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RESEARCH ARTICLE

Characteristic numbers of manifold bundles over spheres and positive curvature via block bundles

with an Appendix by Georg Frenck and Jens Reinhold

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Abstract

Given a simply connected manifold M, we completely determine which rational monomial Pontryagin numbers are attained by fiber homotopy trivial M-bundles over the k-sphere, provided that k is small compared to the dimension and the connectivity of M. Furthermore, we study the vector space of rational cobordism classes represented by such bundles. We give upper and lower bounds on its dimension, and we construct manifolds for which the lower bound is attained. Our proofs are based on the classical approach to studying diffeomorphism groups via block bundles and surgery theory, and we make use of ideas developed by Krannich–Kupers–Randal-Williams.

As an application, we show the existence of elements of infinite order in the homotopy groups of the spaces of positive Ricci and positive sectional curvature, provided that M is Spin, has a nontrivial rational Pontryagin class and admits such a metric. This is done by constructing M-bundles over spheres with nonvanishing $\hat{\mathcal{A}}$-genus. Furthermore, we give a vanishing theorem for generalized Morita–Miller–Mumford classes for fiber homotopy trivial bundles over spheres.

In the appendix coauthored by Jens Reinhold, we investigate which classes of the rational oriented cobordism ring contain an element that fibers over a sphere of a given dimension.

Contents

1. Introduction

Let M be a closed oriented manifold of dimension $d \geq 5$. In this article, we investigate the following question: Given an integer $k \geq 1$ and a universal characteristic class $c \in H^{d+k}(\mathrm{BO};\mathbb{Q})$,[1] does there exist a fiber bundle $M \to E \to S^k$ such that $\langle c(TE), [E]\rangle \neq 0$? If it does, then c is called *spherical for* M. Furthermore, c is called *h-spherical for M* if E can be chosen to be fiber homotopy trivial; that is, E comes equipped with a homotopy equivalence $E \simeq M \times S^k$ over S^k. Obviously, h-spherical classes are spherical. The following is our main result.

Theorem A. *Let M^d be a simply connected, closed manifold and let k be such that $1 \leq k \leq \min(\frac{d-1}{3}, \frac{d-5}{2})$ and $d + k = 4m$.*

(i) *A monomial $p = p_{i_1} \cup \cdots \cup p_{i_n} \neq p_m$ in universal rational Pontryagin classes of total degree $d + k$ is h-spherical for M if and only if there exists an $\ell \in \{1, \ldots, n\}$ such that*

$$p_{i_1}(TM) \cup \cdots \cup \widehat{p_{i_\ell}(TM)} \cup \cdots \cup p_{i_n}(TM) \neq 0.$$

(ii) *Let M be such that $p_i(TM) \neq 0$ for some $i \geq 1$ and let $p_i(TM)$ have the lowest degree among these. Then there exists a fiber homotopy trivial bundle $E \to S^k$ such that*

$$\langle p_i(TE) \cup p_{m-i}(TE), [E]\rangle \neq 0 \neq \langle p_m(TE), [E]\rangle,$$

and these are the only nonzero monomial Pontryagin numbers of E. In particular, the following are equivalent:
   (a) *The class p_m is h-spherical for M.*
   (b) *The class p_m is spherical for M.*
   (c) *M admits some nontrivial rational Pontryagin class.*

(iii) *If $\sum i_j = m \geq 3$ and $i_j < m/2$ for all j, then the monomial $p_{i_1} \cup \cdots \cup p_{i_n}$ is spherical but not h-spherical for $\mathbb{CP}^m$. In particular, the class p_1^m is spherical but not h-spherical for $\mathbb{CP}^m$, provided $m \geq 3$.*

We remark that (iii) of the above theorem follows from Proposition A.8 and Proposition A.10. These propositions go back to a joint project with Jens Reinhold, which now forms the jointly written appendix to this article.

Remark 1.1.

(i) It is known that no characteristic class $c \in H^{d+k}(\mathrm{BO};\mathbb{Q})$ is spherical for any M if $k > 2d$ (cf. [Wie21, Lemma 2.3]). This implies the necessity for a bound on k, even though the one we give in Theorem A might not be optimal. This bound can be improved depending on the connectivity of M: Let M be a d-dimensional, ℓ-connected manifold with $d \geq 5$ and $\ell \geq 1$. We say that $k \geq 1$ is in the *unblocking range for M* if one of the following is satisfied:
   (a) $k \leq \min(\frac{d-1}{3}, \frac{d-5}{2})$
   (b) d is even and $k \leq \min(d-1, 2\ell-1)$.
   (c) d is odd, $(k-1)$ is not divisible by 4 and $k \leq \min(d-3, 2\ell-1)$.
   Theorem A holds for all k in the unblocking range for M by Lemma 2.5. Note that any of the above conditions implicitly enforces $d \geq 5$ if we want $k \geq 1$.

[1] Since $H^*(\mathrm{BO};\mathbb{Q})$ is concentrated in degrees divisible by 4, we restrict to the case $d + k = 4m$ throughout this article.

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(ii) If all rational Pontryagin classes of M vanish, then again no (rational) characteristic class is spherical by [HSS14, Proposition 1.9].[2] In particular, this proves $(b) \Rightarrow (c)$ in Theorem A, (ii): If p_m is spherical, then some rational Pontryagin class of M must be nonzero. Note that $(a) \Rightarrow (b)$ is trivial and $(c) \Rightarrow (a)$ follows from the first half of Theorem A(ii).

Next, let Ω_* denote the oriented bordism ring and let $\mathrm{Fib}^h_{M,k} \subset \Omega_{d+k} \otimes \mathbb{Q}$ denote the set of classes represented by fiber homotopy trivial M-bundles over S^k. Note that $\mathrm{Fib}^h_{M,k}$ is actually a linear subspace since it is given by the image of the homomorphism

$$\pi_k\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right) \otimes \mathbb{Q} \longrightarrow \Omega_{d+k} \otimes \mathbb{Q},$$

where $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ denotes the classifying space for fiber homotopy trivial M-bundles, and the above map is given by sending a pointed map $S^k \to \mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ to the bordism class of the total space of the bundle classified by it. We will now give estimates for the dimension of $\mathrm{Fib}^h_{M,k}$. For this, let $i_{\min}$ be the minimum positive integer i such that $p_i(TM) \neq 0$ and let $n_{\max}$ be the maximum integer $n \geq 1$ such that $p_{i_{\min}}(TM)^n \neq 0$.

Theorem B. *Let M be simply connected and let $k \geq 1$ be in the unblocking range for M such that $d + k = 4m$. Then, for every $1 \leq n \leq n_{\max}$, there exists a fiber homotopy trivial M-bundle $E_n \to S^k$ with the property that for $\ell \geq 1$, we have*

$$\left\langle p_{i_{\min}}(TE_n)^{\ell} \cup p_{m-\ell\cdot i_{\min}}(TE_n), [E_n]\right\rangle \neq 0 \qquad \Longleftrightarrow \qquad n = \ell.$$

Since $\Omega_* \otimes \mathbb{Q}$ is classified by Pontryagin-numbers, we get a lower bound on $\dim \mathrm{Fib}^h_{M,k}$ which we prove to be attained for certain manifolds.

Corollary C. *Let M and k be as in Theorem B.*

(i) *We have* $\dim \mathrm{Fib}^h_{M,k} \geq n_{\max}$.
(ii) *If all Pontryagin classes of M are contained in the truncated polynomial $\mathbb{Q}$-algebra generated by $p_{i_{min}}(TM)$, then* $\dim \mathrm{Fib}^h_{M,k} = n_{\max}$.

Example 1.2. The prototypical examples of manifolds for which this lower bound from Corollary C is attained are $\mathbb{CP}^a$, $\mathbb{HP}^b$ and $\mathbb{OP}^2$ for $a \geq 3$ and $b \geq 2$. If $k_a \equiv 2a \bmod 4$ and $k_a \leq \min(\frac{2a-1}{3}, a - \frac{5}{2})$, then

$$\dim \mathrm{Fib}^h_{\mathbb{CP}^a,k_a} = n_{\max}(\mathbb{CP}^a) = \left\lfloor \frac{a}{2} \right\rfloor.$$

Analogously, for k_b, k_c divisible by 4 and $k_b \leq \min(\frac{4b-1}{3}, 2b - \frac{5}{2})$ (or $k_b \leq 4$) and $k_c \leq 12$, we obtain

$$\dim \mathrm{Fib}^h_{\mathbb{HP}^b,k_b} = b, \qquad \dim \mathrm{Fib}^h_{\mathbb{OP}^2,k_c} = 2.$$

We note that $\dim \mathrm{Fib}^h_{\mathbb{HP}^2,4} = 2$ was already observed in [KKR21, Remark 1].

In order to describe the upper bound on $\dim \mathrm{Fib}^h_{M,k}$, recall that

$$H^*(\mathrm{BO}(d); \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_{\lfloor \frac{d}{2} \rfloor}].$$

Let $p(n)$ be the number of partitions of $n \in \mathbb{N}$ into sums of positive natural numbers and let us fix $m := \frac{d+k}{4}$. The assumption of k being in the unblocking range guarantees that $k \leq d - 2$ which implies $4m = k + d \leq 2d - 2 \leq \deg(p_{\lfloor \frac{d}{2} \rfloor})$, and hence, we have $\dim H^{4m}(\mathrm{BO}(d); \mathbb{Q}) = p(m)$. Furthermore,

[2]Both [Wie21, Lemma 2.3] and [HSS14, Proposition 1.9] are only stated for the $\hat{\mathcal{A}}$-class, but the given proofs apply to any $c \in H^{d+k}(\mathrm{BO}; \mathbb{Q})$.

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for $\ell \in \mathbb{N}$, we define $p(n,\ell)$ to be the number of partitions of n into natural numbers $\leq \ell$. Note that $p(n,n) = p(n)$, $p(n,0) = 0$, $p(n,1) = 1$ and $p(n,2) = 1 + \lfloor n/2 \rfloor$. Furthermore, $p(n,\ell) = \mathcal{O}(n^{\ell-1})$.

We have the following observation concerning an upper bound on $\dim \mathrm{Fib}^h_{M,k}$: If $i_1,\dots,i_r$ is such that $\sum i_j = 4m$ and $i_j < k/4$ for all j, then we have $\langle p_{i_1}(TE)\cdots p_{i_r}(TE), [E]\rangle = 0$ for every fiber homotopy trivial M-bundle $E \to S^k$ by the following argument: By our assumption on (i_j), the degree of $p_{i_1}(TM)\cdots \widehat{p_{i_\ell}(TM)}\cdots p_{i_r}(TM)$ equals $4m - 4i_\ell > d$, and hence, this class vanishes since the corresponding cohomology group of M vanishes. The claim follows from Theorem A, (i). We get the following upper bound, which is a consequence of this observation together with the fact that the signature of a fiber bundle over a sphere vanishes.

Theorem D. *Let M be simply connected and let $k \geq 1$ be in the unblocking range for M. Then for $4m = d + k$, we have $\dim \mathrm{Fib}^h_{M,k} \leq p(m) - p(m, m - \lceil \frac{d+1}{4} \rceil) - 1$. There exists a simply connected manifold M in dimensions $d \equiv 2,3$ (4) for which equality holds.*

The upper bound is an immediate consequence of the above observation together with the fact that $\sigma(E) = 0$. The main difficulty of Theorem D lies in proving sharpness. In order to do so, we construct a manifold M, and for every $I = (i_1,\dots,i_s)$ with $s \geq 2$, $\sum i_j = m$ and $i_j \geq (4m-d)/4$ for some j, we construct a fiber homotopy trivial M-bundle $E_I \to S^k$ such that $\langle p_I(TE_I), [E_I]\rangle$ and $\langle p_m(TE_I), [E_I]\rangle$ are the *only* nontrivial monomial Pontryagin numbers of E_I (Lemma 3.7 together with Lemma 2.5). As $\Omega_{d+k} \otimes \mathbb{Q}$ is classified by Pontryagin numbers, $(E_I)_{I \text{ as above}}$ forms a linearly independent set in $\Omega_{d+k} \otimes \mathbb{Q}$. It follows that

$$\dim \mathrm{Fib}^h_{M,k} \geq \underbrace{\left|\left\{(i_1,\dots,i_s)\colon s \geq 2,\ \sum i_j = m \text{ and } i_j \geq (4m-d)/4\right\}\right|}_{=p(m)-p(m-\lceil \frac{d+1}{4}\rceil)-1}.$$

Remark 1.3. If $d \equiv 0$ (4), there exists a nonconnected manifold M where every component is simply connected such that equality holds. This is proven in Corollary 3.10.

1.1. Outline of the argument and obstructions to unblocking of block bundles

In [KKR21], Krannich–Kupers–Randal-Williams have proven that the class $\hat{\mathcal{A}}_3 \in H^{12}(\mathrm{BO}(8);\mathbb{Q})$ is h-spherical for $\mathbb{HP}^2$. It turns out that their construction delivers an excellent blueprint for our results. Since [KKR21] is written rather densely, we decided to give a more detailed account of their argument in Section 2 before we go on to proving our main results. Let us give an outline of the construction first.

Instead of constructing an actual fiber bundle, one constructs a so-called block bundle (we recall the notion of block bundles and block diffeomorphisms in Section 2). The advantage of working with block bundles is that the k-th homotopy group $\pi_k(\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M))$ of the classifying space for homotopy trivial block bundles is isomorphic to the structure set $\mathcal{S}_\partial(D^k \times M)$ from surgery theory. Since we assumed that $\dim(M) \geq 5$, the latter is accessible through the surgery exact sequence

$$L_{k+d+1}(\mathbb{Z}\pi_1(M)) \longrightarrow \mathcal{S}_\partial(D^k \times M) \longrightarrow \mathcal{N}_\partial(D^k \times M) \xrightarrow{\sigma} L_{k+d}(\mathbb{Z}\pi_1(M)),$$

where $\mathcal{N}_\partial$ denotes the set of normal invariants. We are interested in the case where M is simply connected and $(d+k)$ is divisible by 4, so the L-groups are given by 0 on the left and by $\mathbb{Z}$ on the right. Hence, in order to construct an M-block bundle over S^k, it suffices to construct a normal invariant η with $\sigma(\eta) = 0$. It turns out that the set of normal invariants is (rationally) isomorphic to the reduced real K-theory of $S^k \wedge M_+$ which allows one to construct a normal invariant and hence a (homotopy trivial) block bundle with prescribed Pontryagin classes. Homotopy trivial block bundles over S^k can (rationally) be given the structure of an actual (homotopy trivial) fiber bundle if k is in the unblocking range for M. This follows from a classical result of Burghelea–Lashof (cf. [BL82]) and Morlet's Lemma of disjunction together with work of Krannich, Kupers and Randal-Williams on diffeomorphisms of disks [KR21; KR24]; see Lemma 2.5.

The main work in the present article lies in constructing appropriate normal invariants. This way, we can ensure that certain Pontryagin classes and numbers of the total space of the corresponding block bundle are zero or nonzero.

1.2. Applications

Let us now present a few applications of our main result.

1.2.1. Spaces of metrics of positive curvature

Among all characteristic numbers, there are 2 of particular interest: The signature and the $\hat{\mathcal{A}}$-genus. Using Theorem A, we can construct fiber bundles with nonvanishing $\hat{\mathcal{A}}$-genus; see Proposition 4.1. This can be applied to the study of spaces of Riemannian metrics with lower curvature bounds. Let M be closed and let $\mathcal{R}_{\mathrm{scal}>0}(M)$ denote the space of Riemannian metrics on M of positive scalar curvature, equipped with the Whitney C^∞-topology. Furthermore, let $\mathcal{R}_C(M)$ be a $\mathrm{Diff}^+(M)$-space which admits a $\mathrm{Diff}^+(M)$-equivariant map to $\mathcal{R}_{\mathrm{scal}>0}(M)$ and let $\mathrm{Diff}^+(M,D) \subset \mathrm{Diff}^+(M)$ denote the subgroup of those diffeomorphisms that fix an embedded disk $D \subset M$ point-wise.

Theorem E. *Let M^d be a closed, simply connected* Spin*-manifold that has at least one nonvanishing rational Pontryagin class and let $g \in \mathcal{R}_C(M)$. Let $k \geq 1$ be such that $(d+k)$ is divisible by* 4 *and k is in the unblocking range for M. Then for $k \geq 1$, the image of the map*

$$\pi_{k-1}(\mathrm{Diff}^+(M,D)) \longrightarrow \pi_{k-1}(\mathcal{R}_C(M))$$

induced by the orbit map $f \mapsto f_* g$ contains an element of infinite order.

Theorem E was the original motivation and the main theorem of the first version of the present article. For readers solely interested in the proof of this theorem, we recommend the original version which is considerably more focused and available at 2104.10595v1. The class of manifolds this theorem applies to is quite large; see Example 4.5.

1.2.2. Rationally fibering a cobordism class over a sphere

Given an integer $k \geq 1$, it is a classical problem that goes back to Conner–Floyd [CF65] to decide which cobordism class contains a manifold that fibers over S^k. This has been studied in detail for $k \leq 4$ [Bur66; Neu71; Kah84a; Kah84b]. However, the classical approach relied on identifications like $S^2 \cong \mathbb{CP}^1$ or $S^4 \cong \mathbb{HP}^1$ and does not seem to work for larger k. Theorem D and Remark 1.3 imply that any rational cobordism class in degrees $4m \geq 16$ fibers over S^4, provided that the signature vanishes. In the Appendix, jointly written with Jens Reinhold, we consider the question for bigger values of k building on the methods developed in this paper; see Theorem A.3. We show that in a given dimension $d \geq 32$, every (rational) cobordism class in the kernel of the signature homomorphism fibers over S^k for every $k \leq 8$. We also obtain results for $k \geq 9$; see Theorem A.3.

1.2.3. Sphericity of κ-classes

Our result can also be applied to study κ-classes, also called generalized Morita–Miller–Mumford classes. These are characteristic classes of manifold bundles, and we can employ Theorem A to derive a vanishing result for κ-classes; see Theorem 4.12.

1.2.4. Unblocking of block bundles

As a final application, we obtain a result on block bundles which do not admit the structure of actual fiber bundles. This follows from the fact that our construction of block bundles works regardless of the dimension of the base or fiber as mentioned above. The existence of such bundles has been previously

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observed in [ER14], where the authors construct an $\mathbb{HP}^2$-block bundle $\widetilde{E} \to S^{12}$ with $p_5(T\widetilde{E}) \neq 0$ which cannot be 'unblocked'. We present a more systematic result on unblockable block bundles in Corollary 4.10.

2. Preliminaries

Let M be a closed oriented manifold of dimension d and let $\mathrm{Diff}^+(M)$ denote the group of orientation preserving diffeomorphisms of M. We denote by $B\mathrm{Diff}^+(M)$ the classifying space for fiber bundles $E \to B$ with structure group $\mathrm{Diff}^+(M)$.

2.1. Block diffeomorphisms

In this subsection, we give a short overview of block bundles and diffeomorphisms, and we explain how to compare them to honest fiber bundles and diffeomorphisms. For $p \geq 0$, let Δ^p denote the standard topological p-simplex.[3]

Definition 2.1. A *block diffeomorphism* of $\Delta^p \times M$ is a diffeomorphism of $\Delta^p \times M$ that for each face $\sigma \subset \Delta^p$ restricts to a diffeomorphism of $\sigma \times M$.

The set of all block diffeomorphisms forms a semisimplicial group denoted by $\widetilde{\mathrm{Diff}}^+_\bullet(M)$ whose p-simplices are the block diffeomorphisms of $\Delta^p \times M$. The space $\widetilde{\mathrm{Diff}}^+(M)$ of block diffeomorphisms is defined as the geometric realization of $\widetilde{\mathrm{Diff}}^+_\bullet(M)$, and the associated classifying space is denoted by $B\widetilde{\mathrm{Diff}}^+(M)$. This space classifies block bundles. Let us recall the definition of a block bundle over a simplicial complex.

Definition 2.2 [ER14, Definition 2.4]. Let K be a simplicial complex and let $p\colon E \to |K|$ be continuous. A *block chart* for E over a simplex $\sigma \subset K$ is a homeomorphism $h_\sigma\colon p^{-1}(\sigma) \to \sigma \times M$ which for every face $\tau \subset \sigma$ restricts to a homeomorphism $p^{-1}(\tau) \to \tau \times M$. A *block atlas* is a set $\mathcal{A}$ of block charts, at least one over each simplex of K, such that transition functions are block diffeomorphisms. E is called a block bundle if it admits a block atlas.

By [ER14, Proposition 3.2], a block bundle $\pi\colon E \to B$ has a stable analogue of the vertical tangent bundle (i.e., there exists a stable vector bundle $T^s_\pi E \to E$ which is stably isomorphic to the vertical tangent bundle $T_\pi E$ provided that E is an actual fiber bundle). If furthermore B is a manifold, the total space E is again a manifold and there is a stable isomorphism $T^s_\pi E \oplus \pi^* TB \cong_{\mathrm{st}} TE$ (cf. [ER14, Lemma 3.3]).

Next, let us consider the semisimplicial subgroup $\mathrm{Diff}^+_\bullet(M)$ of those block diffeomorphisms that commute with the projection $\Delta^p \times M \to \Delta^p$. This gives precisely the p-simplices of the singular semisimplicial group $\mathrm{Sing}_\bullet \mathrm{Diff}^+(M)$. We have an inclusion $\mathrm{Sing}_\bullet \mathrm{Diff}^+(M) \subset \widetilde{\mathrm{Diff}}^+_\bullet(M)$, and since the geometric realization of $\mathrm{Sing}_\bullet(X)$ is homotopy equivalent to X for any space X ([Bau95, pp. 8]), we get an induced map

$$B\mathrm{Diff}^+(M) \longrightarrow B\widetilde{\mathrm{Diff}}^+(M).$$

Let $\mathrm{hAut}^+(M)$ denote the group-like topological monoid of (orientation preserving) homotopy equivalences of M with classifying space $B\mathrm{hAut}^+(M)$. Again, let $\widetilde{\mathrm{hAut}}^+(M)$ be the realization of the semisimplicial group of block homotopy equivalences defined analogously to block diffeomorphisms and let $B\widetilde{\mathrm{hAut}}^+(M)$ be the corresponding classifying space. By [Dol63, Thm 6.1], the inclusion $\mathrm{hAut}^+(M) \hookrightarrow \widetilde{\mathrm{hAut}}^+(M)$ is a homotopy equivalence. Consider the following maps induced by inclusions:

[3] We choose to follow [ER14] for this, even though there are more recent expositions on block diffeomorphisms like [Kra22] or [BM20] since the latter do not cover block bundles.

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$$B\widetilde{\mathrm{Diff}}^+(M) \to B\widetilde{\mathrm{hAut}}^+(M) \simeq B\mathrm{hAut}^+(M) \qquad B\mathrm{Diff}^+(M) \to B\mathrm{hAut}^+(M)$$

and let $\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M)$ and $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ denote the respective homotopy fibers. Note that $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ (resp. $\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M)$) classifies M-bundles (resp. M-block bundles) together with a fiberwise (resp. blockwise) homotopy equivalence to the product M-bundle – that is, *fiber homotopy trivial* M-bundles (resp. *blockwise homotopy trivial* M-block bundles). We make the following definition: A characteristic class $c \in H^{d+k}(\mathrm{BO}(d);\mathbb{Q})$ is called *block-spherical* (resp. *block-h-spherical*) if there exists an M-block bundle $E \to S^k$ (resp. a blockwise homotopy trivial one) with $\langle c(TE), [E]\rangle \neq 0$. We have implications:

$$\begin{array}{ccccc} & \nearrow & c \text{ is spherical} & \searrow & \\ c \text{ is } h\text{-spherical} & & & & c \text{ is block-spherical} \\ & \searrow & c \text{ is block-}h\text{-spherical} & \nearrow & \end{array}$$

and the following comparison result which follows from [BL82].

Lemma 2.3. *If* $k \leq \min(\frac{d-1}{3}, \frac{d-5}{2})$*, then the natural map*

$$\pi_k\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right)\left[\frac{1}{2}\right] \longrightarrow \pi_k\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right)\left[\frac{1}{2}\right]$$

is surjective.

Before we dive into the proof, let us recall the stable range: For a manifold M, possibly with boundary, let

$$C(M) := \{f\colon M\times[0,1] \to M\times[0,1] \text{ diffeomorphism} : f|_{M\times\{0\}} = \mathrm{id}\}$$

be the space of *pseudoisotopies*. There is a canonical map $e\colon C(M) \to C(M\times I)$, and we define

$$\phi(d) := \max\{q \in \mathbb{N}\colon e \text{ is } q\text{-connected for all } M \text{ with } \dim(M) \geq d\}$$

which is called the *pseudoisotopy stable range*. By a classical theorem of Igusa, $\phi(d) \geq \min\left(\frac{d-4}{3}, \frac{d-7}{2}\right)$ [Igu88].

Proof of Lemma 2.3. For $\omega \in \mathbb{N}$, let $\mathrm{hAut}^+(M)_\omega$ be the ω-th stage Postnikov tower of $\mathrm{hAut}^+(M)$; that is, the map $\mathrm{hAut}^+(M) \to \mathrm{hAut}^+(M)_\omega$ is ω-connected and higher homotopy groups of $\mathrm{hAut}^+(M)_\omega$ vanish. Furthermore, let $\mathrm{hAut}^+(M)_{2,\omega}$ be the localization away from 2; that is, its homotopy groups are $\mathbb{Z}[\frac{1}{2}]$-modules. We use analogous notations for Diff^+ and $\widetilde{\mathrm{Diff}}^+$. By [BL82, Theorem C7], there is a map $\varphi\colon \mathrm{hAut}^+(M)_{2,\omega} \to (\widetilde{\mathrm{Diff}}^+(M)/\mathrm{Diff}^+(M))_{2,\omega}$, such that the map induced by projection $q\colon \widetilde{\mathrm{Diff}}^+(M) \to \widetilde{\mathrm{Diff}}^+(M)/\mathrm{Diff}^+(M)$ factors as

$$\begin{array}{ccc} \widetilde{\mathrm{Diff}}^+(M)_{2,\omega} & \xrightarrow{q} & \left(\frac{\widetilde{\mathrm{Diff}}^+(M)}{\mathrm{Diff}^+(M)}\right)_{2,\omega} \\ \downarrow & \overset{\varphi}{\nearrow} & \\ \mathrm{hAut}^+(M)_{2,\omega} & & \end{array}$$

where the vertical map is induced by the inclusion $\widetilde{\mathrm{Diff}}^+(M) \to \widetilde{\mathrm{hAut}}^+(M)$, provided that $\omega \leq \phi(d)+1$. After completing this to a square and taking homotopy fibers over the image of the identity, we obtain a map of fiber sequences

$$\begin{array}{ccccc}
\mathrm{Diff}^+(M)_{2,\omega} & \xrightarrow{\iota} & \widetilde{\mathrm{Diff}}^+(M)_{2,\omega} & \xrightarrow{q} & \left(\frac{\widetilde{\mathrm{Diff}}^+(M)}{\mathrm{Diff}^+(M)}\right)_{2,\omega} \\
\downarrow & & \downarrow & & \| \\
\mathrm{hofib}(\varphi) & \longrightarrow & \mathrm{hAut}^+(M)_{2,\omega} & \xrightarrow{\varphi} & \left(\frac{\widetilde{\mathrm{Diff}}^+(M)}{\mathrm{Diff}^+(M)}\right)_{2,\omega}.
\end{array}$$

Note that the left square is a homotopy pullback square since the induced map on its homotopy fibers is a weak equivalence. We map this square to the trivial homotopy pullback square

$$\begin{array}{ccc}
\mathrm{hAut}^+(M)_{2,\omega} & = & \mathrm{hAut}^+(M)_{2,\omega} \\
\| & & \| \\
\mathrm{hAut}^+(M)_{2,\omega} & = & \mathrm{hAut}^+(M)_{2,\omega}
\end{array}$$

and obtain the following homotopy pullback square on homotopy fibers:

$$\begin{array}{ccc}
\Omega\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right)_{2,\omega} & \longrightarrow & \Omega\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right)_{2,\omega} \\
\downarrow & & \downarrow \\
\mathrm{hofib}(\mathrm{hofib}(\varphi) \to \mathrm{hAut}^+(M)) & \longrightarrow & *.
\end{array}$$

Hence, the map $\Omega\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right)_{2,\omega} \to \Omega\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right)_{2,\omega}$ admits a split, and we get a (split-)surjection

$$\pi_\ell\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right)\left[\frac{1}{2}\right] \twoheadrightarrow \pi_\ell\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right)\left[\frac{1}{2}\right]$$

as long as $\ell \leq \phi(d) + 1 = \min(\frac{d-1}{3}, \frac{d-5}{2})$. □

Therefore, an element of $\pi_k(\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M)) \otimes \mathbb{Q}$ yields an M-bundle $E \to S^k$ that is fiber homotopy trivial, provided that the dimension of M is high enough. The advantage of working with $\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M)$ instead of $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ stems from the fact that the former is accessible through surgery theory as we will review in the Section 2.2.

Remark 2.4. Another approach to compare $B\mathrm{Diff}^+(M)$ and $B\widetilde{\mathrm{Diff}}^+(M)$ is by using Morlet's lemma of disjunction as in [KKR21, Lemma]. Consider the following diagram of (homotopy) fibrations:

$$\begin{array}{ccc}
\frac{\widetilde{\mathrm{Diff}}^+_\partial(D^d)}{\mathrm{Diff}^+_\partial(D^d)} & \longrightarrow & \frac{\widetilde{\mathrm{Diff}}^+(M)}{\mathrm{Diff}^+(M)} \\
\downarrow & & \downarrow \\
B\mathrm{Diff}^+_\partial(D^d) & \longrightarrow & B\mathrm{Diff}^+(M) \\
\downarrow & & \downarrow \\
B\widetilde{\mathrm{Diff}}^+_\partial(D^d) & \longrightarrow & B\widetilde{\mathrm{Diff}}^+(M)
\end{array}$$

If M is ℓ-connected with $\ell \leq d-4$, then the induced map on homotopy fibers is $(2\ell-2)$-connected by Morlet's lemma of disjunction (cf. [BLR75, Corollary 3.2 on page 29]). Now $\pi_{k-1}(B\widetilde{\mathrm{Diff}}^+_\partial(D^d)) \cong \pi_0(\mathrm{Diff}^+_\partial(D^{d+k-2}))$ is isomorphic to the finite group of exotic spheres in dimension $(d+k-1)$.

For $k = 1$, we note that both $B\mathrm{Diff}^+(D^d)$ and $B\widetilde{\mathrm{Diff}}^+(D^d)$ are connected and that their fundamental groups are isomorphic.

For $k \geq 2$, we need to distinguish cases depending on the parity of d: If d is even, then $B\mathrm{Diff}^+_\partial(D^d)$ is rationally $(d-1)$-connected by [KR24, Theorem A]. However, if d is odd and $k-1 \leq d-4$ is not divisible by 4, then $\pi_{k-1}(B\mathrm{Diff}^+_\partial(D^d)) \otimes \mathbb{Q}$ is trivial by [KR21, Theorem A]. We note that both [KR24] and [KR21] are generalizations of a classical result due to Farrell–Hsiang [FH78].

Therefore, in all these cases, $\pi_{k-1}(\widetilde{\mathrm{Diff}}^+(D^d)/\mathrm{Diff}^+(D^d)) \otimes \mathbb{Q}$ is trivial, and the same is true for $\pi_{k-1}(\widetilde{\mathrm{Diff}}^+(M)/\mathrm{Diff}^+(M)) \otimes \mathbb{Q}$, provided $k \leq 2\ell - 1$. This implies that the map

$$\pi_k(B\mathrm{Diff}^+(M)) \otimes \mathbb{Q} \to \pi_k(B\widetilde{\mathrm{Diff}}^+(M)) \otimes \mathbb{Q}$$

is surjective. By the five-lemma, the same holds for the induced map

$$\pi_k\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right) \otimes \mathbb{Q} \longrightarrow \pi_k\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right) \otimes \mathbb{Q}.$$

We summarize this discussion about unblocking in the following definition and lemma.

Definition. Let M be ℓ-connected for some $1 \leq \ell \leq d-4$. We say that $k \in \mathbb{N}$ is *in the unblocking range for M* if one of the following holds.

(i) $k \leq \min(\frac{d-1}{3}, \frac{d-5}{2})$.
(ii) d is even and $k \leq \min(d-1, 2\ell-1)$.
(iii) d is odd, $(k-1)$ is not divisible by 4 and $k \leq \min(d-3, 2\ell-1)$.

Lemma 2.5. *Let M be simply connected and let k be in the unblocking range for M. Then the following map is surjective:*

$$\pi_k\left(\frac{\mathrm{hAut}^+(M)}{\mathrm{Diff}^+(M)}\right) \otimes \mathbb{Q} \longrightarrow \pi_k\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right) \otimes \mathbb{Q}.$$

Remark 2.6. Even though this Lemma 2.5 only states surjectivity, this still is enough to completely determine h-sphericity of characteristic classes: Every fiber homotopy trivial bundle is in particular a blockwise homotopy trivial block bundle, and we have analogous definitions for sphericity and h-sphericity for those. Therefore, a characteristic class $c \in H^{k+d}(\mathrm{BO}(d);\mathbb{Q})$ is h-spherical for M if and only if it is block-h–spherical for M, provided that k is in the unblocking range for M.

2.2. Surgery theory

Let X be a simply connected manifold of dimension at least 5 with boundary ∂X. The *structure set* $\mathcal{S}(X, \partial X)$ of $(X, \partial X)$ (sometimes written as $\mathcal{S}_\partial(X)$) is defined to be the set of equivalence classes of tuples $(W, \partial W, f)$ where W is a manifold with boundary ∂W and f is an orientation preserving homotopy equivalence[4] that restricts to a diffeomorphism on the boundary. Two such tuples $(W_0, \partial W_0, f_0)$ and $(W_1, \partial W_1, f_1)$ are equivalent if there exists a diffeomorphism $\alpha\colon W_0 \to W_1$ such that $f_0 = f_1 \circ \alpha$ on the boundary ∂W_0 and on the whole W_0, the map $f_1 \circ \alpha$ is homotopic to f_0 relative to ∂W_0 ([LM24, Definition 11.2]). It is a consequence of the h-cobordism theorem that for $\dim(M) \geq 5$, we have the following isomorphism ([BM13, Section 3.2, pp.33]):

$$\pi_k\left(\frac{\mathrm{hAut}^+(M)}{\widetilde{\mathrm{Diff}}^+(M)}\right) \cong \mathcal{S}_\partial(D^k \times M).$$

The main result of surgery theory is that the structure set $\mathcal{S}_\partial(D^k \times M)$ fits into an exact sequence of sets known as the *surgery exact sequence* (cf. [LM24, Theorem 11.22 and Remark 11.23]):

[4]Since we assume X to be simply connected, every homotopy equivalence is simple and we do not need to require this in the definition.

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$$L_{k+d+1}(\mathbb{Z}) \longrightarrow \mathcal{S}_\partial(D^k \times M) \longrightarrow \mathcal{N}_\partial(D^k \times M) \xrightarrow{\sigma} L_{k+d}(\mathbb{Z}). \tag{1}$$

Here, $\mathcal{N}_\partial(D^k \times M)$ is the set of *normal invariants* which is given by equivalence classes of tuples $(W, f, \hat{f}, \xi)$, where W is a $(d+k)$-dimensional manifold with (stable) normal bundle ν_W, ξ is a stable vector bundle over $D^k \times M$ and $f\colon W \to D^k \times M$ is a map of degree 1 that restricts to a diffeomorphism of the boundary and which is covered by a stable bundle map $\hat{f}\colon \nu_W \to \nu_{D^k \times M} \oplus \xi$. The equivalence relation is given by bordism of manifolds with cylindrical ends of degree 1 normal maps (see [LM24, Definition 11.7]).

Since we only consider simply connected manifolds, the relevant L-groups are 4-periodic and given by (cf. [LM24, Theorem 8.99])

$$L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \ (4) \\ \mathbb{Z}/2 & \text{if } n \equiv 2 \ (4) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the map $\mathcal{S}_\partial(D^k \times M) \hookrightarrow \mathcal{N}_\partial(D^k \times M)$ if $(k+d)$ is even. The map σ in the surgery exact sequence (1) is the so-called *surgery obstruction map*, which in degrees $d + k \equiv 0$ (4) with $k \geq 1$ and for simply connected M is given by

$$\sigma(W, f, \hat{f}, \xi) = \frac{1}{8}\Big(\operatorname{sign}(\underbrace{W \cup (D^k \times M)}_{=:W'}) - \operatorname{sign}(S^k \times M)\Big) = \frac{1}{8}\operatorname{sign}(W'),$$

where sign denotes the signature (cf. [LM24, Theorem 8.173, Exercise 8.191]). The signature of W' can be computed via Hirzebruch's signature theorem, which constructs a power series with rational coefficients

$$\begin{aligned} \mathcal{L}(x_1, x_2, \dots) = 1 &+ s_1 x_1 + \dots + s_i x_i + \dots + s_{i,j} x_i \cdot x_j + \dots \\ &+ s_{i_1,\dots,i_n} x_{i_1} \cdots x_{i_n} + \dots \end{aligned}$$

such that $\operatorname{sign}(W') = \langle \mathcal{L}(p_1(TW'), p_2(TW'), \dots),\ [W'] \rangle$. Here $p_i(TW')$ are the Pontryagin classes of W'.

In order to further analyze $\mathcal{N}_\partial(D^k \times M)$, let us define $G(n) = \{f\colon S^{n-1} \to S^{n-1}$ homotopy equivalence$\}$ and $\mathrm{BG} := \operatorname{colim}_{n\to\infty} \mathrm{BG}(n)$. Note, that the index shift stems from the fact that one wants to have an inclusion $O(n) \subset G(n)$ of the orthogonal group. Analogously, let $\mathrm{BO} := \operatorname{colim}_{n\to\infty} \mathrm{BO}(n)$. Note that BG is the classifying space for stable spherical fibrations, whereas BO is the classifying space for stable vector bundles. The inclusion induces $J(n)\colon\ \mathrm{BO}(n) \hookrightarrow \mathrm{BG}(n)$ which in the colimit yields a map $J\colon\ \mathrm{BO} \to \mathrm{BG}$, and we denote its homotopy fiber by G/O. By [LM24, Equation 11.11], there is an identification

$$\mathcal{N}_\partial(D^k \times M) \cong [(D^k, S^{k-1}) \times M, (\mathrm{G/O}, *)],$$

where $[\_,\_]$ denotes homotopy classes of maps of pairs. For our purpose, we need a more explicit description of this identification; in particular, we want to pay attention to the vector bundle data. We follow [LM24, Theorem 7.10 and above] for this. A map into G/O consists of a map γ into BO and a homotopy h in BG from $J \circ \gamma$ to the constant map. Since $D^k \times M$ is compact, γ and h actually land in finite stages $\mathrm{BO}(\ell)$ and $\mathrm{BG}(\ell)$. We obtain a vector bundle $\gamma^* U_\ell \to (D^k \times M)/(S^{k-1} \times M)$, where $U_\ell \to \mathrm{BO}(\ell)$ is the universal vector bundle and a trivialization $\overline{h}\colon S(\gamma^* U_d) \to \underline{S^{\ell-1}}$ as spherical fibrations. Next, we choose an embedding $D^k \times M \hookrightarrow \mathbb{R}^N_+ := \{(x_1, \dots, x_N)\colon x_1 \geq 0\}$ such that $S^{k-1} \times M$ embeds into $\{x_1 = 0\}$. The relative Pontryagin Thom construction yields a map $(D^N, S^{N-1}) \to (\mathrm{Th}(\underline{S}(\nu_{M\times D^k})), \mathrm{Th}(S(\nu_{M\times S^{k-1}})))$ into the Thom spaces of the respective sphere bundles. Using the map $\overline{h}$ from above, we can define a map

$$
\begin{aligned}
(D^{N+\ell}, S^{N+\ell-1}) = (D^\ell, S^{\ell-1}) \times (D^N, S^{N-1}) &\longrightarrow (D^\ell, S^{\ell-1}) \times \mathrm{Th}(S(\nu_{D^k\times M}))\\
&\longrightarrow \quad (\mathrm{Th}(S(\nu_{D^k\times M}) * \underline{S^{\ell-1}}), \mathrm{Th}(S(\nu_{S^{k-1}\times M}) * \underline{S^{\ell-1}}))\\
&\xrightarrow{\mathrm{Th}(\mathrm{id} * \overline{h}^{-1})} (\mathrm{Th}(S(\nu_{D^k\times M} \oplus \gamma^* U_\ell)), \mathrm{Th}(S(\nu_{S^{k-1}\times M} \oplus \gamma^* U_\ell))),
\end{aligned}
$$

where the middle map is induced by the projection $D^\ell \times (D(\nu_{D^k\times M})/S(\nu_{D^k\times M})) \to D(D^\ell \times \nu_{D^k\times M})/S(D^\ell \times \nu_{D^k\times M})$. The reverse of the Pontryagin–Thom-construction yields an element $(W, \hat{f}, f, \gamma^* U_\ell) \in \mathcal{N}_\partial(D^k \times M)$. Note that the bundle ξ is precisely given by the (stable) vector bundle classified by $\gamma\colon D^k \times M \to \mathrm{BO}(\ell) \to \mathrm{BO}$.

Next, we identify $S^k \wedge M_+ = (S^k \times M_+)/(\{1\} \times M_+ \cup S^k \times \{+\}) \cong (D^k \times M)/(S^{k-1} \times M)$, where M_+ is M with a disjoint base point and $\wedge$ denotes the smash product of pointed spaces. The functor $S^k \wedge (\_)$ is adjoint to the k-fold loop space functor $\Omega^k(\_)$, and so we get $[S^k \wedge M_+, \mathrm{G/O}]_* \cong [M, \Omega^k \mathrm{G/O}]$. Now $\Omega^{k+1}\mathrm{BG}$ is the homotopy fiber of the map $\Omega^k \mathrm{G/O} \to \Omega^k \mathrm{BO}$, and by obstruction theory (cf. [Hat02, p. 418]), the obstructions to the lifting problem

$$
\begin{array}{ccc}
 & & \Omega^k\,\mathrm{G/O}\\
 & \nearrow\!\!\! {}^{\text{- -}} & \downarrow\\
M & \longrightarrow & \Omega^k\,\mathrm{BO}
\end{array}
$$

live in the groups $H^{i+1}(M; \pi_i(\Omega^{k+1}\mathrm{BG})) \cong H^{i+1}(M; \pi_{k+i+1}(\mathrm{BG}))$. The homotopy groups $\pi_{k+i+1}(\mathrm{BG})$ are isomorphic to the shifted stable homotopy groups of spheres π^{st}_{k+i} by [LM24, Equation 6.34]. By Serre's finiteness theorem, these groups are finite for $k + i \geq 1$, and hence, all of our obstruction groups vanish rationally since we assumed that $k \geq 1$. Since $\mathrm{maps}_*(M, \Omega^k \mathrm{BO})$ is an H-space, we see that for every (pointed) map $f\colon M \to \Omega^k \mathrm{BO}$, some multiple of f can be lifted to $\Omega^k \mathrm{G/O}$. Therefore, it suffices for us to specify an element in

$$[(D^k, S^{k-1}) \times M, (\mathrm{BO}, *)] = \mathrm{KO}^0((D^k, S^{k-1}) \times M)$$

in order for a multiple of this element to yield a normal invariant $(W, f, \hat{f}, \xi)$. Furthermore, by the discussion above, the bundle ξ in this normal invariant is precisely given by the multiple of the element in $\mathrm{KO}^0((D^k, S^{k-1}) \times M)$, and ξ is trivial when restricted to $S^{k-1} \times M$. For such a normal invariant, we can hence extend ξ by the trivial bundle to a bundle ξ' over $W' := W \cup_f D^k \times M$, and the maps f and $\hat{f}$ can be extended by the identity to a (stable) degree one normal map $(\hat{f}', f')\colon \nu_{W'} \to \nu_{S^k\times M} \oplus \xi'$.

Next, we consider the isomorphism given by the Pontryagin character

$$
\begin{aligned}
\mathrm{ph}(\_) := \mathrm{ch}(\_ \otimes \mathbb{C})\colon \mathrm{KO}^0((D^k, S^{k-1}) \times M) \otimes \mathbb{Q} &\xrightarrow{\cong} \bigoplus_{i\geq 0} H^{4i}((D^k, S^{k-1}) \times M; \mathbb{Q})\\
&\cong u_k \times \bigoplus_{i\geq 0} H^{4i-k}(M; \mathbb{Q})
\end{aligned}
$$

for u_k the cohomological fundamental class in $H^k(D^k, S^{k-1}) \cong H^k(S^k, *)$. The i-th component of the Pontryagin character is given by

$$
\begin{aligned}
\mathrm{ph}_i(\xi) = \mathrm{ch}_{2i}(\xi \otimes \mathbb{C}) &= \frac{1}{(2i)!}\Big((-2i)c_{2i}(\xi) + f(c_1(\xi), \ldots, c_{2i-1}(\xi))\Big)\\
&= \frac{(-1)^{i+1}}{(2i-1)!} p_i(\xi),
\end{aligned}
$$

where $f(c_1(\xi), \ldots, c_{2i-1}(\xi))$ is a polynomial in Chern classes of ξ homogeneous of degree $2i$ which vanishes since all nontrivial products in $H^*((D^k, S^{k-1}) \times M; \mathbb{Q})$ are trivial. Hence, for any collection

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$(x_i) \in H^{4i-k}(M;\mathbb{Q})$ and $(A_i) \in \mathbb{Q}$ there exists a $\lambda \in \mathbb{Z} \setminus \{0\}$ and a normal invariant $(W, f, \hat{f}, \xi) \in \mathcal{N}_\partial(D^k \times M)$ such that the bundle ξ has the following Pontryagin classes

$$p_i(\xi) = (-1)^{i+1}(2i-1)!\lambda A_i \cdot u_k \times x_i,$$

where u_k denotes the cohomological fundamental class of S^k. We observe that $p_i(\xi) \cup p_j(\xi) = 0$ for $i, j \geq 1$ since $u_k^2 = 0$. Furthermore, $(-1)^{i+1}(2i-1)! \neq 0$ for all choices of i, and hence, after replacing A_i by $\frac{(-1)^{i+1}A_i}{(2i-1)!}$, we may assume that the Pontryagin classes of ξ have the form[5]

$$p_i(\xi) = \lambda A_i \cdot u_k \times x_i. \tag{2}$$

This allows us to construct a normal invariant in the kernel of the signature homomorphism and hence an element of $\pi_k(\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M))$ such that the underlying stable vector bundle has prescribed Pontryagin classes, which we will do in the succeeding section.

With regard to Pontryagin numbers of the extension W' of W, we remark that $p_i(TW') = p_i(-\nu_{W'})$ and $p_i(-(\nu_{S^k \times M} \oplus \xi')) = p_i(T(S^k \times M) \oplus -\xi) = p_i(\mathrm{pr}^* TM \oplus -\xi')$ for $\mathrm{pr}\colon S^k \times M \to M$. Since $\hat{f}'\colon \nu_{W'} \to \nu_{S^k \times M} \oplus \xi'$ covers a map of degree one, any Pontryagin number of W' equals the corresponding Pontryagin number of $\mathrm{pr}^* TM \oplus -\xi'$. Furthermore, as ξ' is trivial on the complement of $W \subset W'$, the Pontryagin numbers of ξ' are obtained from the ones of ξ by replacing the fundamental class in $H^k(D^k, S^{d-1})$ by the corresponding one in $H^k(S^k, *)$ in Equation (2).

3. Prescribing Pontryagin classes

In this section, we will prove the block-analogues of our main results. We call a characteristic class $c \in H^{d+k}(\mathrm{BO}(d);\mathbb{Q})$ *block-spherical (resp. block- h-spherical) for M* if there exists an M-block-bundle $E \to S^k$ (resp. a homotopy trivial one) such that $\langle c(E), [E]\rangle \neq 0$.

3.1. Proof of Theorem A

Part (i) of Theorem A follows from the following 'block-version' in combination with Lemma 2.5 (see also Remark 2.6).

Lemma 3.1. *Let* $4m = d + k$ *and let* $p_m \neq p = p_{i_1} \cdot \ldots \cdot p_{i_s} \in H^{4m}(BSO;\mathbb{Q})$ *be a monomial in universal Pontryagin classes. Then*

$$p \text{ is block-h-spherical} \iff \begin{array}{c} \text{There exists an } \ell \geq 1 \text{ such that} : i_\ell \geq \frac{k}{4} \text{ and} \\ p_{i_1}(TM) \cup \ldots \cup \widehat{p_{i_\ell}(TM)} \cup \ldots \cup p_{i_s}(TM) \neq 0. \end{array}$$

Proof. We first show the '$\Leftarrow$' implication. Let ℓ be such that i_ℓ is maximal with the property above and let $x \in H^*(M;\mathbb{Q})$ be such that

$$p_{i_1}(TM) \cup \ldots \cup \widehat{p_{i_\ell}(TM)} \cup \ldots \cup p_{i_s}(TM) \cup x = u_M \in H^d(M;\mathbb{Q}).$$

By the discussion in Section 2.2 (in particular, see Equation (2)), there exists a normal invariant η such that the underlying (extended) stable vector bundle $\xi \to S^k \times M$ has the following Pontryagin classes:

$$p_0(-\xi) = 1 \qquad p_{i_\ell}(-\xi) = \lambda \cdot u_k \times x$$
$$p_m(-\xi) = \lambda A \cdot u_k \times u_M$$

for u_k the cohomological fundamental class of S^k, $A \in \mathbb{Q}$ to be chosen later and $\lambda \in \mathbb{Z} \setminus \{0\}$ determined by A. All other Pontryagin classes of ξ vanish. Note that by assumption, $i_\ell < m$. Then[6]

[5]Note that λ depends on the collection (A_i), so we cannot absorb it into the A_i's

[6]Since we are only interested in rational Pontryagin classes the Whitney sum formula, $p(V \oplus W) = p(V) \cup p(W)$ holds.

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$$p_{i_1} \cup \ldots \cup p_{i_s}(\mathrm{pr}^* TM \oplus -\xi) = \prod_{j=1}^{s} \sum_{n=0}^{i_j} p_n(-\xi) \cup p_{i_j-n}(\mathrm{pr}^* TM) \tag{3}$$

for $\mathrm{pr}\colon S^k \times M \to M$ the projection. Since $p_i(\mathrm{pr}^* TM) = \mathrm{pr}^* p_i(TM) = 1 \times p_i(TM)$, we will from now on omit 'pr^*' in our computations. Since $i_j < m$ and by our choice of Pontryagin classes of ξ, the expression $p_n(-\xi) \cup p_{i_j-n}(TM)$ is nonzero only if $n = 0$ or if $n = i_\ell$. Furthermore, recall that $p_n(-\xi) \cup p_{n'}(-\xi) = 0$ for $n, n' \geq 1$. We continue the computation

$$\begin{aligned}
(3) &= \prod_{i_j \geq i_\ell} \Big(p_{i_j}(TM) + p_{i_\ell}(-\xi) \cup p_{i_j - i_\ell}(TM) \Big) \cup \prod_{i_j < i_\ell} p_{i_j}(TM) \\
&= \prod_{j=1}^{s} p_{i_j}(TM) \qquad (4) \\
&\quad + \prod_{i_j < i_\ell} p_{i_j}(TM) \cup \sum_{i_j \geq i_\ell} \left(p_{i_\ell}(-\xi) \cup p_{i_j - i_\ell}(TM) \cup \prod_{\substack{q \neq j \\ i_q \geq i_l}} p_{i_q}(TM) \right),
\end{aligned}$$

where the second equality holds after multiplying out and using the fact that there can only be one factor $p_{i_\ell}(-\xi)$ in each summand. The first summand vanishes for degree reasons. If $i_j > i_\ell$, then $\prod_{q \neq j} p_{i_q}(TM) = 0$ because we chose i_ℓ to be maximal such that this product does not vanish. Hence, the latter factor vanishes if $i_j > i_\ell$ and we get

$$\begin{aligned}
(4) &= p_{i_\ell}(-\xi) \cup \prod_{i_j < i_\ell} p_{i_j}(TM) \cup \sum_{j\colon i_j = i_\ell} \left(\prod_{q \neq j} p_{i_q}(TM) \right) \\
&= \underbrace{p_{i_\ell}(-\xi)}_{= \lambda \cdot x \times u_k} \cup \prod_{q \neq \ell} p_{i_q}(TM) \cdot \underbrace{\sum_{j\colon i_j = i_\ell} 1}_{=: a_\ell \neq 0} = \lambda a_\ell \cdot u_k \times u_M \neq 0.
\end{aligned}$$

Finally, we need to choose A, such that the surgery obstruction vanishes. We have

$$p_m(TM \oplus -\xi) = p_m(-\xi) + p_{m - i_\ell}(TM) \cup p_{i_\ell}(-\xi).$$

Consider Hirzebruch's signature formula:

$$\begin{aligned}
\sigma(\eta) &= \operatorname{sign}(W') = \langle \mathcal{L}(W'),\ [W'] \rangle = \big\langle \mathcal{L}(S^k) \cdot \mathcal{L}(TM \oplus -\xi),\ [S^k \times M] \big\rangle \\
&= s_m \cdot \big\langle p_m(TM \oplus -\xi),\ [S^k \times M] \big\rangle \\
&\qquad + \big\langle \mathcal{L}(TM \oplus -\xi) - s_m \cdot p_m(TM \oplus \xi),\ [S^k \times M] \big\rangle \\
&= s_m \cdot \big\langle p_m(-\xi), [S^k \times M] \big\rangle \\
&\quad + \underbrace{\big\langle \mathcal{L}(TM \oplus -\xi) - s_m \cdot p_m(TM \oplus \xi) + s_m \cdot p_{m - i_\ell}(TM) \cup p_{i_\ell}(-\xi) \big\rangle, [S^k \times M]}_{=: z} \\
&= s_m \lambda \cdot A + z,
\end{aligned}$$

where $s_m \neq 0$ is the leading coefficient of $\mathcal{L}$. Note that $z = \lambda \cdot z_0$ for some z_0 which is independent of A. This is true since there appears precisely one factor $p_{i_\ell}(\xi)$ (which is a multiple of λ) in every monomial summand of $(\mathcal{L}(TM \oplus \xi) - s_m p_m(TM \oplus \xi) + s_m \cdot p_{m - i_\ell}(TM) \cup p_{i_\ell}(-\xi))$, while all other factors are coefficients of $\mathcal{L}$ or Pontryagin classes of M which are independent of λ. We choose $A := \frac{z_0}{s_m}$ so that $\sigma(\eta)$ vanishes independently of λ. We hence obtain a normal invariant with vanishing signature and therefore a block bundle with the desired properties.

For the other implication '$\Rightarrow$', let us assume that $p_{i_1}(TM) \cup \ldots \cup \widehat{p_{i_\ell}(TM)} \cup \ldots \cup p_{i_s}(TM) = 0$ for all $i_\ell \geq \frac{k}{4}$.

$$p_{i_1} \cup \ldots \cup p_{i_s}(TM \oplus -\xi) = \prod_{j=1}^{s} \sum_{n=0}^{i_j} p_n(-\xi) \cup p_{i_j-n}(TM)$$

Since $p_n(\xi) \cup p_{n'}(\xi) = 0$ for $n, n' \geq 1$ and $p_{i_1}(TM) \cdots p_{i_s}(TM) = 0$ (for degree reasons), every summand in the above expression must contain precisely one factor $p_n(\xi)$ for some $n \geq 1$. Hence, multiplying out the above delivers

$$p_{i_1} \cup \ldots \cup p_{i_s}(TM \oplus -\xi) = \sum_{j=1}^{s} \sum_{n=1}^{i_j} p_n(-\xi) \cup p_{i_j-n}(TM) \cup \prod_{r \neq j} p_{i_r}(TM).$$

The product $\prod_{r \neq j} p_{i_r}(TM)$ vanishes if $i_j \geq \frac{k}{4}$ by assumption. If $i_j < \frac{k}{4}$, then $p_n(-\xi) = 0$ for all $1 \leq n \leq i_j$ since every higher Pontryagin class of ξ is of the form $u_k \times *$ and hence is of degree at least $k/4$. It follows that there is no normal invariant with $p_{i_1} \cup \cdots \cup p_{i_s}(TW) \neq 0$, and hence, there can be no such block bundle. □

Next, we turn to the proof of Theorem A(ii) and Theorem B which will again follow from block-analogues thereof combined with Lemma 2.5. Recall the following definitions:

$$\begin{aligned} i_{\min} &:= \min\{i \geq 1 \colon p_i(TM) \neq 0\} \\ n_{\max} &:= \max\{n \in \mathbb{N} \colon p_{i_{\min}}(TM)^n \neq 0\}. \end{aligned} \tag{5}$$

We assume that M admits at least on nontrivial rational Pontryagin class, so the set $\{i \geq 1 \colon p_i(TM) \neq 0\}$ is actually nonempty and $n_{\max} \geq 1$.

Lemma 3.2. *For every $\ell = 1, \ldots, n_{\max}$, there exists a normal invariant η_ℓ with underlying (extended) stable vector bundle $\xi_\ell \to S^k \times M$ with the following property:*

$$\left\langle p_{i_{min}}(TM \oplus -\xi_\ell)^r \cup p_{m-r \cdot i_{\min}}(TM \oplus -\xi_\ell),\ [S^k \times M] \right\rangle \neq 0 \quad \Longleftrightarrow \quad \substack{r=\ell \\ or\ r=0}$$

and $\sigma(\eta_\ell) = 0$. For $\ell = 1$, we furthermore have that ξ_1 can be chosen such that

$$\begin{gathered} \left\langle p_{i_{\min}}(TM \oplus -\xi_1) \cup p_{m-i_{\min}}(TM \oplus -\xi_1),\ [S^k \times M] \right\rangle \\ \textit{and} \quad \left\langle p_m(TM \oplus -\xi_1),\ [S^k \times M] \right\rangle \end{gathered}$$

are the only nonvanishing monomial Pontryagin numbers of $TM \oplus -\xi_1$.

Proof. Let $u_M \in H^{4m-k}(M; \mathbb{Q})$ denote the cohomological fundamental class of M. Since the cup product induces a perfect pairing

$$H^{4j}(M; \mathbb{Q}) \times H^{4(m-j)-k}(M; \mathbb{Q}) \to \mathbb{Q},$$

there exists a class $x := x_{n_{\max}} \in H^{4(m-i_{\min} \cdot n_{\max})-k}(M; \mathbb{Q})$ such that $x \cup p_{i_{\min}}(TM)^{n_{\max}} = u_M$. For $r = 0, \ldots, n_{\max}$, we define $x_r := x \cup p_{i_{\min}}(TM)^{n_{\max}-r}$. Then

$$x_r \cup p_{i_{\min}}(TM)^r = x \cup p_{i_{\min}}(TM)^{n_{\max}-r} \cup p_{i_{\min}}(TM)^r = u_M.$$

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By the discussion in Section 2, we know that for every collection $A_0, \dots, A_{n_{\max}} \in \mathbb{Q}$, there exists a $\lambda \in \mathbb{Z} \setminus \{0\}$ and a normal invariant $\eta_\ell = (W_\ell, f_\ell, \hat{f}_\ell, \xi_\ell)$ such that the (extended) stable vector bundle ξ'_ℓ has only the following (rational) Pontryagin classes:

$$\begin{aligned} p_0(-\xi'_\ell) &= 1 \\ p_{m-r\cdot i_{\min}}(-\xi'_\ell) &= \lambda A_r \cdot u_k \times x_r \text{ for } r = 0, \dots, n_{\max}. \end{aligned}$$

Since $r \cdot i_{\min} < m$ and $p_q(TM) = 0$ for all $0 < q < i_{\min}$, we have

$$p_{i_{min}}(TM \oplus -\xi'_\ell) = \sum_{a=0}^{i_{\min}} p_a(TM) \cup p_{i_{\min}-a}(-\xi'_\ell) = p_{i_{\min}}(-\xi'_\ell) + p_{i_{\min}}(TM).$$

We will now distinguish two cases: $m = (s+1) \cdot i_{\min}$ for some $1 \leq s \leq n_{\max}$ and $m \neq (r+1) \cdot i_{\min}$ for all r. In the former case, we have $p_{i_{\min}}(-\xi'_\ell) = \lambda A_s \cdot u_k \times x_s$ and compute

$$\begin{aligned} p_{i_{\min}}(TM\oplus - \xi'_\ell)^s \cup p_{\underbrace{m - s \cdot i_{\min}}_{=i_{\min}}}(TM \oplus -\xi'_\ell) &= p_{i_{\min}}(TM \oplus -\xi'_\ell)^{s+1} \\ &= (p_{i_{\min}}(-\xi'_\ell) + p_{i_{\min}}(TM))^{s+1} = (s+1) \cdot p_{i_{\min}}(TM)^{n_{\max}} \cup p_{i_{\min}}(-\xi'_\ell) \\ &= (s+1)\lambda A_s \cdot \underbrace{p_{i_{\min}}(TM)^s \cup u_k \times x_s}_{=u_k \cdot u_M} \end{aligned}$$

where the third equality follows from multiplying out and the fact that $p_n(\xi) \cup p_{n'}(\xi) = 0$ for $n, n' \geq 1$. For $0 \leq r \neq s$ we have:

$$\begin{aligned} &p_{i_{\min}}(TM \oplus -\xi'_\ell)^r \cup p_{m-i_{\min}\cdot r}(TM \oplus -\xi_\ell) \\ &= \left(p_{i_{\min}}(-\xi'_\ell) + p_{i_{\min}}(TM)\right)^r \cup \sum_{a=0}^{m-i_{\min}\cdot r} p_a(TM) \cup p_{m-i_{\min}\cdot r-a}(-\xi'_\ell) \\ &= \left(p_{i_{\min}}(-\xi'_\ell) + p_{i_{\min}}(TM)\right)^r \\ &\quad \cup \left(p_{m-i_{\min}\cdot r}(-\xi'_\ell) + \sum_{a=i_{\min}}^{m-i_{\min}\cdot r} p_a(TM) \cup p_{m-i_{\min}\cdot r-a}(-\xi'_\ell)\right) \\ &= \underbrace{p_{i_{\min}}(TM)^r \cup \lambda A_r u_k \times x_r}_{=\lambda A_r \cdot u_k \times u_M} + * \cdot u_k \times u_M, \end{aligned} \tag{6}$$

where $*$ is a linear expression in the variables $\lambda A_{r+1}, \dots, \lambda A_{n_{\max}}$. Now for $b, a_1, \dots, a_{n_{\max}} \in \mathbb{Q}$, consider the following system of equations:

$$\begin{aligned} b &= \left\langle p_m(TM \oplus -\xi'_\ell), [S^k \times M] \right\rangle \\ a_1 &= \left\langle p_{i_{\min}}(TM \oplus -\xi'_\ell) \cup p_{m-i_{\min}}(TM \oplus -\xi'_\ell), [S^k \times M] \right\rangle \\ &\vdots \\ a_r &= \left\langle p_{i_{\min}}(TM \oplus -\xi'_\ell)^r \cup p_{m-r\cdot i_{\min}}(TM \oplus -\xi'_\ell), [S^k \times M] \right\rangle \\ &\vdots \\ a_{n_{\max}} &= \left\langle p_{i_{\min}}(TM \oplus -\xi'_\ell)^{n_{\max}} \cup p_{i_{\min}}(TM \oplus -\xi'_\ell), [S^k \times M] \right\rangle. \end{aligned}$$

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By the above computation, this is a linear system of equations in the variables $\lambda A_0, \lambda A_1, \dots, \lambda A_{n_{\max}}$, and it has the following form:

$$\underbrace{\begin{pmatrix} 1 & * & * & * & * \\ 0 & \ddots & * & * & * \\ 0 & 0 & s+1 & * & * \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{=:B} \cdot \begin{pmatrix} \lambda A_0 \\ \vdots \\ \lambda A_{n_{\max}} \end{pmatrix} = \begin{pmatrix} b \\ a_1 \\ \vdots \\ a_{n_{\max}} \end{pmatrix}$$

with $s+1$ in the s-th row. The matrix B is invertible, and hence, we can choose $A_1, \dots A_{n_{\max}}$ such that $a_i = 0$ if and only if $i \neq \ell$. Note that λ is not yet determined, as it also depends on A_0, but the condition of a_i being zero or nonzero is independent of λ. Furthermore, note that since B is triangular, the values of a_i are independent of A_0. Finally, we need to choose A_0, such that the surgery obstruction vanishes. Consider Hirzebruch's signature formula:

$$\begin{aligned} \sigma(\eta_\ell) &= \operatorname{sign}(W'_\ell) = \langle \mathcal{L}(W'_\ell),\ [W'_\ell] \rangle = \langle \mathcal{L}(W'_\ell),\ f_*[S^k \times M] \rangle \\ &= \langle \mathcal{L}(S^k)\mathcal{L}(TM \oplus \xi_\ell),\ [S^k \times M] \rangle = \langle \mathcal{L}(TM \oplus \xi_\ell),\ [S^k \times M] \rangle \\ &= s_m \cdot \lambda \cdot A_0 + \lambda \cdot z_0, \end{aligned}$$

where z_0 is some number independent of A_0 and s_m is the leading coefficient of $\mathcal{L}$ as in the proof of Lemma 3.1. Since $s_m \neq 0$, we can choose $A_0 := \frac{z_0}{s_m}$ so that $\sigma(\eta_\ell)$ vanishes independently of λ.

The case $m \neq (r+1) \cdot i_{\min}$ for all r is very similar. By the same computation as (6), we have for all $r \geq 0$,

$$\begin{aligned} & p_{i_{\min}}(TM \oplus -\xi'_\ell)^r \cup p_{m-i_{\min}\cdot r}(TM \oplus -\xi_\ell) \\ &\quad = \underbrace{p_{i_{\min}}(TM)^r \cup \lambda A_r \cup u_k \times x_r}_{=\lambda A_r \cdot u_k \times u_M} + * \cdot u_k \times u_M. \end{aligned}$$

This implies that the respective matrix B has the same form as above with $s+1$ replaced by 1. The rest of the argument is verbatim to the first case.

If $\ell = 1$, the above system of linear equations reduces to $a_2 = a_3 = \cdots = a_{n_{\max}} = 0$. From the shape of B, this implies that $A_2 = \cdots = A_{n_{\max}}$ must be 0 as well, and hence, bundle ξ'_1 only has three nonvanishing Pontryagin classes – namely,

$$\begin{aligned} p_0(\xi'_1) &= 1 \\ p_{m-i_{\min}}(-\xi'_1) &= \lambda A_1 \cdot u_k \times x \\ p_m(-\xi'_1) &= \lambda A_0 \cdot u_k \times u_M. \end{aligned}$$

As noted above, every Pontryagin number of $TM \oplus -\xi'_1$ contains precisely one Pontryagin class of ξ'_1. Since $p_{i_{min}}(TM)$ is the smallest Pontryagin class of M, we deduce that the only possibly nonvanishing Pontryagin numbers of $TM \oplus -\xi'_1$ are $\langle p_{i_{\min}}(TM \oplus -\xi'_1) \cup p_{m-i_{\min}}(TM \oplus -\xi'_1),\ [S^k \times M] \rangle$ and $\langle p_m(TM \oplus -\xi'_1),\ [S^k \times M] \rangle$. By construction,

$$\langle p_{i_{\min}}(TM \oplus -\xi'_1) \cup p_{m-i_{\min}}(TM \oplus -\xi'_1),\ [S^k \times M] \rangle \neq 0.$$

Since $\sigma(\eta_1) = 0$ by construction, we deduce that $\langle p_m(TM \oplus -\xi'_1),\ [S^k \times M] \rangle$ is nonzero as well since the leading coefficient in the $\mathcal{L}$-polynomial is nontrivial by [Hir95, p. 12]. □

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3.2. *Proof of Corollary C*

(i) Consider the linear homomorphism $\Omega_{d+k} \longrightarrow \mathbb{Q}^{n_{\max}}$ given by

$$[X] \mapsto \left(\left\langle p_{i_{\min}}(TX)^{\ell} \cup p_{\frac{d+k}{4}-\ell \cdot i_{\min}}(TX), [X]\right\rangle\right)_{\ell=1,\dots,n_{\max}}.$$

If $\mathcal{E}$ denotes the vector space generated by $E_1, \dots E_{n_{\max}}$ from Theorem B, the composition

$$\mathcal{E} \to \mathrm{Fib}^h(M,k) \to \Omega_{d+k} \otimes \mathbb{Q} \to \mathbb{Q}^{n_{\max}}$$

is given by a diagonal matrix with nonzero entries on the diagonal by Theorem B. Hence, the first map is forced to be injective.

(ii) Let $p_I[\_] := \langle p_I(\_), [\_]\rangle$ be a monomial Pontryagin number and let $M \xrightarrow{\iota} E^{4m} \xrightarrow{\pi} S^k$ be a fiber homotopy trivial bundle. We will show that $p_I[E]$ is a linear combination of $(p_{i_{\min}}^e \cup p_{m-e\cdot i_{\min}}[E])_{e=0..n_{\max}}$. This implies that $p_I[\_]$ restricted to $\mathrm{Fib}^h_{M,k}$ is a linear combination of $(p_{i_{\min}}^e \cup p_{m-e\cdot i_{\min}}[\_])_{e=0..n_{\max}}$. Again, since Ω_* is classified by Pontryagin numbers, this implies that $\dim \mathrm{Fib}^h_{M,k} \leq n_{\max}+1$. Furthermore, the signature of any fiber bundle $E \to S^k$ is trivial. Since every coefficient in the $\mathcal{L}$-polynomial is nontrivial by [BB18], this gives one nontrivial linear relation among the functionals $(p_{i_{\min}}^e \cup p_{m-e\cdot i_{\min}}[\_])_{e=0\dots n_{\max}}$ restricted to $\mathrm{Fib}^h_{M,k}$, implying our claim:

$$\dim \mathrm{Fib}^h_{M,k} = \dim(\mathrm{Fib}^h_{M,k})^* \leq n_{\max}.$$

Now, let us consider the Wang-sequence:

$$\cdots \longrightarrow H^n(M) \xrightarrow{\delta} H^{k+n}(E) \xrightarrow{\iota^*} H^{k+n}(M) \longrightarrow \dots.$$

Furthermore,

$$\iota^* p_i(TE) = \iota^* p_i(\pi^* TS^k \oplus T_\pi E) = \iota^* p_i(T_\pi E) = p_i(\iota^* T_\pi E) = p_i(TM)$$

for $T_\pi E$ the vertical tangent bundle of E. The fiber homotopy trivialization $h\colon E \to M \times S^k$ yields a retraction $s := \mathrm{pr}_M \circ h\colon E \to M$ of ι, in particular, $(s\circ\iota)^*TM = TM$. Therefore, $p_i(TM) = \iota^* p_i(s^*TM)$, and by the above computation, it follows for all i that $(p_i(TE) - p_i(s^*TM)) = \delta(x_i)$ for some $x_i \in H^{4i-k}(M)$. Since products of elements in the image of δ vanish by [Whi78, p. 337, Corollary 3.3], we can multiply out

$$p_I(TE) = \prod_{i\in I}(\delta(x_i) + p_i(s^*TM)) = s^* p_I(TM) + \sum_{i\in I} \delta(x_i) \cup s^* p_{I\setminus\{i\}}(TM).$$

The first summand vanishes for degree reasons, and by our assumption on M, there exist $a_i \in \mathbb{Q}$ such that for $n_i = \frac{m-i}{i_{\min}}$ we have

$$p_{I\setminus\{i\}}(s^*TM) = a_i \cdot p_{i_{\min}}(s^*TM)^{n_i} = a_i \cdot (p_{i_{\min}}(TE) - \delta(x_{i_{\min}}))^{n_i}.$$

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Again using that products in the image of δ vanish, we compute

$$\begin{aligned} p_I(TE) &= \sum_{i\in I} a_i \cdot \delta(x_i) \cup (p_{i_{\min}}(TE) - \delta(x_{i_{\min}}))^{n_i} \\ &= \sum_{i\in I} a_i \cdot \delta(x_i) \cup p_{i_{\min}}(TE)^{n_i} \\ &= \sum_{i\in I} a_i \cdot (p_i(TE) - p_i(s^*TM)) \cup p_{i_{\min}}(TE)^{n_i} \\ &= \sum_{i\in I} a_i \cdot p_i(TE) \cup p_{i_{\min}}(TE)^{n_i} - \sum_{i\in I} a_i \cdot p_i(s^*TM) \cup p_{i_{\min}}(TE)^{n_i}. \end{aligned} \tag{7}$$

For the second sum in this formula, we note that there exist $b_i \in \mathbb{Q}$ such that for $m_i = \frac{i}{i_{\min}}$ we have

$$\begin{aligned} &\sum_{i\in I} a_i \cdot p_i(s^*TM) \cup p_{i_{\min}}(TE)^{n_i} \\ &\quad = \sum_{i\in I} a_i b_i p_{i_{\min}}(s^*TM)^{m_i} \cup \left(\delta(x_{i_{\min}}) + p_{i_{\min}}(s^*TM)\right)^{n_i} \\ &\quad = \sum_{i\in I} a_i b_i \binom{n_i}{1} \delta(x_{i_{\min}}) \cup p_{i_{\min}}(s^*TM)^{m_i+n_i-1} \\ &\quad\quad + \sum_{i\in I} a_i b_i \binom{n_i}{1} p_{i_{\min}}(s^*TM)^{m_i+n_i}. \end{aligned}$$

Note that $m_i + n_i = \frac{m}{i_{\min}} =: \ell$ is independent of i and the second summand hence vanishes for degree reasons. We compute

$$\begin{aligned} &\sum_{i\in I} a_i b_i \binom{n_i}{1} \delta(x_{i_{\min}}) \cup p_{i_{\min}}(s^*TM)^{m_i+n_i-1} \\ &\quad = \delta(x_{i_{\min}}) \cup p_{i_{\min}}(s^*TM)^{\ell-1} \sum_{i\in I} a_i b_i n_i \\ &\quad = \left(\delta(x_{i_{\min}}) + p_{i_{\min}}(s^*TM)\right)^{\ell} \underbrace{\frac{1}{\ell} \sum_{i\in I} a_i b_i n_i}_{=:\lambda} = \lambda \cdot p_{i_{\min}}(TE)^{\ell}. \end{aligned}$$

Using that $m - n_i \cdot i_{\min} = m - \frac{m-i}{i_{\min}} \cdot i_{\min} = i$, we can combine this with (7) to obtain

$$\begin{aligned} p_I(TE) &= \sum_{i\in I} a_i \cdot p_i(TE) \cup p_{i_{\min}}(TE)^{n_i} - \lambda \cdot p_{i_{\min}}(TE)^{\ell} \\ &= \sum_{i\in I} a_i \cdot p_{m-n_i\cdot i_{\min}}(TE) \cup p_{i_{\min}}(TE)^{n_i} - \lambda \cdot p_{i_{\min}}(TE)^{\ell}. \end{aligned}$$

Since a_i and λ do not depend on E but only on M and I, the restriction of the functional $p_I[\_]$ to $\mathrm{Fib}^h_{M,k}$ is contained in the linear span of the functionals $(p^e_{i_{\min}} \cup p_{m-e\cdot i_{\min}}[\_])_{e=0..n_{\max}}$, as claimed.

3.3. *Upper bounds on* $\dim \mathrm{Fib}^h_{M,k}$

Let $\widetilde{\mathrm{Fib}}^h_{M,k} \subset \Omega_{d+k} \otimes \mathbb{Q}$ denote the linear subspace spanned by homotopy trivial M-block bundles. By Lemma 2.5, we have $\widetilde{\mathrm{Fib}}^h_{M,k} \subset \mathrm{Fib}^h_{M,k}$ provided that k is in the unblocking range for M. Furthermore,

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by Lemma 3.2, $\dim(\widetilde{\mathrm{Fib}}^h_{M,k}) \geq n_{\max}$. In this section, we prove sharpness of the upper bound as claimed in Theorem D or rather its block-analogue.

Proposition 3.3. *Let $p = p_{i_1} \cup \ldots \cup p_{i_s} \in H^{4m}(BO(d);\mathbb{Q})$. If $i_j < m - \frac{d}{4}$ for all $j \in \{1, \ldots, s\}$, $p_{i_1}(TE) \cup \cdots \cup p_{i_s}(TE) = 0$ for all blockwise homotopy trivial M-block bundles E.*

Proof. If $p_{i_1}(TE) \cup \cdots \cup p_{i_s}(TE)$ were nonzero, we would get an i_j such that $p_{i_1}(TM) \cup \ldots \cup \widehat{p_{i_j}(TM)} \cup \ldots \cup p_{i_s}(TM) \neq 0$ by Lemma 3.1. However, the degree of this product is $4(m - i_j) > d$, and hence, the product has to vanish because the cohomology of M vanishes above degree d, leading to a contradiction. □

Recall that $p(n)$ is defined to be the number of partitions and the number of those partitions into natural numbers $\leq n'$ is $p(n, n')$. Proposition 3.3 yields the following upper bound.

Lemma 3.4. $\dim \widetilde{\mathrm{Fib}}^h_{M,4m-d} \leq p(m) - p(m, m - \lceil \frac{d+1}{4} \rceil) - 1.$

Achieving the upper bound

We will now show that this upper bound is sharp (i.e., there exists a manifold for which equality holds).

Definition 3.5. A manifold is said to be *P-large* if all monomials in rational Pontryagin classes of TM are linearly independent.

If $\tau\colon M \to \mathrm{BO}(d)$ is the classifying map for the tangent bundle TM, then M is P-large if and only if the induced map $\tau^*\colon H^*(\mathrm{BO}(d);\mathbb{Q}) \to H^*(M;\mathbb{Q})$ is injective for $* \leq d$. In the example below, we construct a P-large manifold, which can be made simply connected in dimensions $d \equiv 2, 3(4)$.

Example 3.6. For $n \geq 1$, let $M^n_1, \ldots, M^n_{s_n}$ be a basis for $\Omega_{4n} \otimes \mathbb{Q}$ with the property that each of those only has one nontrivial monomial Pontryagin number. Since $\Omega_* \otimes \mathbb{Q}$ is generated by $\{\mathbb{CP}^{2n}\}_{n\in\mathbb{N}}$, we may choose M^n_i to be simply connected. Consider the following d-dimensional manifold:

$$M \coloneqq \coprod_{n=1}^{\frac{d}{4}} \coprod_{j=1}^{s_n} M^n_j \times S^{d-4n}.$$

This manifold has all possible products of Pontryagin classes, and they are all linearly independent since $H^*(M;\mathbb{Q}) = \bigoplus_{n=1}^{d/4} \bigoplus_{j=1}^{s_n} H^*(M^n_j \times S^{d-4n})$ and for every $I = (i_1, \ldots, i_s)$, there is a unique $j \in \{1, \ldots, s_{|I|}\}$ such that $p_I(TM^{|I|}_j) \neq 0$. Note that for $d \not\equiv 1\ (4)$, every component of M is simply connected. If $d \equiv 2, 3\ (4)$, we can even assume M to be simply connected by performing connected sums. If $d \equiv 0, 1\ (4)$, this is not possible since Pontryagin products of top degree would then live in the 1-dimensional space $H^d(M;\mathbb{Q})$ (resp. in the 0-dimensional space $H^{d-1}(M;\mathbb{Q})$).

For $I = (i_1, \ldots, i_s)$, let $|I| := \sum i_j$, and we introduce the short notation

$$p_I = p_{i_1} \cup \ldots \cup p_{i_s}.$$

Lemma 3.7. *Let M be simply connected and P-large and let $I = \{i_1, \ldots, i_s\} \neq \{m\}$ with $|I| = m$ and $i_j \geq m - \frac{d}{4}$ for some j. Then there exists a normal invariant $\eta \in \mathcal{N}_\partial(D^{4m-d} \times M)$ with underlying (extended) stable vector bundle ξ' such that*

$$\left\langle p_I(TM \oplus -\xi'), [S^{4m-d} \times M] \right\rangle \quad \text{and} \quad \left\langle p_m(TM \oplus -\xi'), [S^{4m-d} \times M] \right\rangle$$

are the only *nonvanishing monomial Pontryagin numbers of $TM \oplus -\xi'$ and $\sigma(\eta) = 0$.*

Proof. Since the cup product induces a perfect pairing and all monomials in Pontryagin classes of M are linearly independent, there exist elements $x_J \in H^{d-4|J|}(M)$ for every collection $J = (j_1, \dots, j_t)$ with $|J| < \frac{d}{4}$ such that $x_J \cup p_{J'}(TM) = u_M \in H^d(M)$ is a generator if $J = J'$ and $x_J \cup p_{J'}(TM) = 0$ for $J \neq J'$.

Without loss of generality, let i_1 be the biggest element of I and let a_1 be the number of elements in I equal to i_1. By assumption, $i_1 \geq m - \frac{d}{4}$ and $p_{I\setminus\{i_1\}}(TM) \neq 0$, since M is P-large. By Section 2.2, there exists a normal invariant η such that the underlying stable vector bundle $\xi \longrightarrow S^k \times M$ has the following Pontryagin classes:

$$\begin{aligned} p_0(-\xi') &= 1 \\ p_{i_1}(-\xi') &= \lambda \cdot u_{4m-d} \times x_{I\setminus\{i_1\}} \\ p_i(-\xi') &= 0 && \text{for } i < i_1 \\ p_i(-\xi') &= \lambda \cdot u_{4m-d} \times \sum_{J\colon |J|=m-i} A_J x_J && \text{for } i > i_1 \end{aligned}$$

for a generator $u_{4m-d} \in H^{4m-d}(S^{4m-d})$ and A_J to be determined later. Note that for $|J| = m - i$, the degree of x_J is $d - 4|J| = 4i - (4m - d)$. The same computation as the first one in the proof of Lemma 3.1 that

$$p_I(TM \oplus -\xi') = \lambda a_1 \cdot u_{4m-d} \times u_M \neq 0.$$

It remains to show that we can choose A_J such that all other monomial Pontryagin numbers are trivial. Now let $I' = (i'_1, \dots, i'_t)$ be different collection with again $|I'| = m$, i'_1 the maximum and a'_1 the number of elements in I' equal to i'_1. Then

$$\begin{aligned} p_{I'}(TM \oplus -\xi') &= \prod_{j=1}^{t} p_{i'_j}(TM \oplus -\xi') = \prod_{j=1}^{t} \sum_{a=0}^{i'_j} p_a(-\xi') \cup p_{i'_j - a}(TM) \\ &= \prod_{j=1}^{t} \left(p_{i'_j}(TM) + \sum_{a=i_1}^{i'_j} p_a(-\xi') \cup p_{i'_j - a}(TM) \right). \end{aligned}$$

If $i'_j < i_1$ for all j, then the sum on the right vanishes and for degree reasons so does the entire expression. If $i'_1 = i_1$, then we get

$$\begin{aligned} p_{I'}(TM \oplus -\xi') &= \prod_{j\colon i'_j < i'_1} p_{i'_j}(TM) \cup \prod_{j\colon i'_j = i'_1} \left(p_{i'_j}(TM) + p_{i'_j}(-\xi') \right) \\ &= \underbrace{p_{I'}(TM)}_{=0} + \prod_{j\colon i'_j < i'_1} p_{i'_j}(TM) \cup \left(\sum_{j\colon i'_j = i'_1} p_{i_1}(-\xi') \cup p_{i_1}(TM)^{a'_1 - 1} \right) \\ &= p_{i_1}(-\xi') \cup p_{I'\setminus\{i_1\}}(TM) \\ &= \lambda \cdot u_{4m-d} \times x_{I\setminus\{i_1\}} \cup p_{I'\setminus\{i_1\}}(TM), \end{aligned}$$

where the second equality follows from the observation that $p_n(\xi) \cup p_{n'}(\xi) = 0$ for $n, n' \geq 1$. By the choice of x_J, we have that $x_{I\setminus\{i_1\}} \cup p_{I'\setminus\{i_1\}}(TM) = 0$, and therefore, $p_{I'}(TM \oplus -\xi') = 0$ unless $I = I'$. It remains to investigate the case $i'_1 > i_1$. The strategy is to choose the coefficients A_J by downwards induction with respect to $|J|$. Note that we have already chosen A_J for $|J| \geq m - i_1$ to be either 0 or 1. Let $J = (j_1, \dots, j_r)$ with $|J| \geq 1$ and let us assume that $A_{J'}$ is already chosen for all J' with $|J'| > |J|$. By the choice of the Pontryagin classes of $-\xi'$, this implies that $p_i(-\xi')$ is already determined for all $i < 4(m - |J|) =: i_J$. If there exists a $j \in J$ such that $j > i_J$, we set $A_J = 0$. If not, let $I' := \{i_J, J\} =: \{i'_1, \ldots, i'_t\}$ and note that by assumption, $|I'| = 4m$ and i'_1 is the largest entry of I'. We again denote the number of indices agreeing with i'_1 by a'_1. We compute

$$\begin{aligned} p_{I'}(TM \oplus -\xi') &= \prod_{\ell=1}^{t}\left(p_{i'_\ell}(TM) + \sum_{a=i_1}^{i'_\ell} p_a(-\xi') \cup p_{i'_\ell - a}(TM)\right) \\ &= \prod_{\ell:\, i'_\ell = i'_1}\left(p_{i'_\ell}(TM) + p_{i'_\ell}(-\xi') + \sum_{a=i_1}^{i'_\ell - 1} p_a(-\xi') \cup p_{i'_\ell - a}(TM)\right) \\ &\quad \cup \prod_{\ell:\, i'_\ell < i'_1}\left(p_{i'_\ell}(TM) + \sum_{a=i_1}^{i'_\ell} p_a(-\xi') \cup p_{i'_\ell - a}(TM)\right). \end{aligned} \tag{8}$$

Extracting all summands containing $p_{i'_1}(-\xi)$, we obtain

$$\begin{aligned} (8) &= a'_1 p_{i'_1}(-\xi') \cup \underbrace{p_{I' \setminus \{i'_1\}}(TM)}_{= p_J(TM)} \\ &+ \underbrace{\left(\prod_{\ell:\, i'_\ell = i'_1} p_{i'_\ell}(TM) + \sum_{a=i_1}^{i'_\ell - 1} p_a(-\xi') \cup p_{i'_\ell - a}(TM)\right) \cup \prod_{\ell:\, i'_\ell < i'_1}\left(\cdots\right)}_{=: \lambda \cdot B_{I'}}. \end{aligned}$$

Note that the highest index of a Pontryagin class of $-\xi'$ appearing in $B_{I'}$ is strictly smaller than $i'_1 = i_J$. Hence, it is only dependent on $A_{J'}$ with $|J'| < |J|$, and by our assumption, this summand is already determined. We get

$$\begin{aligned} p_{I'}(TM \oplus -\xi') &= a'_1 \lambda \cdot u_{4m-d} \times \sum_{\tilde{J}:\, |\tilde{J}| = m - i'_1} A_{\tilde{J}} \underbrace{x_{\tilde{J}} \cup p_J(TM)}_{= \begin{cases} 0 & \text{if } \tilde{J} \neq J \\ u_M & \text{if } \tilde{J} = J \end{cases}} + \lambda \cdot B_{I'} \\ &= \lambda \cdot (a'_1 A_J \cdot u_{4m-d} \times u_M + B_{I'}). \end{aligned}$$

Therefore, we can choose A_J for all J with $|J| = m - i_J$ such that $p_{I'}(TM \oplus -\xi') = 0$ for all I' with $i'_1 = i_J$.[7] It remains to specify $A_{\{0\}}$ and hence $p_m(-\xi)$. By the same argument as in the proof of Lemma 3.2, we can choose $A_{\{0\}}$ such that $\sigma(\eta) = 0$ which finishes the proof. □

Corollary 3.8. *For a simply connected, P-large manifold M of dimension $d \geq 5$, we have*

$$\dim \widetilde{\mathrm{Fib}}^h_{M,4m-d} = p(m) - p\left(m, m - \left\lceil \frac{d+1}{4} \right\rceil\right) - 1.$$

Proof. Since the oriented cobordism ring is classified by Pontryagin numbers, the functionals given by evaluating monomial Pontryagin numbers are all linearly independent. Let $\mathcal{I}_{m,d} := \{I = \{i_1, \ldots i_s\} : I \neq \{m\},\ |I| = m \text{ and } i_j \geq m - d/4 \text{ for some } j\}$. By Lemma 3.7, there exists for every $I \in \mathcal{I}_{m,d}$ an

[7] Note that $I' \neq I$ since $i'_1 > i_1$ by assumption.

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M-block bundle $E_I \to S^k$ such that $\langle p_J(E_I), [E]\rangle \neq 0$ if and only if $I = J$ for all $J \in \mathcal{I}_{m,d}$. Therefore, $(E_I)_{I\in\mathcal{I}_{m,d}}$ is also linearly independent, and the claim follows from

$$|\mathcal{I}_{m,d}| = p(m) - p(m, m - \left\lceil \frac{d+1}{4} \right\rceil) - 1$$

and from the upper bound in Lemma 3.4. □

The proof above also shows the following lemma, which states a similar result but does not require the set of all possible Pontryagin classes to be linearly independent.

Lemma 3.9. *Let M be a manifold such that the set of all nontrivial Pontryagin classes of M are linearly independent and let I be a partition of $\ell \leq d = \dim(M)$ such that $p_I(TM) \neq 0$ is the only nontrivial monomial Pontryagin class of degree $\geq \ell$. Then there exists a normal invariant $\eta \in \mathcal{N}_\partial(D^{4m-d} \times M)$ with underlying stable vector bundle ξ such that*

$$\begin{aligned} &\left\langle p_I(TM \oplus -\xi) \cup p_{4m-|I|}(TM \oplus -\xi), [S^{4m-d} \times M] \right\rangle \\ &\textit{and} \quad \left\langle p_m(TM \oplus -\xi), [S^{4m-d} \times M] \right\rangle \end{aligned}$$

are the only nonvanishing monomial Pontryagin numbers of $TM \oplus -\xi$ and $\sigma(\eta) = 0$.

This lemma can be used to prove the following.

Corollary 3.10. *There exists a manifold M of dimension $d \geq 8$ divisible by 4 which is nonconnected but has simply connected components such that*

$$\dim \widetilde{\mathrm{Fib}}^h_{M,4m-d} = p(m) - p(m, m - \left\lceil \frac{d+1}{4} \right\rceil) - 1.$$

Proof. For $\ell \leq m$, let P_ℓ denote the set of partitions of ℓ and let $P := \cup_{\ell\geq 1} P_\ell$. We now construct for each $I \in P$ a manifold M_I satisfying the hypothesis for Lemma 3.9. This will be done by induction on $|I|$.

For $|I| = 1$, we have $I = (1)$, and hence, $\mathbb{CP}^2 \times S^{d-4}$ does the trick.

If $|I| \geq 2$, let $N_I \times S^{d-|I|}$ be a simply connected manifold of dimension $|I|$ whose only nontrivial monomial Pontryagin number is p_I. Now, we can make all Pontryagin classes of M_I of lower degree linearly independent, and we take connected sums with M_J for $|J| < |I|$.

We define $M = \amalg_{I\in P} M_I$. An application of Lemma 3.9 now yields a that for each partition $I \cup \{m - |I|\}$ of m, there exists a manifold M_I and an M_I-block bundle such that $p_I \cup p_{m-|I|}$ and p_m are its only nontrivial Pontryagin numbers. Taking the trivial bundle for all other components of M hence yields an M-block bundle with the same property. □

Remark 3.11. If $\dim(M) \geq 8$ is divisible by four, then we can even assume that every component of M is 3-connected. This is clear for all components arising as products of $4n$-manifolds and spheres if $n \geq 2$, as one can choose them to be spin and simply perform surgeries to make them 3-connected. Finally, one needs to replace $\mathbb{CP}^2 \times S^{d-4}$ by the product $X \times S^{d-8}$ for X a 3-connected, nullbordant 8-manifold with nonvanishing first Pontryagin class. Such a manifold can be constructed, for example, as the linear S^4-bundle over S^4 for which the associated vector bundle $V \to S^4$ of rank 5 has nontrivial first Pontryagin class.

3.4. Pontryagin numbers for topological bundles

Let us now investigate what happens if we leave the smooth world and ask about ($h-$)sphericity of Pontryagin classes for topological bundles. Obviously, if a class is h-spherical, then it is h-spherical for topological bundles as well.

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Now, let $\pi\colon E \longrightarrow S^k$ be a fiber homotopy trivial, topological M-bundle and let $\rho\colon M \times S^k \longrightarrow E$ be a homotopy equivalence over S^k. Then we obtain an isomorphism $\rho^*\colon H^*(E) \xrightarrow{\cong} H^*(M \times S^k)$ of rational cohomology rings. Therefore, $H^*(E) \cong H^*(M) \oplus u_k \cdot H^{*-k}(M)$ for u_k the cohomological fundamental class of S^k. Therefore, we have $\rho^* p_i(E) = \mathrm{pr}_M^* \, p_i(M) + u_k \cdot a_i$ for some $a_i \in H^*(M)$. Since $u_k^2 = 0$, we have

$$\rho^*(p_{i_1} \cdots p_{i_r}(E)) = \prod_{\ell=1}^{r} (\mathrm{pr}^* \, p_{i_\ell}(M) + u_k \cdot a_{i_\ell}) = \sum_{\ell=1}^{r} p_{i_1} \cdots \widehat{p_{i_\ell}} \cdots p_{i_r}(M) \cdot u_k \cdot a_{i_\ell}.$$

If $p_{i_1} \cdots \widehat{p_{i_\ell}} \cdots p_{i_r}(M) = 0$ for all $\ell \in \{1 \dots r\}$, then this enforces $p_{i_1} \cdots p_{i_r}(E) = 0$, and we obtain the same necessary criterion for sphericity as in Theorem A.

Therefore, if M admits a smooth structure, we have that Theorem A holds for topological M-bundles as well. However, if M is only a topological manifold, one would need a more thorough analysis of topological block bundles and their unblocking in order to prove the analogue of Theorem A.

4. Applications

4.1. The space $\mathcal{R}_C(M)$

Let us now explain the first application of our result to spaces of metrics satisfying positive curvature conditions.

It is well known that the $\hat{\mathcal{A}}$-genus is not multiplicative in fiber bundles: In [HSS14, Proposition 1.10], Hanke–Schick–Steimle constructed a fiber bundle $E \longrightarrow S^k$ with $\hat{\mathcal{A}}(E) \neq 0$. Their construction however ‘is based on abstract existence results in differential topology [and] does not yield an explicit description of the diffeomorphism type of the [. . .] manifold’ [loc. cit. p. 3]. This has been resolved by Krannich–Kupers–Randal-Williams in [KKR21], where they constructed a fiber bundle $\mathbb{HP}^2 \longrightarrow E \longrightarrow S^4$ with $\hat{\mathcal{A}}(E) \neq 0$. Employing part (ii) of Theorem A together with the computational result [FR21, Lemma 2.5], we can go far beyond their result.

Proposition 4.1. *Let M be simply connected and let $k \geq 1$ be in the unblocking range for M such that $d + k = 4m$. Then the following are equivalent:*

(i) *$\hat{\mathcal{A}}_m$ is h-spherical for M.*
(ii) *$\hat{\mathcal{A}}_m$ is spherical for M.*
(iii) *M admits a nontrivial rational Pontryagin class.*

Proof. Let M be an oriented, simply connected manifold with at least one nontrivial Pontryagin class. By the last part of Lemma 3.2 together with Lemma 2.5, there exists an M-bundle $E \longrightarrow S^k$ that has only two nonvanishing monomial Pontryagin numbers – namely, p_m and $p_{i_{\min}} \cup p_{m-i_{\min}}$. By [FR21, Lemma 2.5], this implies that $\hat{\mathcal{A}}(E) \neq 0$, proving the implication $(iii) \Rightarrow (i)$. The observation $(i) \Rightarrow (ii)$ is trivial, and the proof for $(ii) \Rightarrow (iii)$ is the same as for Theorem A; see Remark 1.1 (ii). □

Remark 4.2. This recovers [HSS14, Theorem 1.4] and provides an upgrade: Not only are we able to identify the diffeomorphism type of the fiber, but our result states that it is correct for generic manifolds.

Next, let us investigate if the bundles we constructed have cross-sections with trivial normal bundle. This is sometimes desirable for applications to positive curvature, as it allows for fiber-wise connected sums. We have the following result.

Lemma 4.3. *If $i_{\min} < d/4$, then the bundle from Proposition 4.1 has a cross-section with trivial normal bundle.*

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Proof. Let triv: $S^k \hookrightarrow S^k \times M$ be the trivial section. Since the bundle E from Proposition 4.1 is fiber homotopy trivial via $f\colon S^k \times M \simeq E$, we get a section $s := f \circ \text{triv}\colon S^k \to E$. We have

$$s^* p_n(TE) = \text{triv}^* \left(\sum_{i=0}^{n} p_i(TM) \cup p_{n-i}(-\xi') \right)$$
$$= \sum_{i=0}^{n} \underbrace{\text{triv}^* p_i(TM)}_{=0 \text{ for } i \geq 1} \cup \text{triv}^* p_{n-i}(-\xi') = \text{triv}^* p_n(-\xi')$$

with ξ' as in the proof of Lemma 3.2. Recall that the only nonvanishing Pontryagin classes of ξ' are $p_{m-i_{\min}}$ and p_m and let ν_s denote the normal bundle of s. Since the rank of this bundle is bigger than k, the bundle ν_s is stable in the sense that it is classified by an element in

$$\pi_k(\text{BO}) = \text{KO}^{-k}(\text{pt}) \cong \begin{cases} \mathbb{Z} & \text{for } k \equiv 0\ (4) \\ \mathbb{Z}/2 & \text{for } k \equiv 1, 2\ (8) \\ 0 & \text{otherwise.} \end{cases}$$

Since we are only interested in the problem rationally, it suffices to consider the case $k \equiv 0$ (4). It follows that ν_s is trivial if $p_{k/4}(\nu_s) = 0$, and as $p(S^k) = 1$, the Pontryagin class $p_{k/4}$ of ν_s satisfies

$$p_{k/4}(\nu_s) = p_{k/4}(s^* TE) = s^* p_{k/4}(TE) = \text{triv}^* p_{k/4}(\xi) = 0$$

since by our assumption, $k/4 < \frac{d+k}{4} - i_{\min} = m - i_{\min}$ and $p_{m-i_{\min}}$ and p_m are the only Pontryagin classes of ξ. □

Remark 4.4. If $d \not\equiv 0$ (4), the requirement from the lemma is automatically fulfilled. If $d \equiv 0$ (4) and $i_{\min} = d/4$, then M has only one nonvanishing Pontryagin number – namely, $\langle p_{d/4}(TM), [M] \rangle$. Since all coefficients in the $\hat{\mathcal{A}}$-polynomial are nonzero by [BB18], we have $\hat{\mathcal{A}}(M) = a \cdot \langle p_{d/4}(TM), [M] \rangle \neq 0$ for some $a \in \mathbb{Z} \setminus \{0\}$. If additionally M admits a Spin-structure, then by the Lichnerowicz-formula and the Atiyah–Singer index theorem [AS63; Lic63], M does not support a metric of positive scalar curvature. Hence, for a Spin-manifold of positive scalar curvature, we have $i_{\min} < d/4$ and Lemma 4.3 applies.

4.1.1. Spin-structures and positive (scalar) curvature

Let M be Spin and let $B\text{Diff}^{\text{Spin}}(M)$ be the classifying space for M-bundles with a Spin-structure on the vertical tangent bundle.[8] By [Ebe06, Lemma 3.3.6], the homotopy fiber of the forgetful map $B\text{Diff}^{\text{Spin}}(M) \to B\text{Diff}^+(M)$ is a $K(\mathbb{Z}/2, 1)$ if M is simply connected. Therefore, the induced map

$$\pi_n(B\text{Diff}^{\text{Spin}}(M)) \otimes \mathbb{Q} \longrightarrow \pi_n(B\text{Diff}^+(M)) \otimes \mathbb{Q}$$

is an isomorphism, and we may assume without loss of generality that the bundles from Section 3 carry a Spin-structure on the vertical tangent bundle and hence on the total space, provided that M admits one. Theorem E then follows from Proposition 4.1 by a standard argument that goes back to Hitchin [Hit74] (see [HSS14, Remark 1.5] or [FR21, Proposition 3.7]).

Example 4.5.

(i) The class of manifolds to which Theorem E is applicable contains $\mathbb{CP}^{2n+1}$, $\mathbb{HP}^n$, $\mathbb{OP}^2$, as well as iterated products and connected sums of these with *arbitrary* Spin-manifolds.

(ii) The most interesting examples of spaces $\mathcal{R}_C(M)$ are the ones of positive or nonnegative sectional curvature and positive Ricci curvature metrics. Theorem E implies that for M and $k \geq 2$ as above, we

[8] $B\text{Diff}^{\text{Spin}}(M)$ is given by the homotopy quotient $\text{Bun}(TN, \theta^* U_d) /\!\!/ \text{Diff}^+(M)$ for $\theta\colon B\text{Spin}(d) \to BSO(d)$ the 2-connected cover, $U_d \to BSO(d)$ the universal oriented vector bundle and $\text{Bun}(\_, \_)$ the space of bundle maps.

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have $\pi_{k-1}(\mathcal{R}_C(M)) \otimes \mathbb{Q} \neq 0$ for $C = \{\mathrm{Sec} > 0, \mathrm{Ric} > 0, \mathrm{Sec} \geq 0\}$,provided the respective spaces are nonempty. The case of nonnegative sectional curvature follows from a Ricci-flow argument; see [FR21, Proposition 3.3].

According to [Zil14], the only known examples of positively curved manifolds in dimensions $4k+3$ for $k \geq 2$ are spheres. Also, all 7-dimensional examples have finite fourth cohomology (cf. [Esc92; Goe14; GKS04]). Therefore, the answer to the following question appears to be unknown. A positive answer would yield the first example of a closed manifold that admits infinitely many pairwise non-isotopic metrics of positive sectional curvature.

Question 4.6. *Is there a positively curved manifold of dimension $4k+3$, $k \geq 1$ with a nonvanishing rational Pontryagin class?*

For positive Ricci and nonnegative sectional curvature, lots of examples for such manifolds are known; see Example 4.5.

Applying Theorem E, we get the following classification for the push-forward action on metrics of positive scalar curvature which uses rigidity results from [ER22] and [Fre21].

Corollary 4.7. *Let M be a 2-connected,*[9] *d-dimensional* Spin*-manifold of positive scalar curvature and let k be in the unblocking range for M.*

(i) *Then the orbit map $\pi_{k-1}\,\mathrm{Diff}^+(M, D) \longrightarrow \pi_{k-1}(\mathcal{R}_{\mathrm{scal}>0}(M))$ factors through a finite group if and only if $(d+k)$ is not divisible by four or all rational Pontryagin classes of M vanish.*
(ii) *If $k = 1$, then the same holds for $\mathrm{Diff}^+(M)$ instead of $\mathrm{Diff}^+(M, D)$.*

Proof. The orbit maps

$$\pi_{k-1}(\mathrm{Diff}^+(M, D)) \longrightarrow \pi_{k-1}(\mathcal{R}_{\mathrm{scal}>0}(M))$$

factor through finite groups if all Pontryagin classes of M vanish by [ER22, Theorem F]. Furthermore, $\pi_0(\mathrm{Diff}^+(M)) \longrightarrow \pi_0(\mathcal{R}_{\mathrm{scal}>0}(M))$ factors through a finite group by [Fre21, Theorem A]. The rest follows from Theorem E. □

Theorem E also allows to recover the main result from [HSS14] (loc.cit. Theorem 1.1 a) and is actually slightly more precise on the dimension restriction.

Corollary 4.8. *Let $k \geq 1$ and let N be a* Spin*-manifold of positive scalar curvature such that $d+k \equiv 0$ (4) and k is in the unblocking range for N. Then the group $\pi_{k-1}(\mathcal{R}_{\mathrm{scal}>0}(N))$ contains an element of infinite order (resp. is infinite if $k = 1$).*

Proof. Let K be a $K3$-surface. Then for $n := d-4 \geq 2$, the manifold $K \times S^n$ satisfies the hypothesis of Theorem E and there is a $K \times S^n$-bundle $E \longrightarrow S^k$ that has nonvanishing $\hat{\mathcal{A}}$-genus and admits a cross-section with trivial normal bundle. Gluing in the trivial $N \setminus D^d$-bundle along this cross-section yields an $N\#(K \times S^{d-4})$-bundle over S^k with nonvanishing $\hat{\mathcal{A}}$-genus. Hence, the group $\pi_{k-1}(\mathcal{R}_{\mathrm{scal}>0}(N\#(K \times S^{d-4})))$ contains an element of infinite order. Since N is cobordant to $N\#(K \times S^{d-4})$ in $\Omega^d_{\mathrm{Spin}}(B\pi_1(N))$, the corresponding spaces of positive scalar curvature metrics are homotopy equivalent by [EF21, Theorem 1.5]. □

Remark 4.9. A more general result without any dimension restriction has been proven by Botvinnik–Ebert–Randal-Williams [BER17]. The methods from loc.cit. are, however, not constructive and do not give a way to decide if the obtained elements arise from the orbit of the action $\mathrm{Diff}^+(M) \curvearrowright \mathcal{R}_{\mathrm{scal}>0}(M)$. Furthermore, it is unclear if those elements originate from the spaces $\mathcal{R}_{\mathrm{Ric}>0}(M)$ or $\mathcal{R}_{\mathrm{Sec}>0}(M)$.

[9]The assumption of 2-connectedness stems from employing [ER22, Theorem F], which works for manifolds W where the inclusion $\partial W \hookrightarrow W$ is 2-connected.

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4.2. Obstructions to unblocking

One observation from the proofs in Section 3 is that the construction of block bundles works regardless of the dimension k of the base sphere. The same cannot be true for actual fiber bundles over spheres since the tangent bundle of the total space is stably isomorphic to its vertical tangent bundle which is a vector bundle of rank d. Therefore, all Pontryagin classes of degree $* > 2d$ vanish. Let $I = (i_1, \dots, i_s)$ be such that $|I| := i_1 + \dots + i_s < m$ and let M be a manifold such that $p_I(TM) = p_{i_1} \cdot \dots \cdot p_{i_s}(TM) \neq 0$. If $I = (0)$, assume instead that M has some nonvanishing rational Pontryagin class. Let

$$\widetilde{P_{M,I}} : \pi_{4m-d}(B\widetilde{\mathrm{Diff}}^+(M)) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$

be the map sending a block bundle E classified by f to the Pontryagin number $\langle p_I(TE) \cup p_{m-|I|}(TE), [E] \rangle$. The following corollary follows from the proof of Theorem A and Theorem B in Section 3.

Corollary 4.10.

(i) *The map $\widetilde{P_{M,I}}$ is surjective.*
(ii) *If $m > \frac{d+2|I|}{2}$, then the following composition is trivial:*

$$\pi_{4m-d}(B\mathrm{Diff}^+(M)) \otimes \mathbb{Q} \longrightarrow \pi_{4m-d}(B\widetilde{\mathrm{Diff}}^+(M)) \otimes \mathbb{Q} \xrightarrow{\widetilde{P_{M,I}}} \mathbb{Q}.$$

(iii) *For $1 \leq n \leq n_{\max}$ (cf. Equation (5)) and $m > \frac{d+2n\cdot i_{\min}}{2}$, there exists an n-dimensional subspace $\mathcal{N} \subset \pi_{4m-d}(B\widetilde{\mathrm{Diff}}^+(M)) \otimes \mathbb{Q}$ of block-bundles that do not admit the structure of actual fiber bundles.*

Remark 4.11. The same is true for fiber homotopy trivial bundles; that is, the same corollary holds if $B\mathrm{Diff}^+(M)$ and $B\widetilde{\mathrm{Diff}}^+(M)$ are replaced by $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ and $\mathrm{hAut}^+(M)/\widetilde{\mathrm{Diff}}^+(M)$.

Proof of Corollary 4.10.

(i) This is precisely Lemma 3.1.
(ii) By assumption, $m - |I| > \frac{d+2|I|}{2} - |I| = \frac{d}{2}$, and if $E \to S^{4m-d}$ is a fiber bundle, then $p_{m-|I|}(TE) = p_{m-|I|}(T_\pi E) = 0$ since $T_\pi E$ is a vector bundle of rank d and therefore its highest possible Pontryagin class is $p_{\lfloor d/2 \rfloor}$.
(iii) By Lemma 3.2, there exist (homotopy trivial) block bundles E_ℓ for $1 \leq \ell \leq n$ with

$$\langle p_{i_{\min}}(TE_\ell)^r \cup p_{m-r\cdot i_{\min}}(TE_\ell), [E_\ell] \rangle \neq 0 \qquad \Longleftrightarrow \qquad r = \ell$$

which yield linearly independent classes of $\Omega_{4m} \otimes \mathbb{Q}$ and hence in $\pi_{4m-d}(B\widetilde{\mathrm{Diff}}^+(M)) \otimes \mathbb{Q}$. Furthermore, by assumption, $m - r \cdot i_{\min} > \frac{d+2n\cdot i_{\min}}{2} - n \cdot i_{\min} = \frac{d}{2}$. The same argument as in (ii) shows that these block bundles do not admit the structure of actual fiber bundles.

□

4.3. Sphericity of κ-classes

Next, consider $\mathrm{Diff}^+(M)$ the group of orientation preserving diffeomorphisms of M equipped with the Whitney C^∞-topology and $B\mathrm{Diff}^+(M)$ the associated classifying space. Since M is assumed to be closed, for any M-bundle $\pi: E \to B$ with structure group $\mathrm{Diff}^+(M)$, there is a map

$$H^{4m}(\mathrm{BO}(d); \mathbb{Q}) \to H^{4m-d}(B; \mathbb{Q})$$

sending a characteristic class $c \in H^{4m}(\mathrm{BO}(d); \mathbb{Q})$ to $\kappa_c(E) := \pi_!(c(T_\pi E))$ where $\pi_!: H^*(E) \to H^{*-d}(B)$ is the Gysin-homomorphism and $T_\pi E$ is the vertical tangent bundle of π. $\kappa_c(E)$ is called the *generalized Miller-Morita-Mumford class* or simply *κ-class* associated to c. For the universal M-bundle $\pi_M : E_M \to B\mathrm{Diff}^+(M)$, we hence get universal κ-classes

$$\kappa_c := \kappa_c(E_M) \in H^{4m-d}(B\mathrm{Diff}^+(M); \mathbb{Q}).$$

Let $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$ be the classifying space for fiber homotopy trivial M-bundles. We define the maps $\Psi_{M,m}$ and $\Psi^h_{M,m}$ as follows:

$$\begin{array}{ccl} H^{4m}(\mathrm{BO}(d);\mathbb{Q}) & \xrightarrow{(\pi_M)_!} & H^{4m-d}(B\mathrm{Diff}^+(M);\mathbb{Q}) \cong \hom(H_{4m-d}(B\mathrm{Diff}^+(M));\mathbb{Q}) \\ & & \qquad \downarrow \mathrm{hur}^* \\ & \overset{\Psi_{M,m}}{\dashrightarrow} & \hom(\pi_{4m-d}(B\mathrm{Diff}^+(M));\mathbb{Q}) \\ & & \qquad \downarrow \\ & \overset{\Psi^h_{M,m}}{\dashrightarrow} & \hom(\pi_{4m-d}(\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M));\mathbb{Q}) \end{array}$$

where hur denotes the Hurewicz-homomorphism. Obviously, $\ker(\Psi_{M,m}) \subset \ker(\Psi^h_{M,m})$. Since the signature of a fiber bundle over S^k vanishes, the Hirzebruch $\mathcal{L}$-class lies in $\ker(\Psi_{M,m})$ for all M and m. The maps $\Psi^h_{M,m}$ and $\Psi_{M,m}$ are both given by $c \mapsto (f \mapsto \langle f^*\kappa_c, [S^{4m-d}]\rangle)$, and we have

$$\begin{aligned} \langle f^*\kappa_c, [S^{4m-d}]\rangle &= \langle \kappa_c(E), [S^{4m-d}]\rangle = \langle \pi_! c(T^s_\pi E), [S^{4m-d}]\rangle \\ &= \langle c(T^s_\pi E), [E]\rangle = \langle c(\pi^* TS^{4m-d} \oplus T^s_\pi E), [E]\rangle = \langle c(TE), [E]\rangle \end{aligned}$$

since TS^{4m-d} is stably parallelizable and hence $c(TS^{4m-d}) = 1$. With this, Theorem A translates to the following.

Theorem 4.12. *Let M be simply connected and let $\tau\colon M \to \mathrm{BO}(d)$ be a classifying map for TM. Furthermore, let $k \geq 1$ be in the unblocking range for M and let $d + k = 4m$. Then*

(i) *If $p = p_{i_1} \cdots p_{i_r} \neq p_m$ is a monomial in universal Pontryagin classes of degree $4m$, then*

$$p \in \ker(\Psi^h_{M,m}) \iff \tau^*(p_{i_1} \cdots \widehat{p_{i_\ell}} \cdots p_{i_r}) = 0 \text{ for all } \ell.$$

(ii) *The following are equivalent:*
  (a) $p_m \in \ker(\Psi^h_{M,m})$
  (b) $p_m \in \ker(\Psi_{M,m})$
  (c) $\tau^* p_i = 0$ *for all* $i \geq 1$

(iii) *If $\sum i_j = m \geq 3$ and $i_j < m/2$ for all j, then*

$$p_{i_1} \cdots p_{i_n} \in \ker(\Psi^h_{\mathbb{CP}^m,m}) \setminus \ker(\Psi_{\mathbb{CP}^m,m}).$$

Furthermore, we obtain the following bounds on the dimension of $\ker(\Psi^h_{M,m})$: For $n_{\max}$ and $p(m,\ell)$ as above, we have

$$p(m, m - \left\lceil \frac{d+1}{4} \right\rceil) + 1 \leq \dim \ker(\Psi^h_{M,m}) \leq p(m) - n_{\max},$$

where the upper bound is attained for manifolds for which all Pontryagin classes are contained in the polynomial algebra generated by $p_{i_{\min}}(TM)$.

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A. Rationally fibering a cobordism class over a sphere (with Jens Reinhold)

This appendix promotes the problem of studying the ideal of oriented cobordism classes that have a representative fibering over a sphere of fixed dimension. Such a class also fibers over any manifold of smaller dimension; see Proposition A.5. An answer thus has consequences for other bases, too. We are only interested in the rational version. It turns out that the results from the present paper can be used to say something new about this problem, which has been solved (even integrally) for dimensions at most 4 some time ago: in this case, the rational answer is that a cobordism class fibers over S^k for $k \leq 4$ if and only if its signature vanishes [Bur66; Neu71; Kah84a; Kah84b]. A variant of the analogous problem without orientations was originally introduced by Connor and Floyd [CF65]. We describe a construction that goes beyond the way bundles arise in the preceding paper. Unfortunately, it seems as if even both ideas combined are not sufficient to solve the problem completely unless $k \leq 8$. We first outline a more concrete version of the problem. Let Ω_* denote the (graded) oriented cobordism ring.

Definition A.1.

(i) An oriented cobordism class $\alpha \in \Omega_*$ is said to *fiber over* a manifold B if there is an oriented smooth fiber bundle $M \to E^d \to B$ such that $[E] = \alpha$.
(ii) For $k \geq 1$, let $A^k_* \subset \Omega_*$ denote the graded subgroup spanned by cobordism classes that fiber over S^k and $\widetilde{A}^k_*$ the respective subgroup spanned by block-fibering classes.
(iii) For given $k, m \geq 1$, define $c^k(m) \in \mathbb{Z}$ by

$$c^k(m) := \dim_{\mathbb{Q}}(\Omega_{4m} \otimes \mathbb{Q}) - \dim_{\mathbb{Q}}(A^k_{4m} \otimes \mathbb{Q}) - 1$$
$$\widetilde{c}^k(m) := \dim_{\mathbb{Q}}(\Omega_{4m} \otimes \mathbb{Q}) - \dim_{\mathbb{Q}}(\widetilde{A}^k_{4m} \otimes \mathbb{Q}) - 1.$$

Since any fiber bundle is also a block bundle, we have $\widetilde{c}^k(m) \geq c^k(m)$. Forming disjoint unions and products, we see that A^k_* and $\widetilde{A}^k_*$ are ideals in Ω_* and we may ask what how ideals depend on k. As the signature of any manifold that (block-)fibers over a sphere vanishes, the two maps

$$A^k_* \otimes \mathbb{Q} \hookrightarrow \Omega_* \otimes \mathbb{Q} \xrightarrow{\mathcal{L}_*} \mathbb{Q}$$

compose to 0 as do the corresponding ones for $\widetilde{A}^k_*$. Since there exist manifolds of nonzero signature in any dimension divisible by 4, this implies $c^k(m)$ and $\widetilde{c}^k(m)$ are both nonnegative. We may ask if the above sequence is exact in sufficiently high degrees, or equivalently (see part (ii)):

Problem A.2.

(i) Describe the ideals $A^k_* \subset \Omega_*$ for all values of k.
(ii) Is $c^k(m) = 0$ for fixed k and sufficiently large m?

We will see below that we (at least) need to restrict to degrees $m \geq k/2$ for (ii) to be true: there are more constraints than the vanishing of the signature in lower degrees; see Proposition A.7. The following is our contribution toward an answer the block-version of Problem A.2.

Theorem A.3. *Let $k \geq 1$ be fixed.*

(i) *We have $\widetilde{c}^k(m) = \dim \Omega_{4m} \otimes \mathbb{Q} - 1$ for $m < \frac{3k}{8}$, and $\widetilde{c}^k(m) \geq 1$ for $m < \frac{k}{2}$.*
(ii) *For $5 \leq k \leq 8$, we have $\widetilde{c}^k(m) = 0$ for $m \geq k$.*
(iii) *For $9 \leq k \leq 12$, we have $\widetilde{c}^k(m) \leq 6$ in degrees $m \geq k$.*

Regarding the last item, we note that from computer-aided calculations, we know that $c^k(m) = 0$ for $k \leq 12$ and $m \leq 1250$; see Remark A.11.

We prove Theorem A.3 below. Before doing so, let us elaborate on the consequences of the preceding paper regarding a partial answer to Problem A.2 for bigger values of k: sharpness of the upper bound from Theorem D can be reformulated as $\widetilde{c}^k(m) \leq p(m, \lfloor (k-1)/4 \rfloor)$. Note that $p(n, \ell) = p(n - \ell, \ell) +$

$p(n, \ell - 1)$. Using $p(n, 1) = 1$, a simple induction shows that $p(n, \ell) = \mathcal{O}(n^{\ell-1})$, which yields the following consequence of Theorem D (see also Remark 1.3).

Corollary A.4. *For $k \geq 1$, $k \not\equiv 1(4)$ and $m > k$, we have*

$$c^k(m) \leq p(m, \lfloor (k-1)/4 \rfloor) = \mathcal{O}(m^{\lfloor (k-1)/4 \rfloor - 1}).$$

Note that for $k \equiv 1(4)$, we also have $c^k(m) = \mathcal{O}(m^{\lfloor (k-1)/4 \rfloor - 1})$ by the observation in Proposition A.5. The rest of the appendix is devoted to proving Theorem A.3.

Elementary observations

We first collect some elementary facts about the ideals $A_*^k \subset \Omega_*$.

Proposition A.5. *A cobordism class that fibers over S^k fibers over any $(k-1)$-manifold B.*

Proof. Cutting out a $(k+1)$-disk from an oriented nullbordism of $B \times S^1$, we see that S^k and $B \times S^1$ can be joined by a connected oriented cobordism W. Applying obstruction theory to a relative CW-decomposition of (W, S^k) and using that the obstructions lie in $H^j(W, S^k; \pi_{j-1}(S^k)) = 0$ for all j, we see that W retracts onto S^k. For any smooth oriented bundle $M \to E \to S^k$, we can thus extend the classifying map $S^k \to B\,\mathrm{Diff}^+(M)$ to W. Restricting this extension to the other end of the cobordism gives an oriented bundle $M \to E' \to B \times S^1$ with $[E'] = [E] \in \Omega_{4m}$. Since E' clearly also fibers over B, this finishes the proof. □

Remark A.6. Replacing $B\mathrm{Diff}^+(M)$ by $\mathrm{hAut}^+(M)/\mathrm{Diff}^+(M)$, the same proof yields that a fiber homotopy trivial M-bundle over S^k is cobordant to a fiber homotopy trivial $M \times S^1$-bundle over B.

Proposition A.5 implies that $(A_*^k)_{k \geq 1}$ forms a decreasing chain of ideals of Ω_*. We next prove part (i) of Theorem A.3.

Proof. (cf. [Wie21, Lemma. 2.3]) We need to show that for any smooth bundle $\pi\colon E^{4m} \to S^k$ with $d := (4m-k)$-dimensional fiber M such that $4m < \frac{3}{2}k$, we have $[E] = 0 \in \Omega_{4m} \otimes \mathbb{Q}$.

Since the tangent bundle TE is stably isomorphic to the vertical tangent bundle $T_\pi E$ whose dimension is d, we deduce that only Pontryagin classes p_i with $i \leq 2d$ can be nonzero.

Analyzing the Serre spectral sequence of the fibration $M \to E \to S^k$ yields that E has no cohomology in degrees $d < * < k$, and hence, $p_i(TE) = 0$ for $i > d$. If $p(TE) := p_{i_1}(TE) \cup \cdots \cup p_{i_s}(TE)$ now is a monomial in Pontryagin classes, then $i_j \leq d$ for all j. Hence, if the degree of $p(TE)$ is at least k, some sub-product of $p(TE)$ has degree in the range $(d, 2d] \subset (d, k)$ and hence vanishes. Therefore, all composite Pontryagin numbers of E are zero, and we deduce that E is rationally nullbordant.

The second part of the assertion immediately follows from Proposition A.7 that we state and prove next. □

Proposition A.7. *Let $\pi\colon E^{4m} \to S^k$ be a fiber bundle with $4m < 2k$. Then $p_i(TE) = 0$ for all $i > 2m - \frac{k}{2}$. (The inequality ensures this number is smaller than m.)*

Proof. Let $T_\pi E$ be the vertical tangent bundle of π. We have

$$p_i(TE) = p_i(T_\pi E \oplus TS^k) = p_i(T_\pi E).$$

Now $\mathrm{rank}(T_\pi E) = 4m - k$, and so any $p_i(T_\pi E)$ with $i \geq \frac{1}{2}(4m-k)$ vanishes. □

Bundles that are trivial as fibrations

Note that the construction from Section 3 yield bundles that are trivial as fibrations. For such bundles, the following vanishing result holds, which implies that the analogue of Problem A.2 (ii) for fiber-homotopically trivial bundles has a negative answer.

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Proposition A.8. *For a fiber-homotopically trivial bundle $M \to E^{4m} \to B$ whose base space B is 4ℓ-connected and $p \in H(BSO(4m); \mathbb{Q})$ a monomial in Pontryagin classes p_i with $i \leq \ell$, the Pontryagin number $p(E)$ vanishes.*

Proof. From the assumption that the bundle is trivial as a fibration, we deduce that $E \simeq M \times B$. In particular, we get a retraction $E \to M$ for the inclusion of a fiber. But B is 4ℓ-connected, and hence, all Pontryagin classes p_i with $i \leq \ell$ pull back along this map. Since $H^{4m}(M) = 0$, this implies the assertion. □

Constructing a bundle that is non-trivial as a fibration

In this section, we construct for any $m \geq 1$ a bundle $\mathbb{C}P^m \to E \to S^{2m}$ so that $p_1^m(E) \neq 0$ if $m \geq 3$. We have seen in Proposition A.8 that the latter is not possible for bundles that are trivial as fibrations.

Construction A.9. *Let $m \geq 1$. We construct a smooth $\mathbb{C}P^m$-bundle over S^{2m} as follows. The topological group $\mathrm{GL}_m(\mathbb{C})$ acts on*

$$\mathbb{C}P^m = \{[z_0 : z_1 : \ldots : z_m] \mid z_i \in \mathbb{C} \text{ not all } 0\}$$

by acting linearly on the last m projective coordinates. This action fixes the point $* := [1 : 0 : \ldots : 0]$ and induces a map

$$B\mathrm{GL}_m(\mathbb{C}) \to B\mathrm{Diff}(\mathbb{C}P^m, *).$$

The action of a differential on the tangent space of this fixed point produces a map

$$B\mathrm{Diff}(\mathbb{C}P^m, *) \to B\mathrm{GL}_{2m}(\mathbb{R}),$$

and it is evident that the composition of these two maps is the canonical map $B\mathrm{GL}_m(\mathbb{C}) \to B\mathrm{GL}_{2m}(\mathbb{R})$ induced from seeing $\mathbb{C}$ as a 2-dimensional real vector space. We now choose a complex m-dimensional vector bundle over S^{2m}, classified by a map $S^{2m} \to B\mathrm{GL}_m(\mathbb{C})$, whose underlying $2m$-dimensional real vector bundle ξ has a nonzero Euler number. When composed with the previous map, we obtain a map classifying a smooth bundle $\mathbb{C}P^m \to E \to S^{2m}$. Note that this bundle admits a section $s\colon S^{2m} \to E$ since it has a fixed point and the normal bundle of s is precisely given by ξ.

Proposition A.10. *If $m \geq 3$, then the bundle from Construction A.9 satisfies and we have*

$$\begin{aligned} p_1(E)^m &\neq 0 \\ p_i(E) &= \binom{m+1}{i} \cdot \left(\frac{1}{m+1}\right)^i \cdot p_1(E)^i \neq 0 \quad \text{for } 2i < m. \end{aligned}$$

In particular, $p_{i_1} \cup \cdots \cup p_{i_s}(E) \neq 0$ if $\sum i_\ell = m$ and $i_\ell < m/2$ for all ℓ.

Proof. Choose a generator $\alpha \in H^2(\mathbb{C}P^m)$. Then there exists a unique class $\beta \in H^2(E)$ that pulls back to α along the inclusion $j\colon \mathbb{C}P^m \to E$ of the fiber. The Serre spectral sequence of the bundle $\mathbb{C}P^m \to E \xrightarrow{\pi} S^{2m}$ collapses since the E^2 page is supported in even degrees. Since we are considering cohomology with rational coefficients, there is no extension problem to solve, and we see that β^m is Poincaré dual to a nonzero multiple of $s_*[S^{2m}]$, where $[S^{2m}] \in H_{2m}(S^{2m})$ denotes the fundamental class and $s\colon S^{2m} \to E$ is the cross-section. We thus get that $\beta^{2m} \in H^{4m}(E)$ is Poincaré dual to the self-intersection number of S^{2m} in E which equals the Euler number of its normal bundle ξ, and hence is nonzero by construction. Since S^{2m} has a trivial tangent bundle, we deduce $j^* p_1(E) = p_1(\mathbb{C}P^m) = (m+1)\alpha^2$; hence, $p_1(E) = (m+1)\beta^2$. Hence, indeed, $p_1^m(E) = (m+1)^{2m}\beta^{2m} \neq 0$.

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The class $p_i(E)$ is computed as follows:

$$j^*p_i(E) = p_i(\mathbb{CP}^{2m}) = \binom{m+1}{i}a^{2i} = \binom{m+1}{i}\left(\frac{1}{m+1}\right)^i((m+1)a^2)^i$$
$$= \binom{m+1}{i}\left(\frac{1}{m+1}\right)^i p_1(\mathbb{CP}^{2m})^i = \binom{m+1}{i}\left(\frac{1}{m+1}\right)^i j^*p_1(E)^i.$$

Since j^* is injective in degrees smaller than $2m-1$, the claim follows. The final part follows from

$$p_{i_1}\cup\cdots\cup p_{i_s}(E) = \prod_{\ell=1}^{s}\binom{m+1}{i_\ell}\left(\frac{1}{m+1}\right)^{i_\ell} p_1(E)^{i_\ell}$$
$$= p_1(E)^m \cdot \underbrace{\prod_{\ell=1}^{s}\binom{m+1}{i_\ell}\left(\frac{1}{m+1}\right)^{i_\ell}}_{\neq 0} \neq 0. \qquad \square$$

Proof of Theorem A (iii). Consider the bundle $\mathbb{CP}^k \longrightarrow E \longrightarrow S^{2k}$ constructed in Proposition A.10. Taking the product with $\mathbb{CP}^{2\ell}$, we obtain a fiber bundle $\tilde{E} := E\times\mathbb{CP}^{2\ell} \longrightarrow S^{2k}$ with fiber $\mathbb{CP}^{2\ell}\times\mathbb{CP}^n$. We have

$$p_1(T\tilde{E})^{k+\ell} = (p_1(T\mathbb{CP}^n)\times 1 + 1\times p_1(TE))^{k+\ell}$$
$$= \binom{k+\ell}{k} p_1(T\mathbb{CP}^n)^\ell \times p_1(TE)^k \neq 0. \qquad \square$$

We next prove part (ii) of Theorem A.3.

Proof. Assume that $5\le k\le 8$ and $m>k$. Then Corollary A.4 says that $\widetilde{c}^k(m)\le 1$. We want to improve this to $\widetilde{c}^k(m)=0$. To do so, observe that the proof of Corollary A.4, which was simply a reformulation of (the sharpness of the upper bound of) Theorem D, only involved bundles that are trivial as fibrations. For any such bundle, we know from Proposition A.8 that $p_1^m(E)=0$. However, the bundle arising from Construction A.9 satisfies $p_1^m(E)\neq 0$; we have thus found another element in A_{4m}^k which is not fiber homotopy trivial, and so we have finished the proof. $\square$

Finally, we prove part (iii) of Theorem A.3. Before doing so, let us observe that for $k\le 12$, Theorem D yields bundles $E\longrightarrow S^k$ with all possible monomial Pontryagin classes except for $p_1^a\cup p_2^j$. By Proposition A.8, these classes cannot be realized by fiber-homotopically trivial bundles and are hence the ones we need to construct.

Proof. For $i=1,\dots,m$, let $E_i\longrightarrow S^{2i}$, denote the $\mathbb{CP}^i$-bundle from Construction A.9. First, note that for $i\ge 5$, the map $j^*|_{H^8(E)}$ is invertible, and we have

$$p_2(E_i) = (j^*)^{-1}p_2(\mathbb{CP}^{2i}) = \frac{i}{2}(j^*)^{-1}p_1(\mathbb{CP}^{2i})^2 = \frac{i}{2}p_1(E_i)^2.$$

Next, let Q_i be a manifold of dimension $4(m-i)$ such that $p_1^{m-i}(Q_i)\neq 0$ is the only nonvanishing Pontryagin number and let $X_i := E_i\times Q_i$. Note that X_i is a fiber bundle with fiber $\mathbb{CP}^{2i}\times Q_i$ over S^{2i}. We consider the following matrix:

$$B^m := \left(p_2^j(X_i)\cdot p_1^{m-2j}(X_i)\right)_{\substack{i=6\dots m\\ j=0\dots\lfloor\frac{m}{2}\rfloor}}$$

Note that the rank of B^m determines the size of the subspace spanned by the X_i's in $\Omega_{4m}\otimes\mathbb{Q}$. Since X_i is not fiber homotopy trivial, this subspace is complementary to the one spanned by the bundles constructed in Section 3. Therefore, if $\mathrm{rank}(B^m) \geq \lfloor \frac{m}{2} \rfloor + 1 - a$ and $k \leq 12$, then

$$\begin{aligned} \widetilde{c}^k(m) &= \dim(\Omega_{4m} \otimes \mathbb{Q}) - \dim(A^k_{4m} \otimes \mathbb{Q}) - 1 \\ &= p(m, \left\lfloor \frac{k-1}{4} \right\rfloor) - \mathrm{rank}(B^m) - 1 \\ &\leq p(m,2) - \lfloor m/2 \rfloor + 1 - a - 1 = a. \end{aligned}$$

Now, let us embark the computation of the entries of B^m:

$$\begin{aligned} p_2^j(X_i) \cdot p_1^{m-2j}(X_i) &= p_2^j(E_i \times Q_i) \cdot p_1^{m-2j}(E_i \times Q_i) \\ &= \Big(p_2(E_i) + p_1(E_i)p_1(Q_i) + p_2(Q_i)\Big)^j \cdot \Big(p_1(E_i) + p_1(Q_i)\Big)^{m-2j}. \end{aligned}$$

By our choice of Q_i, any product containing $p_2(Q_i)$ will vanish, and therefore, we can go on with our computation.

$$\begin{aligned} &= \Big(p_2(E_i) + p_1(E_i)p_1(Q_i)\Big)^j \cdot \Big(p_1(E_i) + p_1(Q_i)\Big)^{m-2j} \\ &= p_1(Q_i)^{m-i} \cdot \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} p_2(E_i)^n \cdot p_1(E_i)^{j-n} \cdot p_1(E_i)^{i-(j+n)} \cdot \binom{j}{n}\binom{m-2j}{i-(j+n)} \\ &= p_1(Q_i)^{m-i} \cdot \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} \left(\frac{i}{2}\right)^n \cdot \binom{j}{n}\binom{m-2j}{i-(j+n)} p_1(E_i)^{2n} \cdot p_1(E_i)^{j-n} \cdot p_1(E_i)^{i-(j+n)} \\ &= \underbrace{\frac{p_1(Q_i)^{m-i} \cdot p_1(E_i)^i}{2^m}}_{\neq 0} \cdot \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} 2^{m-n}\, i^n \binom{j}{n}\binom{m-2j}{i-(j+n)} \end{aligned}$$

and hence, it suffices to compute or estimate the rank of the following matrix:

$$A^m = (A^m_{ij})_{\substack{i = 6 \dots m \\ j = 0 \dots \lfloor \frac{m}{2} \rfloor}} := \left(\sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} 2^{m-n}\, i^n \binom{j}{n}\binom{m-2j}{i-(j+n)} \right)_{\substack{i = 6 \dots m \\ j = 0 \dots \lfloor \frac{m}{2} \rfloor}}.$$

Note that for $j > i$, we have $\binom{m-2j}{i-(j+n)} = 0$ for all $n \geq 0$, and hence, $A_{ij} = 0$. Therefore, the matrix A has the following form, where the asterisks represent nonzero entries:

$$A = \begin{pmatrix} * & \dots & * & 0 & \dots & 0 \\ & & & \ddots & \ddots & \vdots \\ \vdots & & & & * & 0 \\ & & & & & * \\ \vdots & & & & & \vdots \\ * & & \dots\dots & & & * \end{pmatrix}$$

In the first row, there are 7 nonzero entries, so the rank of A is at least $\lfloor \frac{m}{2} \rfloor - 5$. □

Remark A.11. Computer calculations, for which we thank Marek Kaluba, have shown that the matrix A^m and hence the matrix B^m as well have rank equal to $\lfloor \frac{m}{2} \rfloor + 1$ for $m \leq 1250$. This implies that

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$\widetilde{c}^k(m) = 0$ for $k \leq 12$ and $m \leq 1250$ which can be rephrased in the following way: For every $k \leq 12$ and any oriented manifold M of dimension at most 5000 with vanishing signature, there exists a $\lambda \in \mathbb{N}$ such that the λ-fold connected sum of M with itself is cobordant to a fiber bundle $E \to S^k$.

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