SCHLICHT DIRICHLET SERIES

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1. Introduction. For power series

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots \tag{1.1}$$

for which

$$\sum_2^{\infty} n|a_n| \leqslant 1, \tag{1.2}$$

it has been known for four decades **(1)** that $f(z)$ is regular and univalent or schlicht in $|z| < 1$. This theorem, due to J. W. Alexander, has more recently been studied by Remak **(5)** who has shown that $w = f(z)$, under the hypothesis (1.2), maps $|z| < 1$ onto a star-like region, and if (1.2) is not satisfied $f(z)$ need not be univalent in $|z| < 1$ for a proper choice of the amplitudes of the coefficients a_n.

We may recast the theorem of Alexander in the following form. Let the power series

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots \tag{1.3}$$

have a radius of convergence $R > 0$, and let ρ be the largest positive number, $0 < \rho \leqslant R$, for which

$$\sum_2^{\infty} n|a_n|\rho^{n-1} \leqslant 1. \tag{1.4}$$

Then $f(z)$ is univalent and star-like with respect to the origin in $|z| < \rho$.

For Dirichlet series

$$f(s) = -e^{-\lambda_1 s} + \sum_{n=2}^{\infty} a_n e^{-\lambda_n s}, \qquad s = \sigma + it, \tag{1.5}$$

whose abscissa of absolute convergence is $\bar{\sigma}$, $-\infty \leqslant \bar{\sigma} < \infty$, there is a smallest real number τ, $\bar{\sigma} \leqslant \tau < \infty$, for which

$$\sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \tau} \leqslant 1. \tag{1.6}$$

Working by analogy with power series one might guess that under the hypothesis (1.6), $f(s)$ would be univalent in the half-plane $\Re s > \tau$. However, this is not the case, as the simple example

$$f(s) = -e^{-\lambda_1 s}$$

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shows, and because of the almost periodic character of the functions $f(s)$ in general.

Functions $f(z)$ given by power series (1.1) which satisfy (1.2) are said to be of Hurwitz class **(5)**. Similarly, those functions $f(s)$ given by Dirichlet series (1.5) which satisfy (1.6) will be said to be of class τ.

Recalling certain concepts of univalency introduced by Montel **(3)**, we say that $f(z)$ is *locally univalent* in a region D if $f(z)$ is regular in D and if, for every closed domain $D^* \subset D$ and for every point z_0 of D^*, there exists a positive number ρ independent of z_0 such that $f(z)$ is univalent in every circle $|z - z_0| < \rho$ lying within D. Moreover, if there is a class of functions $\{f(z)\}$ regular in the region D we shall say that the functions $f(z)$ of the class are *uniformly locally univalent* in D whenever $f(z)$ is locally univalent in D and ρ has the same value for each member $f(z)$ of the class.

We shall show that the functions $f(s)$ given by a Dirichlet series (1.5) of class τ are uniformly locally univalent in a half-plane. If $\tau < (\log \lambda_1)/\lambda_1$, the half-plane is the one for which $\Re s > \tau$. If $\tau \geqslant (\log \lambda_1)/\lambda_1$ the half-plane is the one for which

$$\Re s > \frac{\lambda_q \tau - \log \lambda_1}{\lambda_q - \lambda_1} \geqslant \tau,$$

where q is the suffix of the first non-vanishing coefficient a_q of the numbers a_n, $n \geqslant 2$. The theorem is a best possible one. More explicitly we prove

THEOREM 1. *Let*

$$f(s) = -e^{-\lambda_1 s} + \sum_{n=q}^{\infty} a_n e^{-\lambda_n s}, \; a_q \neq 0, \qquad s = \sigma + it, \tag{1.7}$$

have $\bar{\sigma}$ as its abscissa of absolute convergence, $-\infty \leqslant \bar{\sigma} < \infty$. Let $f(s)$ be of class τ. Let ϵ be an arbitrary real number in the range $0 < \epsilon < 1$. Then $f(s)$ is univalent in every circle $|s - s_0| \leqslant (1 - \epsilon)\pi/\lambda_1$ for $\Re s_0 > \alpha$ where

$$\alpha = \max\left\{\tau + \frac{(1-\epsilon)\pi}{\lambda_1},\; \frac{3\log(2-\epsilon) - \log(\lambda_1 \epsilon) + \lambda_q \tau + (1-\epsilon)\pi\lambda_q/\lambda_1}{\lambda_q - \lambda_1}\right\}. \tag{1.8}$$

The factor π/λ_1 in the radius $(1 - \epsilon)\pi/\lambda_1$ cannot be replaced by a larger one.

An application is made to the Riemann Zeta-function $\zeta(s)$ which is shown to be locally schlicht in the half-plane $\Re s > 6.32$.

The radius of univalency of the function $e^{-\lambda_1 s}$ about any point s_0 is exactly π/λ_1 and the function has a period $2\pi i/\lambda_1$. Since this function is also univalent in every strip of width $2\pi/\lambda_1$ parallel to the real axis, this suggests that perhaps semi-infinite strips would form more natural domains in which to investigate properties of univalency for functions represented in half-planes by Dirichlet series. Accordingly, in §4 we obtain several results applicable to strip domains. The following theorem is proved.

THEOREM 3. *Let*

$$f(s) = -e^{-\lambda_1 s} + \sum_{n=2}^{\infty} a_n e^{-\lambda_n s}, \qquad s = \sigma + it, \tag{1.9}$$

be absolutely convergent for $\sigma > \bar{\sigma}$, $-\infty \leqslant \bar{\sigma} < \infty$, *and let* $f(s)$ *be of class* $\tau \leqslant \tau_0$ *where*

$$\tau_0 = \frac{\log \lambda_1}{\lambda_1} - \frac{\log 2}{2\lambda_1} = \frac{\log \lambda_1 - \frac{1}{2}\log 2}{\lambda_1}. \tag{1.10}$$

Let k be an arbitrary integer and let

$$t_0 = \frac{1}{\lambda_1} \operatorname{arc\,cos}\left(\frac{e^{\lambda_1 \tau}}{\lambda_1}\right), \qquad 0 < t_0 < \frac{\pi}{2\lambda_1}. \tag{1.11}$$

Let D_k *denote the strip of the s-plane defined by*

$$\sigma \geqslant \tau, \quad \left| t - \frac{2k\pi}{\lambda_1} \right| \leqslant t_0.$$

Then $W = f(s)$ *is univalent in* D_k *and maps the interior of* D_k *onto a bounded region* Δ_k *which is star-shaped with respect to the point* $\sigma = +\infty$ *at an end of the real axis, this region being convex in the direction of the real axis. The theorem is not true in general if the strip is enlarged, or if* τ *exceeds* τ_0. *If* τ_0 *is replaced by* $\tau^*_0 = (\log \lambda_1)/\lambda_1$ $f(s)$ *is still univalent in* D_k, *but* Δ_k *is in general no longer convex in the direction of the real axis. Again, the theorem is not true if* τ *exceeds* τ^*_0.

2. Preliminary lemmas. Let $f(s)$ be defined by a Dirichlet series, and normalized as in (1.5), with $\bar{\sigma}$ as abscissa of absolute convergence, $-\infty \leqslant \bar{\sigma} < \infty$, and where λ_n is a given sequence

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots, \lambda_n \to \infty.$$

We shall suppose that not all the coefficients a_n are zero.

It is well-known that the derived series

$$f'(s) = \lambda_1 e^{-\lambda_1 s} - \sum_{n=2}^{\infty} \lambda_n a_n e^{-\lambda_n s} \tag{2.1}$$

also converges absolutely for $\sigma > \bar{\sigma}$. If

$$\sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \bar{\sigma}} \tag{2.2}$$

diverges to $+\infty$ and $\bar{\sigma}$ is finite there exists a unique real number τ, $\bar{\sigma} < \tau < \infty$, for which

$$\sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \tau} = 1. \tag{2.3}$$

This follows since

$$g(\sigma) = \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \sigma} \tag{2.4}$$

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is a strictly decreasing continuous function of σ for $\sigma > \bar{\sigma}$ which assumes arbitrarily large positive values for σ near $\bar{\sigma}$, $\sigma > \bar{\sigma}$, and which assumes arbitrarily small positive values for sufficiently large positive values of σ. Since $\bar{\sigma}$ was assumed finite, there are an infinite number of coefficients a_n different from zero.

The same conclusion about τ in (2.3) may be drawn if the series (2.2) converges to a positive number not less than 1. In this case $\bar{\sigma} \leqslant \tau < \infty$. If the series (2.2) converges to a positive number less than 1 we define $\tau = \bar{\sigma}$ so that in this case (2.3) is replaced by

$$\sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \tau} < 1. \tag{2.5}$$

If $\bar{\sigma} = -\infty$, $g(\sigma)$ assumes arbitrarily large positive values for τ sufficiently small (algebraically) and negative so that again there exists a unique τ for which (2.3) holds. In all cases $\bar{\sigma} \leqslant \tau < \infty$. We shall call τ the "class" of the Dirichlet series (1.5). Thus we have the lemma:

LEMMA 1. *Let $f(s)$ be defined by the Dirichlet series* (1.5) *with $\bar{\sigma}$ as its abscissa of absolute convergence, $-\infty \leqslant \bar{\sigma} < \infty$. Then there exists a smallest real number, τ, $\bar{\sigma} \leqslant \tau < \infty$, for which*

$$\sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \tau} \leqslant 1. \tag{2.6}$$

LEMMA 2. *Let s_1 and s_2 be any two distinct points in the circle $|s| \leqslant r$. Let λ be a positive number. Then*

$$\left| \frac{e^{-\lambda s_2} - e^{-\lambda s_1}}{s_2 - s_1} \right| \leqslant \lambda e^{r\lambda}. \tag{2.7}$$

Lemma 2 follows immediately from the expansion

$$\left| \frac{e^{-\lambda s_2} - e^{-\lambda s_1}}{s_2 - s_1} \right| = \lambda \left| 1 - \frac{(s_2 + s_1)}{2!} \lambda + \frac{(s_2^2 + s_2 s_1 + s_1^2)}{3!} \lambda^2 - \dots \right| \tag{2.8}$$
$$\leqslant \lambda \left(1 + \frac{r\lambda}{1!} + \frac{r^2 \lambda^2}{2!} + \dots + \frac{r^k \lambda^k}{k!} + \dots \right) = \lambda e^{r\lambda}.$$

LEMMA 3. *Let s_1 and s_2 be any two distinct points on the circle $|z| = r$. Let λ be a positive number. Let ϵ be an arbitrary positive number less than one. Then for $r = (1 - \epsilon)\pi/\lambda$*

$$\left| \frac{e^{-\lambda s_2} - e^{-\lambda s_1}}{\lambda s_2 - \lambda s_1} \right| \geqslant \frac{\epsilon}{(2 - \epsilon)^3} > \frac{\epsilon}{8} > 0. \tag{2.9}$$

To prove Lemma 3 we observe that if

$$F(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} + \dots + \frac{b_n}{\zeta^n} + \dots \tag{2.10}$$

is regular and schlicht for $|\zeta| > 1$, and if ζ_1 and ζ_2 are two distinct points for which $|\zeta_1| = |\zeta_2| = R > 1$ then it is known **(2)** that

$$\left|\frac{F(\zeta_2) - F(\zeta_1)}{\zeta_2 - \zeta_1}\right| \geqslant 1 - R^{-2}. \tag{2.11}$$

From (2.11) it follows that if

$$f(z) = z + b_2 z^2 + \ldots + b_n z^n + \ldots \tag{2.12}$$

is regular and univalent in $|z| < 1$, and if z_1 and z_2 are any two distinct points on $|z| = \rho < 1$, then

$$\left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| \geqslant \frac{1 - \rho}{(1 + \rho)^3}. \tag{2.13}$$

(2.13) follows from (2.11) if we define $F(\zeta) = \{f(\zeta^{-1})\}^{-1}$ and use the well-known inequality for univalent functions (2.12):

$$\left|\frac{f(z)}{z}\right| \geqslant (1 + \rho)^{-2}, \; |z| = \rho < 1. \tag{2.14}$$

Since e^z is univalent in $|z| < \pi$, it follows from (2.13) that

$$\left|\frac{e^{z_2} - e^{z_1}}{z_2 - z_1}\right| \geqslant \frac{\pi^2(\pi - \rho)}{(\pi + \rho)^3}, \; |z_1| = |z_2| = \rho < \pi \tag{2.15}$$

$$\left|\frac{e^{-\lambda s_2} - e^{-\lambda s_1}}{\lambda s_2 - \lambda s_1}\right| \geqslant \frac{\pi^2(\pi - \lambda r)}{(\pi + \lambda r)^3}, \; |s_1| = |s_2| = r. \tag{2.16}$$

Choosing $r = (1 - \epsilon)\pi/\lambda$ in (2.16) we obtain (2.9). This completes the proof of Lemma 3.

3. Proof of Theorem 1. Let $s_0 = \sigma_0 + it_0$ be a complex number for which $\sigma_0 = \Re s_0 \geqslant \bar{\sigma} + (1 - \epsilon)\pi/\lambda_1$, $0 < \epsilon < 1$. Let s_1, s_2 be any two distinct values of s in the circle $|s - s_0| \leqslant r$ where $r < \sigma_0 - \bar{\sigma}$. Let $s_1' = s_1 - s_0$, $s_2' = s_2 - s_0$ so that $|s_i'| \leqslant r$. For an appropriate r we shall show that $f(s)$, given by the Dirichlet series (1.5) which is of class τ, is univalent in $|s - s_0| \leqslant r$, provided σ_0 is sufficiently large. In proving

$$\frac{f(s_2) - f(s_1)}{s_2 - s_1} \neq 0 \tag{3.1}$$

it will be sufficient to assume $|s_1'| = |s_2'| = r$. This follows from the fact that if the image curve of the circle $|s - s_0| = r$ by the mapping function bounds a simply connected region, the mapping function is schlicht in the interior when it is schlicht on the boundary. Choose $r = (1 - \epsilon)\pi/\lambda_1$. We now have

$$\begin{aligned} \frac{f(s_2) - f(s_1)}{s_2 - s_1} &= -\left(\frac{e^{-\lambda_1 s_2} - e^{-\lambda_1 s_1}}{s_2 - s_1}\right) + \sum_{n=2}^{\infty} a_n \left(\frac{e^{-\lambda_n s_2} - e^{-\lambda_n s_1}}{s_2 - s_1}\right) \\ &= -e^{-\lambda_1 s_0}\left(\frac{e^{-\lambda_1 s_2'} - e^{-\lambda_1 s_1'}}{s_2' - s_1'}\right) \\ &\qquad + \sum_{n=2}^{\infty} a_n e^{-\lambda_n s_0}\left(\frac{e^{-\lambda_n s_2'} - e^{-\lambda_n s_1'}}{s_2' - s_1'}\right) \end{aligned} \tag{3.2}$$

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(3.3) $$\left|\frac{f(s_2)-f(s_1)}{s_2-s_1}\right| \geqslant e^{-\lambda_1\sigma_0}\left|\frac{e^{-\lambda_1 s_2'}-e^{-\lambda_1 s_1'}}{s_2'-s_1'}\right| - R_q$$

where

(3.4) $$R_q = \sum_{n=q}^{\infty} |a_n| e^{-\lambda_n\sigma_0}\left|\frac{e^{-\lambda_n s_2'}-e^{-\lambda_n s_1'}}{s_2'-s_1'}\right|$$

and a_q is the first non-vanishing coefficient a_n, $n \geqslant 2$. By Lemma 2, we have for $r < \sigma_0 - \tau \leqslant \sigma_0 - \bar{\sigma}$

(3.5) $$\begin{aligned} R_q &\leqslant \sum_{n=q}^{\infty} \lambda_n|a_n|e^{\lambda_n(r-\sigma_0)} \\ &\leqslant \sum_{n=q}^{\infty} \lambda_n|a_n|e^{-\lambda_n\tau}\cdot e^{-\lambda_n(\sigma_0-r-\tau)} \\ &\leqslant e^{-\lambda_q(\sigma_0-r-\tau)}\cdot\sum_{n=2}^{\infty}\lambda_n|a_n|e^{-\lambda_n\tau} \\ &\leqslant e^{-\lambda_q(\sigma_0-r-\tau)}, \end{aligned}$$

where we have used the inequality (1.6) for functions $f(s)$ of class τ. From (3.3) and (3.5) we have for $r < \sigma_0 - \tau$.

(3.6) $$\begin{aligned} \left|\frac{f(s_2)-f(s_1)}{s_2-s_1}\right| &\geqslant \lambda_1 e^{-\lambda_1\sigma_0}\left|\frac{e^{-\lambda_1 s_2'}-e^{-\lambda_1 s_1'}}{\lambda_1 s_2'-\lambda_1 s_1'}\right| - e^{-\lambda_q(\sigma_0-r-\tau)} \\ &\geqslant \lambda_1 e^{-\lambda_1\sigma_0}\cdot m(\lambda_1 r) - e^{-\lambda_q(\sigma_0-r-\tau)}, \end{aligned}$$

where

(3.7) $$m(r) = \min_{|z_1|=|z_2|=r}\left|\frac{e^{z_2}-e^{z_1}}{z_2-z_1}\right|.$$

If $r < \pi$, $m(r) > 0$ since e^z is univalent in $|z| < \pi$. In spite of the fact that e^z is a simple elementary function, the problem of finding $m(r)$ as a function of r appears to be far from simple. It can be shown that

(3.8) $$m(r) = \min_{0\leqslant x\leqslant r<\pi} e^{-(r^2-x^2)^{\frac{1}{2}}}\cdot\frac{\sin x}{x}.$$

We shall take $r = (1-\epsilon)\pi/\lambda_1$, where ϵ is an arbitrary number in the range $0 < \epsilon < 1$. We require the value of $m(\lambda_1, r) = m((1-\epsilon)\pi)$. For small values of ϵ,

$$m((1-\epsilon)\pi) \geqslant \frac{\epsilon}{\pi} + o(\epsilon).$$

However, we require a lower bound for $m((1-\epsilon)\pi)$ which holds uniformly for all ϵ in $0 < \epsilon < 1$. A positive lower bound of the correct order in ϵ is furnished in a simple way by the use of Lemma 3, which gives

(3.9) $$m((1-\epsilon)\pi) \geqslant \frac{\epsilon}{(2-\epsilon)^3}, \qquad 0 < \epsilon < 1.$$

Thus, for $r = (1-\epsilon)\pi/\lambda_1$, $\sigma_0 > \tau + (1-\epsilon)\pi/\lambda_1$, we have

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$$\begin{aligned}(3.10)\qquad \left|\frac{f(s_2)-f(s_1)}{s_2-s_1}\right| &\geqslant e^{-\lambda_q\sigma_0}\{\lambda_1 e^{(\lambda_q-\lambda_1)\sigma_0}\,.\,m(\lambda_1 r) - e^{\lambda_q(r+\tau)}\}\\ &\geqslant e^{-\lambda_q\sigma_0}\left\{\frac{\lambda_1\epsilon}{(2-\epsilon)^3}\,.\,e^{(\lambda_q-\lambda_1)\sigma_0} - e^{\lambda_q(\tau+(1-\epsilon)\pi/\lambda_1)}\right\}\\ &> 0,\end{aligned}$$

provided we choose σ_0 so that $\sigma_0 \geqslant \tau + (1-\epsilon)\pi/\lambda_1$, and

$$(3.11)\qquad e^{(\lambda_q-\lambda_1)\sigma_0} > \frac{(2-\epsilon)^3}{\lambda_1\epsilon}\cdot e^{\lambda_q(\tau+(1-\epsilon)\pi/\lambda_1)},$$

that is

$$(3.12)\qquad \sigma_0 > \frac{3\log(2-\epsilon) - \log(\lambda_1\epsilon) + \lambda_q(\tau+(1-\epsilon)\pi/\lambda_1)}{\lambda_q-\lambda_1}.$$

We observe that the number π/λ_1, appearing in the radius $(1-\epsilon)\pi/\lambda_1$ cannot be replaced by a larger one since the radius of univalency of the function

$$-e^{-\lambda_1 s},$$

which is the first term of the Dirichlet series (1.7), is exactly π/λ_1. We remark that for functions of the same class τ the value of α in (1.8) is independent of the function $f(s)$ once the sequence $\{\lambda_n\}$ has been selected. This completes the proof of Theorem 1.

It is by means of Theorem 1 that we are now able to establish the uniformly locally univalent property for all normalized Dirichlet series of the same class τ in a half-plane $\Re s > \beta$ where β has the value given in the following theorem.

THEOREM 2. *Let the class of functions* $\{f(s)\}$ *where*

$$f(s) = -e^{-\lambda_1 s} + \sum_{n=q}^{\infty} a_n e^{-\lambda_n s}, \quad a_q \neq 0,\ s = \sigma + it,$$

be of the same class τ. *Then the functions* $f(s)$ *are uniformly locally univalent in the half-plane* $\Re s > \beta$ where

$$\beta = \begin{cases} \tau, & \text{if } \tau < \dfrac{\log\lambda_1}{\lambda_1},\\[2ex] \dfrac{\lambda_q\tau - \log\lambda_1}{\lambda_q-\lambda_1} \geqslant \tau, & \text{if } \tau \geqslant \dfrac{\log\lambda_1}{\lambda_1}.\end{cases}$$

The functions $f(s)$ *of class* $(\log\lambda_1)/\lambda_1$ *are not uniformly locally univalent in* $\Re s > \tau - \eta$ *for arbitrarily small* $\eta > 0$, *and the functions* $f(s)$ *of class* $\tau > (\log\lambda_1)/\lambda_1$, *are not uniformly locally univalent in* $\Re s > \tau$ *while for an arbitrary* $\eta_1 > 0$, *the functions of a sub-class are uniformly locally univalent in* $\Re s > \tau + \eta_1$.

Before proving Theorem 2 we remark that Theorem 1 shows that the functions $f(s)$ of the same class τ are uniformly locally univalent in the half-plane $\Re s > \alpha - (1-\epsilon)\pi/\lambda_1$ at least. As $\epsilon \to 1$ we have

(3.13) $$\lim_{\epsilon\to 1} \alpha = \max\left(\tau, \frac{\lambda_q\tau - \log\lambda_1}{\lambda_q - \lambda_1}\right).$$

(3.14) $$\lim_{\epsilon\to 1} \alpha = \tau \quad \text{if } \tau \leqslant \frac{\log\lambda_1}{\lambda_1}.$$

(3.15) $$\lim_{\epsilon\to 1} \alpha = \frac{\lambda_q\tau - \log\lambda_1}{\lambda_q - \lambda_1} \quad \text{if } \tau \geqslant \frac{\log\lambda_1}{\lambda_1}.$$

In (1.8) we have $\alpha = \tau + (1 - \epsilon)\pi/\lambda_1$ provided

(3.16) $$\tau \leqslant \frac{-1}{\lambda_1}\log\frac{(2-\epsilon)^3}{\lambda_1\epsilon} - \frac{(1-\epsilon)\pi}{\lambda_1}.$$

Suppose now that $\tau < (\log\lambda_1)/\lambda_1$. Then (3.16) is true for a range of ϵ,

$$0 < 1 - \frac{\delta\lambda_1}{\pi} \leqslant \epsilon < 1,$$

since

(3.17) $$\frac{\log\lambda_1}{\lambda_1} \leqslant \frac{-1}{\lambda_1}\log\frac{(2-\epsilon)^3}{\lambda_1\epsilon} - \frac{(1-\epsilon)\pi}{\lambda_1}$$

for $\epsilon = 1$, but for no value of ϵ in the range $0 < \epsilon < 1$. For $\sigma_0 > \tau + \delta$, $f(s)$ is univalent in $|s - s_0| \leqslant \delta$ if $\tau < (\log\lambda_1)/\lambda_1$ since (3.16) is verified. Thus the functions $f(s)$ are uniformly locally univalent in $\Re s > \tau + \delta$ for arbitrarily small $\delta > 0$. It follows that the functions $f(s)$ are uniformly locally univalent in $\Re s > \tau$ whenever $\tau < (\log\lambda_1)/\lambda_1$.

Again, if $\tau \geqslant (\log\lambda_1)/\lambda_1$ we have

(3.18) $$\alpha = [3\log(2-\epsilon) - \log(\lambda_1\epsilon) + \lambda_q\tau + (1-\epsilon)\pi\lambda_q/\lambda_1]/(\lambda_q - \lambda_1)$$

provided

(3.19) $$\tau \geqslant -[3\log(2-\epsilon) - \log(\lambda_1\epsilon) + (1-\epsilon)\pi]/\lambda_1$$

It is readily seen that (3.19) is verified for all ϵ, $0 < \epsilon < 1$, when $\tau \geqslant (\log\lambda_1)/\lambda_1$. Then for

(3.20) $$\sigma_0 > \frac{\lambda_q\tau - \log\lambda_1}{\lambda_q - \lambda_1} + \delta, \qquad \delta > 0,$$

$f(s)$ is univalent in every circle $|s - s_0| \leqslant (1-\epsilon)\pi/\lambda_1$ provided

(3.21) $$\frac{\lambda_q\tau - \log\lambda_1}{\lambda_q - \lambda_1} + \delta > \alpha$$

where α is given by (3.18). But since when $\epsilon = 1$, α has the value given by $\lim \alpha$ in (3.15) we see that for each given $\delta > 0$ there exists a range of values of ϵ, $0 < 1 - \delta_1 \leqslant \epsilon < 1$ for which (3.21) is verified. Since δ may be taken arbitrarily small it follows that the functions $f(s)$ are uniformly locally univalent for

(3.22) $$\Re s > \frac{\lambda_q\tau - \log\lambda_1}{\lambda_q - \lambda_1} \text{ if } \tau \geqslant \frac{\log\lambda_1}{\lambda_1}.$$

If $\tau \geqslant (\log \lambda_1)/\lambda_1$, there exist functions $f(s)$ which are not locally univalent in $\Re s > \tau$ although they are locally univalent in $\Re s > \tau + \eta_1$ for a given $\eta_1 > 0$. For example, let $f(s)$, defined as

$$(3.23) \qquad f(s) = -e^{-\lambda_1 s} + \sum_{n=q}^{\infty} a_n e^{-\lambda_n s}, \qquad a_n > 0 \text{ for } n \geqslant q,$$

be of class $\tau \geqslant (\log \lambda_1)\lambda_1$ and choose q sufficiently large so that

$$(3.24) \qquad \frac{\lambda_q \tau - \log \lambda_1}{\lambda_q - \lambda_1} < \tau + \eta_1, \qquad \eta_1 > 0.$$

The size of the coefficients a_n determine the value of τ,

$$(3.25) \qquad f'(\tau) = \lambda_1 e^{-\lambda_1 \tau} - \sum_{n=q}^{\infty} \lambda_n a_n e^{-\lambda_n \tau} = \lambda_1 e^{-\lambda_1 \tau} - 1.$$

If $\tau = (\log \lambda_1)/\lambda_1, f'(\tau) = 0$. If $\tau > (\log \lambda_1)/\lambda_1, f'(\tau) < 0$, whereas $f'(\sigma) > 0$ for large values of σ, since

$$\lim_{\sigma \to +\infty} e^{\lambda_1 \sigma} f'(\sigma) = \lambda_1 > 0.$$

Thus $f'(\sigma)$, being continuous, must vanish for at least one value $\sigma = \sigma_1 > \tau$. But $\sigma_1 < \tau + \eta_1$ since $f(s)$ is locally univalent at least for $\Re s > \tau + \eta_1$, and $f'(\sigma)$ can not vanish for $\sigma > \tau + \eta_1$. It follows that $f(s)$ is not schlicht in the neighbourhood of σ_1. Thus, if τ exceeds $(\log \lambda_1)/\lambda_1$, $f(s)$ need not be locally univalent in $\Re s > \tau$ even though it is for $\Re s > \tau + \eta_1$. It is also seen that if $\tau = (\log \lambda_1)/\lambda_1$, $f(s)$ need not be locally univalent in $\Re s > \tau - \eta$, $\eta > 0$. This completes the proof of Theorem 2.

We shall now make an application of Theorem 2 to the Riemann zeta-function $\zeta(s)$.

$$(3.26) \qquad 1 - \zeta(s) = -\sum_{n=1}^{\infty} e^{-s \log(n+2)} = -\sum_{n=1}^{\infty} (n+1)^{-s}.$$

Now $1 - \zeta(s)$ is of class τ where

$$(3.27) \qquad \sum_{n=2}^{\infty} \frac{\log (n+1)}{(n+1)^{\tau}} = 1, \quad \zeta'(\tau) + 2^{-\tau} \log 2 + 1 = 0.$$

Since

$$\frac{\log \lambda_1}{\lambda_1} = \frac{\log \log 2}{\log 2} < 0$$

and $\tau > 1$, we see that $\zeta(s)$ is locally univalent for

$$(3.28) \quad \Re s > \frac{\tau \lambda_2 - \log \lambda_1}{\lambda_2 - \lambda_1} = \frac{\tau \log 3 - \log \log 2}{\log 3 - \log 2} = 2.70749\, \tau + 0.90428,$$

where τ is the solution of the equation (3.27). Since

$$(3.29) \qquad \sum_{n=2}^{\infty} \frac{\log (n+1)}{(n+1)^{\tau}} < \int_2^{\infty} \frac{\log x}{x^{\tau}} dx = \frac{(\tau - 1) \log 2 + 1}{(\tau - 1)^2\, 2^{\tau - 1}} = 1$$

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for a value $\tau = \tau_0$ in the range $1.9 < \tau_0 < 2.0$ it follows that $1 - \zeta(s)$ is of class $\tau < 2$. Also, since

$$\sum_{n=2}^{\infty} \frac{\log (n+1)}{(n+1)^{\tau}} > \frac{\log 3}{3^{\tau}} + \int_3^{\infty} \frac{\log x}{x^{\tau}} dx = \frac{\log 3}{3^{\tau}} + \frac{(\tau - 1) \log 4 + 1}{(\tau - 1)^2 4^{\tau - 1}} = 1 \tag{3.30}$$

for a value of $\tau = \tau_1$ in the range $1.9 < \tau_1 < 2.0$ it follows that $1 - \zeta(s)$ is of class $\tau > 1.9$. Hence, the class of $1 - \zeta(s)$ lies in the range $1.9 < \tau < 2.0$. We conclude that $\zeta(s)$ is locally schlicht in a half-plane $\Re s > c$ where $c < 6.32$.

4. Proof of Theorem 3. Univalency in strips. Instead of examining $f(s)$, given by (1.5) and of class τ, for univalency in circles $|s - s_0| \leqslant r$, we shall turn now to a similar task for strips. Let D_k denote the strip of the $s = \sigma + it$ plane defined by $\sigma \geqslant \tau$, where $\tau < \tau_0$ in the notation of (1.10), and $-t_0 \leqslant t - 2k\pi/\lambda_1 \leqslant t_0$, where k is an arbitrary integer and

$$t_0 = \frac{1}{\lambda_1} \arc\cos\left(\frac{e^{\lambda_1 \tau}}{\lambda_1}\right), \qquad 0 < t_0 \leqslant \pi/2\lambda_1. \tag{4.1}$$

Let C_k denote the boundary of D_k and consist of the three line segments α_k, β_k, γ_k defined as follows.

α_k: that part of C_k which lies on $t = t_0 + 2k\pi/\lambda_1$,

β_k: that part of C_k which lies on $\sigma = \tau$,

γ_k: that part of C_k which lies on $t = -t_0 + 2k\pi/\lambda_1$.

Let D^*_k denote the rectangular sub-domain of D_k whose boundary C^*_k consists of the parts of the two line segments α_k and γ_k for which $\tau \leqslant \sigma \leqslant \tau^*$, together with β_k and δ_k, where δ_k denotes the line segment $\sigma = \tau^* > \tau$, $-t_0 + 2k\pi/\lambda_1 \leqslant t \leqslant t_0 + 2k\pi/\lambda_1$.

We shall show that $f(s)$ is univalent in the domains D_k and that $w = f(s)$ maps C_k onto a simple, closed Jordan curve Γ_k which is convex in the direction of the real axis, which is to say that the region bounded by Γ_k is star-shaped with respect to the point at infinity at an end of the real axis. Since $\lim_{\sigma \to +\infty} f(\sigma) = 0$, it follows that the only zero $f(s)$ has in D_k corresponds to the point of D_k at infinity. Thus Γ_k passes through the origin in the w-plane. If w_1 and w_2 are any two distinct points of the image of D_k by $w = f(s)$ for which $\Im w_1 = \Im w_2$, it will follow that the line segment joining w_2 and w_1 lies entirely within the region encompassed by Γ_k. If w_1 and w_2 are any two points interior to Γ_k, they must also lie interior to Γ^*_k, the image of C^*_k, if τ^* is taken sufficiently large. Thus it is sufficient to prove that the region bounded by Γ^*_k is convex in the direction of the real axis for every $\tau^* > \tau$.

On β_k we have $\sigma = \tau$ and

$$f(\tau + it) = -e^{-\lambda_1(\tau+it)} + \sum_{n=2}^{\infty} a_n e^{-\lambda_n(\tau+it)}. \tag{4.2}$$

Because $f(s)$ is of class τ, $f(s)$ and $f'(s)$ are continuous on $\sigma = \tau \geqslant \bar{\sigma}$, and we have

$$\Im f(\tau + it) = e^{-\lambda_1\tau} \sin (\lambda_1 t) - \sum_{n=2}^{\infty} \{\alpha_n \sin (\lambda_n t) - \beta_n \cos (\lambda_n t)\} e^{-\lambda_n\tau} \tag{4.3}$$

where $a_n = \alpha_n + i\beta_n$, α_n and β_n real, and

$$\frac{\partial}{\partial t} \Im f(\tau + it) = \lambda_1 e^{-\lambda_1\tau} \cos (\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n \{\alpha_n \cos (\lambda_n t) + \beta_n \sin (\lambda_n t)\} e^{-\lambda_n\tau}. \tag{4.4}$$

Since

$$|\, \alpha_n \cos \theta + \beta_n \sin \theta \,| \leqslant (\alpha_n^2 + \beta_n^2)^{\frac{1}{2}} = |a_n| \tag{4.5}$$

for all θ we have

$$\begin{aligned} \frac{\partial}{\partial t} \Im f(\tau + it) &\geqslant \lambda_1 e^{-\lambda_1 t} \cos (\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n\tau} \\ &\geqslant \lambda_1 e^{-\lambda_1 t} \cos (\lambda_1 t) - 1 \geqslant 0 \end{aligned} \tag{4.6}$$

for

$$-t_0 \leqslant t - \frac{2k\pi}{\lambda_1} \leqslant t_0,\ 0 < t_0 = \frac{1}{\lambda_1} \operatorname{arc\,cos}\left(\frac{e^{\lambda_1\tau}}{\lambda_1}\right) \leqslant \frac{\pi}{2\lambda_1}.$$

Thus, $\Im f(s)$ is a monotonically increasing function of t on β_k.

A similar proof holds on δ_k where $\sigma = \tau^* > \tau$ with a slight modification. Here we have

$$\begin{aligned} \frac{\partial}{\partial t} \Im f(\tau^* + it) &\geqslant \lambda_1 e^{-\lambda_1\tau^*} \cos (\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n\tau^*} \\ &= e^{-\lambda_1\tau^*}\left\{\lambda_1 \cos (\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-(\lambda_n-\lambda_1)\tau^*}\right\} \\ &\geqslant e^{-\lambda_1\tau^*}\left\{\lambda_1 \cos (\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-(\lambda_n-\lambda_1)\tau}\right\} \\ &\geqslant e^{-\lambda_1\tau^*}\{\lambda_1 \cos (\lambda_1 t) - e^{\lambda_1\tau}\} \geqslant 0 \end{aligned} \tag{4.7}$$

for $|t - 2k\pi/\lambda_1| \leqslant t_0$. Thus $\Im f(s)$ is a monotonically increasing function of t on δ_k.

On α_k we have $t = t_k = t_0 + 2k\pi/\lambda_1$, $\tau \leqslant \sigma \leqslant \tau^*$, and

$$(4.8) \quad \Im f(\sigma + it_k) = e^{-\lambda_1\sigma}\sin(\lambda_1 t_0) - \sum_{n=2}^{\infty} \{\alpha_n \sin(\lambda_n t_k) - \beta_n \cos(\lambda_n t_k)\} e^{-\lambda_n \sigma},$$

$$\frac{\partial \Im}{\partial \sigma} f(\sigma + it_k) = -\lambda_1 e^{-\lambda_1 \sigma}\sin(\lambda_1 t_0)$$

$$+ \sum_{n=2}^{\infty} \lambda_n \{\alpha_n \sin(\lambda_n t_k) - \beta_n \cos(\lambda_n t_k)\} e^{-\lambda_n \sigma}$$

$$\leqslant -\lambda_1 e^{-\lambda_1\sigma}\sin(\lambda_1 t_0) + \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \sigma}$$

$$(4.9) \quad \leqslant e^{-\lambda_1\sigma}\left\{-\lambda_1 \sin(\lambda_1 t_0) + \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-(\lambda_n - \lambda_1)\sigma}\right\}$$

$$\leqslant e^{-\lambda_1\sigma}\left\{-\lambda_1 \sin(\lambda_1 t_0) + \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-(\lambda_n - \lambda_1)\tau}\right\}$$

$$\leqslant e^{-\lambda_1\sigma}\{-\lambda_1 \sin(\lambda_1 t_0) + e^{\lambda_1 \tau}\}$$

$$= e^{-\lambda_1\sigma}\{-(\lambda_1^2 - e^{2\lambda_1\tau})^{\frac{1}{2}} + e^{\lambda_1\tau}\} \leqslant 0,$$

provided, in the notation of (1.10),

$$(4.10) \qquad \tau \leqslant \tau_0, \qquad \sigma \geqslant \tau.$$

Thus, $\Im f(s)$ is a monotonically decreasing function of σ on α_k.

A similar argument shows that $\Im f(s)$ is a monotonically increasing function of σ on γ_k (t_0 is replaced by $-t_0$).

Combining the above results we have shown that, as three sides of the rectangle C^*_k are traversed in the counter-clockwise direction beginning at the point of intersection of β_k and γ_k and ending at the point of intersection of α_k and β_k, the corresponding arc of the curve Γ^*_k has the property that every horizontal straight line (parallel to the real axis) cuts it in at most one point since $\Im f(s)$ is non-decreasing. Similarly, the image of β_k also has the property that every horizontal line cuts it in at most one point. Thus, the region bounded by Γ^*_k is convex in the direction of the real axis for every $\tau^* > \tau$. Since Γ^*_k has therefore no double points $f(s)$ must be univalent in D^*_k, and consequently univalent in D_k as well.

We next see that there exist functions $f(s)$ and certain sequences $\{\lambda_n\}$ for which the theorem is not true if the strip D_k is enlarged by keeping the sides parallel to the axes of reference. Let $\epsilon > 0$ be chosen arbitrarily. Choose λ_2 so that

$$e^{(\lambda_2 - \lambda_1)\epsilon} > 2^{\frac{1}{2}}, \qquad \lambda_2 > \lambda_1.$$

Choose the coefficients a_n of (1.9) positive and so that $f(s)$ is of class $\tau = \tau_0$. Then for $\sigma = \tau - \epsilon$, $t = 0$ we have

$$\frac{\partial}{\partial t}\Im f(\tau-\epsilon+it) = \lambda_1 e^{-\lambda_1(\tau-\epsilon)} - \sum_{n=2}^{\infty}\lambda_n a_n e^{-\lambda_n(\tau-\epsilon)} \tag{4.11}$$
$$< \lambda_1 e^{-\lambda_1\tau}.\, e^{\lambda_1\epsilon} - e^{\lambda_2\epsilon}$$
$$= 2^{\frac{1}{2}}\, e^{\lambda_1\epsilon} - e^{\lambda_2\epsilon} < 0.$$

Thus $\Im f(s)$ in this case is not steadily increasing as t increases on $\tau = \tau_0 - \epsilon$. This shows that we cannot enlarge the strip D_0 horizontally and have Theorem 3 valid for all functions $f(s)$ of the class considered.

Next we shall show that the strip may not be enlarged vertically. Choose $\lambda_1 > 0$ and $\epsilon > 0$ arbitrarily, and, for $n \geqslant 2$, choose $\lambda_n = (4n+1)\pi/(2t_0+2\epsilon)$ where t_0 is defined as in (1.11) and where

$$\tau < \tau_0.$$

We choose the coefficients a_n of (1.9) so that for $n \geqslant 2$, $\alpha_n = \Re a_n = 0$, $\beta_n = \Im\, a_n > 0$, with a proper choice of magnitude of β_n so that $f(s)$ is of the given class τ. Then for $t = t_0 + \epsilon$, $\epsilon > 0$, ϵ small, and $\sigma = \tau$,

$$\frac{\partial}{\partial t}\Im f(\tau+it)\Big]_{t=t_0+\epsilon} = \lambda_1 e^{-\lambda_1\tau}\cos\lambda_1(t_0+\epsilon) - \sum_{n=2}^{\infty}\lambda_n\beta_n e^{-\lambda_n\tau} \tag{4.12}$$
$$\leqslant \lambda_1 e^{-\lambda_1\tau}\cos\lambda_1(t_0+\epsilon) - 1$$
$$< \lambda_1 e^{-\lambda_1\tau}\cos(\lambda_1 t_0) - 1$$
$$= 0.$$

It follows that $\Im f(\tau+it)$ is not monotonically increasing for $|t| < t_0+\epsilon$, $\epsilon > 0$, for this function of class τ. Therefore the strip D_0 cannot be enlarged vertically for all functions considered in Theorem 3.

It is also seen that no larger value of τ than the one given by (1.10) is permissible. Theorem 3 is, therefore, a "best possible" one. This completes the proof of the first part of Theorem 3 where τ is restricted as in (1.10).

On the other hand, we may increase the range of τ slightly if mere univalency is demanded in the strips D_k. We assume that the inequality $\tau \leqslant \tau_0$ where τ_0 is defined in (1.10) is replaced by $\tau \leqslant \tau^*_0 = (\log\lambda_1)/\lambda_1$ and shall show that $f(s)$ is still univalent in D_k. We make use of Noshiro's Theorem **(4)** and show first that $\Re f'(s) > 0$ in D_k. Since

$$\Re f'(s) = \lambda_1 e^{-\lambda_1\sigma}\cos(\lambda_1 t) - \sum_{n=2}^{\infty}\lambda_n\{\alpha_n\cos(\lambda_n t) - \beta_n\sin(\lambda_n t)\}e^{-\lambda_n\sigma}, \tag{4.13}$$

$$\Re f'(s) \geqslant \lambda_1 e^{-\lambda_1\sigma}\cos(\lambda_1 t) - \sum_{n=2}^{\infty}\lambda_n|a_n|e^{-\lambda_n\sigma}$$
$$\geqslant e^{-\lambda_1\sigma}\left\{\lambda_1\cos(\lambda_1 t) - \sum_{n=2}^{\infty}\lambda_n|a_n|e^{-(\lambda_n-\lambda_1)\sigma}\right\} \tag{4.14}$$
$$\geqslant e^{-\lambda_1\sigma}\{\lambda_1\cos(\lambda_1 t) - e^{\lambda_1\tau}\} > 0$$

if s is in D_k and $\tau < \tau^*_0$.

Since we have shown that $\Re f'(s) > 0$ in D_k and since D_k is convex it follows at once by Noshiro's Theorem that $f(s)$ is univalent in D_k.

No larger value than τ^*_0 for τ is permissible in general. Indeed if $a_n > 0$ for $n \geqslant 2$, and $\tau > \tau^*_0$ we have

$$f'(\tau) = e^{-\lambda_1 \tau}(\lambda_1 - e^{\lambda_1 \tau}) < 0 \tag{4.15}$$

if

$$\sum_{n=2}^{\infty} \lambda_n a_n e^{-\lambda_n \tau} = 1. \tag{4.16}$$

But

$$\begin{aligned} f'(\sigma) &= \lambda_1 e^{-\lambda_1 \sigma} - \sum_{n=2}^{\infty} \lambda_n a_n e^{-\lambda_n \sigma} \\ &> 0 \end{aligned} \tag{4.17}$$

for σ sufficiently large. Thus $f'(s)$ vanishes in the strip D_0 in this case. In this case $f(s)$ is not univalent.

It should be noticed also that if $\tau < \tau^*_0$ then

$$\frac{\partial}{\partial \sigma} \Re f(\sigma + it) \geqslant 0$$

on $t =$ constant, $|t| \leqslant t_0$, $\sigma \geqslant \tau$. For

$$\begin{aligned} \frac{\partial}{\partial \sigma} \Re f(\sigma + it) &= \lambda_1 e^{-\lambda_1 \sigma} \cos(\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n \{\alpha_n \cos(\lambda_n t) - \beta_n \sin(\lambda_n t)\} e^{-\lambda_n \sigma} \\ &\geqslant \lambda_1 e^{-\lambda_1 \sigma} \cos(\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \sigma} \\ &= e^{-\lambda_1 \sigma} \left\{ \lambda_1 \cos(\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-(\lambda_n - \lambda_1)\sigma} \right\} \\ &\geqslant e^{-\lambda_1 \sigma} \left\{ \lambda_1 \cos(\lambda_1 t) - \sum_{n=2}^{\infty} \lambda_n |a_n| e^{-(\lambda_n - \lambda_1)\tau} \right\} \\ &\geqslant e^{-\lambda_1 \sigma} \{\lambda_1 \cos(\lambda_1 t) - e^{\lambda_1 \tau}\} \\ &\geqslant 0 \end{aligned} \tag{4.18}$$

for $|t| \leqslant t_0$, $\tau < \tau^*_0$. Thus, if $\tau < \tau^*_0$, D_k is mapped into Δ_k by $w = f(s)$, and part of the boundary of Δ_k is convex in the direction of the imaginary axis while the remaining part of the boundary is convex in the direction of the real axis. These parts correspond to sides of D_k regardless of which function $f(s)$ of class τ is used.

We have completed the proof of Theorem 3, and the following corollary is a consequence of the preceding remarks.

COROLLARY 1. *If* $\{f_n(s)\}$ *is a sequence of functions defined by Dirichlet series*

$$f_n(s) = -e^{-\lambda_1 s} + \sum_{m=2}^{\infty} a_m^{(n)} e^{-\lambda_m^{(n)} s}, \tag{4.19}$$

relative to the sequence $\{\lambda_m^{(n)}\}$,

$$(4.20) \qquad 0 < \lambda_1 < \lambda_2^{(n)} < \lambda_3^{(n)} < \ldots < \lambda_m^{(n)} < \ldots, \qquad \lambda_m^{(n)} \to \infty,$$

and if each $f_n(s)$ *is of the same class* τ, *so that*

$$(4.21) \qquad \sum_{m=2}^{\infty} \lambda_m^{(n)} |a_m^{(n)}| e^{-\lambda_m^{(n)} s} \leqslant 1, \quad \tau < \frac{\log \lambda_1}{\lambda_1},$$

then, for each sequence $\{A_n\}$ *of positive real numbers for which*

$$(4.22) \qquad \phi(s) = \sum_{n=1}^{\infty} A_n f_n(s)$$

converges uniformly to $\phi(s)$ *in* $\Re s > \alpha$, $\alpha < \tau$, *we have* $\phi(s)$ *analytic in* $\Re s > \alpha$, *and* $\phi(s)$ *is univalent in each strip* D_k *of Theorem* 3.

COROLLARY 2. *Let*

$$(4.23) \qquad f(s) = -e^{-\lambda_1 s} + \sum_{n=2}^{\infty} a_n e^{-\lambda_n s}, \qquad s = \sigma + it,$$

be absolutely convergent for $\sigma > \bar{\sigma}$, $-\infty \leqslant \bar{\sigma} < \infty$, *and let* $f(s)$ *be of class* $\tau < (\log \lambda_1)/\lambda_1$. *Let*

$$(4.24) \qquad t_0 = \frac{1}{\lambda_1} \operatorname{arc\,cos}\left(\frac{e^{\lambda_1 \tau}}{\lambda_1}\right), \quad 0 < t_0 < \frac{\pi}{2\lambda_1}.$$

Then $f(s)$ *is univalent in every semi-infinite strip* D *of width* $2t_0$ *which is parallel to the real axis and lies in the half-plane* $\Re s \geqslant \tau$.

In order to see that Corollary 2 follows from Theorem 3 we observe that if t' is an arbitrary real number the function

$$(4.25) \quad F(s) = e^{i\lambda_1 t'}.f(s + it') = -e^{-\lambda_1 s} + \sum_{n=2}^{\infty} a_n e^{-i(\lambda_n - \lambda_1)t'} \cdot e^{-\lambda_n s}$$

is a Dirichlet series of the same class τ as the class of $f(s)$. Applying Theorem 3 to $F(s)$ we find that $F(s)$ is univalent in each strip D_k of width $2t_0$. Hence $f(s)$ is univalent in a strip obtained by a translation vertically of the strip D_k by the arbitrary value t'.

If $\tau \leqslant \tau_0$ as in (1.10), we can conclude further that $f(s)$ is convex in some one direction in each strip of width $2t_0$, $\sigma \geqslant \tau$, parallel to the real axis. The direction of convexity varies with the position of each strip in general. As in Theorem 3 Corollary 2 is sharp.

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