \cdots, q$), (z_k) $(\lambda_k = 0$, $k = 1, \cdots, r)$ where $z_k = Ay_k$. Then tere exist non-negative Hermitian linear operators H, H_1 such that the nullspace of H is $[z_i]$ and that of H_1 is $[Hy_i]$; H symmetrises A to the left and H_1 symmetrises A^* to the left.

REMARK 3.5. The symmetrising operator H in the above argument is defined by means of the bi-orthonormal sets (g_s) and (g_s^*). Since the normalisation of these can be done in infinitely many ways we see that for any symmetrisable A the symmetrising H is not unique.

It is clear that if we restricted ourselves to operators symmetrisable by positive Hermitian H the spectral theorems would be simpler. However, at least for operators in $\mathfrak{U}_n$ it is worthwhile to summarise the more general case. This can be done in the following

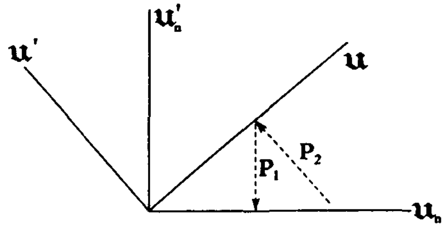
THEOREM 3.3. The symmetrisable operators in $\mathfrak{U}_n$ are those and only those whose eigenvalues are real, whose non-zero eigenvalues are of index 1 and whose zero eigenvalues are of index 2 at most. If and only if all eigenvalues are of index 1 the symmetrising operator H can be taken as positive definite. An operator is Hermitian (self-adjoint) if and only if it is symmetrisable by a positive H and it has a complete orthogonal system of eigenvectors, i.e. if it is symmetrisable by I.

The only statement in the above theorem that requires any comment is that all eigenvalues of A are of index 1 if and only if there exists a positive symmetrising operator. If H is positive then by statement (iii) of Theorem (3.1) $A^2y = 0$ implies $HAy = 0$ implies $Ay = 0$. On the other hand if all eigenvalues of A are of index 1 then $\mathfrak{Z} = [z_i] = [0]$ and the nullspace of the symmetrising H of Theorem (3.2) is $[0]$.

4. The relationship between symmetrisable operators in $\mathfrak{U}_n$, and certain Hermitian operators

Theorem (3.3) shows that the main difference between Hermitian and symmetrisable operators is the orthogonality of the eigenvectors. This suggests that it might be possible to arrive at symmetrisable operators in $\mathfrak{U}_n$ by projection from Hermitian operators defined in a space of $2n$ dimensions; the projective relationship sought projects eigenvectors of one operator onto eigenvectors of the other operator, the eigenvalues being the same for both. It will be shown that this relationship exists even in the case when the symmetrising operator is not positive definite. However, the positive case is much simpler and will be dealt with more fully to explain the procedure.

Let $\mathfrak{U}_{2n}$ be a $2n$-dimensional unitary space containing $\mathfrak{U}_n$. The orthogonal complement of $\mathfrak{U}_n$ in $\mathfrak{U}_{2n}$ is denoted by $\mathfrak{U}_n'$. $\mathfrak{U}$ is an n-dimensional subspace which contains a complete orthonormal set (g_i''). (The Hermitian operators we are looking for are defined on $\mathfrak{U}$). $\mathfrak{U}$ is the range of an isometric operator V_1 with domain $\mathfrak{U}_n$; $\mathfrak{U}$ only has the zero element in common with $\mathfrak{U}_n'$ and is defined in terms of a symmetrising H and independently of A. A schematic representation of the subspaces of $\mathfrak{U}_{2n}$ is given in the figure.



We can prove the following

THEOREM 4.1. If A is a symmetrisable operator in $\mathfrak{U}_n$ and the symmetris-

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ing operator H is strictly positive then without loss of generality we can take 1 as the lower bound for H. Further we can define an isometric operator V_1 with domain $\mathfrak{U}_n$ and range $\mathfrak{U}$ and extend it to a unitary operator V such that $V(\mathfrak{U}_n) = \mathfrak{U}$. Then in $\mathfrak{U}$ we can define a Hermitian operator H_0 such that for any $f \in \mathfrak{U}$,

$$Ax = PH_0 f = PH_0 P_1^{-1} x \tag{4.1}$$

where $Pf = x$, P is the projector onto $\mathfrak{U}_n$, P_1 is the contraction of P to domain $\mathfrak{U}$ and incidentally $VH^{\frac{1}{2}} = P_1^{-1}$ so that $A = PH_0VH^{\frac{1}{2}} = PH_0P_1^{-1}$. Conversely if H_0 is a Hermitian operator in $\mathfrak{U}$ the relationship (4.1) defines an operator A which is symmetrisable by $\mathrm{H} = H^{\frac{1}{2}}V^* P_1^{-1}$. A^*, the adjoint of A, is related to H_0 by

$$A^*x = H^{\frac{1}{2}}V^*H_0 f = P_2^{-1}H_0P_{\mathfrak{U}}x$$

where $f = P_{\mathfrak{U}}x$, $P_{\mathfrak{U}}$ is the projector onto $\mathfrak{U}$ and P_2 its contraction to domain $\mathfrak{U}_n$; also $P_2^{-1} = H^{\frac{1}{2}}V_1^*$ so that $A^* = H^{\frac{1}{2}}V^*H_0P_{\mathfrak{U}}$. A^* is symmetrisable to the right by H and to the left by H^{-1} which is bounded above by 1.

Since $A^{**} = A$ the roles of A and A^* are interchanged if H is bounded above by 1.

REMARK: The operators $VH^{\frac{1}{2}}$ and $H^{\frac{1}{2}}V^*$ are seen to be "inverse projectors".

PROOF: We construct the H with desired lower bound by renormalising the bi-orthonormal set (g_s, g_s^*) defined in the previous section. Let (e_s) be a complete orthonormal set of eigenvectors of H and (μ_s) the corresponding eigenvalues. Suppose μ_n is the smallest eigenvalue. We put $g_s' = \sqrt{\mu_n} g_s$. Then the bi-orthonormal set $(g_s'^*)$ is defined by $g_s'^* = (1/\sqrt{\mu_n}) g_s^*$ so that $H' g_s' = (1/\sqrt{\mu_n}) g_s^*$ where H' is a new symmetrising operator. Since

$$e_s = \sum_t (e_s, g_t^*) g_t = \sum_t \frac{1}{\sqrt{\mu_n}} (e_s, g_t^*) g_t'$$

$$H' e_s = \sum_t \frac{1}{\sqrt{\mu_n}} (e_s, g_t^*) \frac{1}{\sqrt{\mu_n}} g_t^* = \frac{1}{\mu_n} H \sum_t (e_s, g_t^*) g_t = \frac{\mu_s}{\mu_n} e_s.$$

Hence 1 is the lowest eigenvalue of H' and thus its "greatest" lower bound.

Without loss of generality therefore we can assume H to have lower bound 1 and we proceed to define all the operators required. Let (e_s) be a complete orthonormal set of eigenvectors and (μ_s) the corresponding set of eigenvalues of H, let (e_s') be any orthonormal set in $\mathfrak{U}_n'$ then we define a unitary operator V by

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$$(4.1) \qquad \begin{aligned} Ve_s = e_s'' &= \frac{1}{\sqrt{\mu_s}}\, e_s + \sqrt{\frac{\mu_s - 1}{\mu_s}}\, e_s' \\ Ve_s' = e_s''' &= -\sqrt{\frac{\mu_s - 1}{\mu_s}}\, e_s + \frac{1}{\sqrt{\mu_s}}\, e_s'. \end{aligned}$$

The sets (e_s'') and (e_s''') are complete orthonormal sets in $\mathfrak{U}$ and its orthogonal complement $\mathfrak{U}'$ respectively. By V_1 we denote the contraction of V to domain $\mathfrak{U}_n$. Clearly also $V^* = V^{-1}$ is defined by

$$(4.2) \qquad \begin{aligned} V^* e_s'' = e_s &= \frac{1}{\sqrt{\mu_s}}\, e_s'' - \sqrt{\frac{\mu_s - 1}{\mu_s}}\, e_s''' \\ V^* e_s''' = e_s' &= \sqrt{\frac{\mu_s - 1}{\mu_s}}\, e_s'' + \frac{1}{\sqrt{\mu_s}}\, e_s'''. \end{aligned}$$

Now since (g_s) is a complete set of eigenvectors of A, $(H^{\frac{1}{2}} g_s)$ is a complete orthonormal set in $\mathfrak{U}_n$. Hence $(V_1 H^{\frac{1}{2}} g_s)$ is a complete orthonormal set in $\mathfrak{U}$. Hence if (λ_s) are the eigenvalues of A, we define as H_0 the operator in $\mathfrak{U}$ with (λ_s) and $(V_1 H^{\frac{1}{2}} g_s)$ as eigenvalues and eigenvectors respectively. This operator is Hermitian by Theorem 3.3. If we can show that $(V_1 H^{\frac{1}{2}})^{-1}$ is the contraction of a projector to domain $\mathfrak{U}$ with range $\mathfrak{U}_n$ then H_0 and A are related in the required manner. Let P be the projector onto $\mathfrak{U}_n$ — i.e. the Hermitian operator which maps $\sum \alpha_s e_s + \sum \beta_s e_s'$ onto $\sum \alpha_s e_s$. Hence any element in $\mathfrak{U}$, i.e.

$$\sum \alpha_s e_s'' = \sum \alpha_s \left(\frac{1}{\sqrt{\mu_s}}\, e_s + \sqrt{\frac{\mu_s - 1}{\mu_s}}\, e_s' \right)$$

is mapped on

$$\sum \alpha_s \frac{1}{\sqrt{\mu_s}}\, e_s.$$

Now from the definition of V_1 and H we have $V_1^{-1} \sum \alpha_s e_s'' = \sum \alpha_s e_s$ and

$$(H^{\frac{1}{2}})^{-1} \sum \alpha_s e_s = \sum \frac{\alpha_s}{\sqrt{\mu_s}}\, e_s,$$

as required, i.e. $P_1 = (V_1 H^{\frac{1}{2}})^{-1} = H^{-\frac{1}{2}} V_1^*$.

The converse problem is also solved because for given subspaces $\mathfrak{U}$ and $\mathfrak{U}_n$ in $\mathfrak{U}_{2n}$ the projector P is always defined and V_1^{-1} can be defined analogously to the definition of V given in (4.1) or (4.2). This defines $H = (V_1^{-1} P_1^{-1})^2$ and hence it is immaterial whether we start with given subspaces or given operators V and H. Now it is immediate that every Hermitian operator will give rise to a symmetrisable operator because

$$A = PH_0VH^{\frac{1}{2}} = H^{-\frac{1}{2}}V^*H_0VH^{\frac{1}{2}}. \tag{4.3}$$

Hence

$$HA = H^{\frac{1}{2}}V^*H_0VH^{\frac{1}{2}}$$

is Hermitian as required.

The statements about the adjoints are proved as follows. Let $P_{\mathfrak{U}}$ be the orthogonal projector onto $\mathfrak{U}$ and P_2 its contraction to domain $\mathfrak{U}_n$. Then

$$P_2e_s = \frac{1}{\sqrt{\mu_s}}e_s''$$

$$V^*P_2e_s = \frac{1}{\sqrt{\mu_s}}e_s$$

shows that the "inverse projection" P_2^{-1} is defined by $H^{\frac{1}{2}}V^*$ with the property $P_2^{-1}P_2 = I$ (where I is the identity in $\mathfrak{U}_n$). Correspondingly $P_2 = VH^{-\frac{1}{2}}$. The operator B in $\mathfrak{U}_n$ is defined by

$$Bx = P_2^{-1}H_0P_2x \quad \text{all } x \in \mathfrak{U}_n. \tag{4.4}$$

By substituting explicit expressions for P_2 and P_2^{-1} it follows that

$$B = H^{\frac{1}{2}}V^*H_0VH^{-\frac{1}{2}}.$$

On comparing this with (4.3) it is seen that $B = A^*$.

It is worth observing that since A^* is symmetrisable by H^{-1} the relation (4.4) defines a one-one correspondence between operators in $\mathfrak{U}_n$ symmetrisable by positive operators with upper bound 1 and Hermitian operators in $\mathfrak{U}$.

We now drop the restriction that the symmetrising operator be strictly positive.

THEOREM 4.2. To every symmetrisable operator A in $\mathfrak{U}_n$ there corresponds an Hermitian operator H_0 in $\mathfrak{U}_{2n}$ such that if P is the orthogonal projector onto $\mathfrak{U}_n$

$$Ax = PH_0f \quad \text{where} \quad Pf = x. \tag{4.5}$$

The proof of this theorem is given in full in [4] and will only be sketched here. It is based on the construction of H_0. It is done by first extending the definition of the unitary operators V and V^* given in (4.1) and (4.2) by using the same formulae except that the sets (μ_i) and (e_i) are now the eigenvalues and eigenvectors of K the operator which maps the set (g_i) on (g_i^*). Without loss of generality the lower bound of K is 1.

It is fairly easy to see that H_0 must now be defined on $\mathfrak{U}'$ as well as $\mathfrak{U}$ since principal vectors could not be coped with by any H_0 only defined on $\mathfrak{U}$. We proceed by first subdividing $\mathfrak{U}$ into subspaces $\mathfrak{D}$, $\mathfrak{F}$, $\mathfrak{R}$, $\mathfrak{L}$ in the same way as $\mathfrak{U}_n$ is subdivided into $\mathfrak{X}$, $\mathfrak{W}$, $\mathfrak{Y}$, $\mathfrak{Z}$ in Remark 3.3. Explicitly we define bases spanning these spaces as follows:

$$Pd_i = x_i \qquad (i = 1, 2, \cdots, p)$$
$$Pf_j = w_j \qquad (j = 1, 2, \cdots, q)$$
$$Pk_k = y_k \qquad (k = 1, 2, \cdots, r)$$
$$Pl_k = z_k \qquad (k = 1, 2, \cdots, r)$$

thus $\mathfrak{D} = [d_i]$ etc. Further if P_1 is the contraction of P to domain $\mathfrak{U}$ we now have $P_1^{-1} = VK^{\frac{1}{2}}$ so that for instance $d_i = P_1^{-1}x_i = VK^{\frac{1}{2}}x_i$. By virtue of the fact that (g_s) and (g_s^*) are bi-orthonormal and $Kg_s = g_s^*$ it follows that $(P_1^{-1}g_s, P_1^{-1}g_t) = (VK^{\frac{1}{2}}g_s, VK^{\frac{1}{2}}g_t) = (g_s, Kg_t) = \delta_{st}$,* i.e. the vectors (d_i, f_j, k_k, l_k) form an orthonormal basis in $\mathfrak{U}$. Further we define the operator H_0 on $\mathfrak{U}$ by

$$H_0 d_i = \lambda_i d_i \qquad (i = 1, \cdots, p)$$
$$H_0 f_j = 0 \qquad (j = 1, \cdots, q)$$
$$H_0 k_k = k_k + l_k + p'_k \qquad (k = 1, \cdots, r)$$
$$H_0 l_k = k_k + p'_k \qquad (k = 1, \cdots, r)$$

where p'_k is a vector in $\mathfrak{U}'$ such that $P_{p'_k} = -y_k$ so that $PH_0k_k = z_k$ as required; explicitly we have $p'_k = \sum_s (k_k, e''_s)e'''_s/\sqrt{(\mu_s - 1)}$.

We now extend the definition of H_0 to the whole of $\mathfrak{U}' + \mathfrak{U}$. Let $\mathfrak{K}'$ denote the subspace of $\mathfrak{U}'$ spanned by the p'_k and (k'_k) be an orthonormal basis in $\mathfrak{K}'$, then we put

$$H_0 k'_k = \sum \alpha_{ks} k_s + \sum \alpha_{ks} l_s \quad \text{where} \quad \alpha_{ks} = (k'_k, p'_s)$$

and for all f' in the orthogonal complement of $\mathfrak{K}'$ in $\mathfrak{U}'$ we put

$$H_0 f' = 0.$$

It can be easily verified that H_0 is Hermitian and its definition has made certain that it satisfies the requirements of the theorem.

5. Symmetrisable operators in Hilbert space

(Summary of results of Parts II and III)

As is usual when the product of two not necessarily bounded operators is involved, we must start with a discussion of domains of definition. Since we are interested in deducing properties of A from properties of HA it is clearly necessary to require $D_H \supset R_A$ i.e. $D_A \subset D_{HA}$. Now if HA is symmetric

* δ_{st} here is the Kronecker delta.

with possible symmetric extension, we immediately face the problem of possible extensions of A or even of H in the case when it is not self-adjoint.

The conditions (2.1) and (2.2) are critically examined and the following facts are revealed: $\overline{\mathfrak{N}}_A \supset \mathfrak{N}_H$ is not in general strong enough to avoid pathological cases and $\mathfrak{N}_A \supset \mathfrak{N}_H$ is necessary. However it can be proved that for a wide class of operators, i.e. all J-operators as defined by Dixmier [1], all symmetrisable operators have closed nullspaces.

Lemma (3.2) will be generalised to prove that the symmetrising operator H can be assumed to have a nullspace contained in $\overline{\mathfrak{N}}_A \cap \overline{\mathfrak{R}}_A$. The fact that HA is a closed operator is in particular cases responsible for a number of consequences including the following: Let $\tilde{A}$ be the operator whose graph is the closure of the graph of A then if $\tilde{A}^{-1}$ is compact any symmetrising H has a bounded inverse, a case which is shown to be easy to deal with.

The question whether extensions of HA throw light on extensions of A is discussed and the following facts are revealed: If $H(D_A)$ is dense then A has a closed extension $\tilde{A}$. If $D_{\tilde{A}}$ and $R_{\tilde{A}}$ are contained in D_H then $H\tilde{A}$ is symmetric. If A is symmetrised by a strictly positive H and $H(D_A)$ is dense (e.g. if H is bounded) then A is closed.

The following results will be proved for the spectrum of symmetrisable operators A:

Eigenvalues are real and eigenvectors with different eigenvalues are H-orthogonal. If H is non-negative, eigenvalues of A other than 0 are of index 1, 0 of index 2 at most. When H is strictly positive all eigenvalues of A are of index 1. When H has a bounded inverse the whole spectrum of A is real. Under fairly general conditions it is shown that the residual spectrum of A is real or void. An example is constructed to show that the continuous spectrum of A may be complex. It is proved, however, that if $A = BH$ where B, H self-adjoint, H positive, then the continuous spectrum of A is real and the residual spectrum empty.

The construction relating symmetrisable operators and symmetric operators defined in a larger space is shown to generalise in the case when the symmetrising operator is bounded. This can be used to prove, for instance, that the residual spectrum of A must be real in this case. The construction can also be used to prove sufficient conditions for the continuous spectrum to be real and the residual spectrum empty.

Part III will conclude with a discussion of symmetrisable operators in an extended Hilbert space in which they are symmetric.

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