ON THE RATE OF CONVERGENCE OF PROBABILITIES OF MODERATE DEVIATIONS

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1. Introduction

Let $\{X_n : n \geqq 1\}$ be a sequence of independent random variables and write $S_n = \sum_{k=1}^n X_k$. Let

$$EX_i = 0, \quad EX_i^2 = \sigma_i^2 \tag{1}$$

and let

$$s_n^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2, \qquad 0 < a \leqq s_n \leqq A < \infty. \tag{2}$$

Suppose that $n^{-\frac{1}{2}} s_n^{-1} S_n$ converges in law to the standard normal distribution (see [5, 280] for necessary and sufficient conditions). Let $\{x_n\}$ be a monotonic sequence of positive real numbers such that $x_n \to \infty$ as $n \to \infty$. Then $x_n^{-1} n^{-\frac{1}{2}} s_n^{-1} S_n \to 0$ in probability. In particular, choose $x_n = \sqrt{\log n}$. Then

$$\Pr\left\{\left|\frac{S_n}{n}\right| > \varepsilon s_n \sqrt{\frac{\log n}{n}}\right\} \to 0 \tag{3}$$

as $n \to \infty$ for all $\varepsilon > 0$. In [6] Rubin and Sethuraman call probabilities of the form $\Pr\{|S_n| > \varepsilon s_n \sqrt{n \log n}\}$ probabilities of moderate deviations and obtain asymptotic forms for such probabilities under appropriate moment conditions.

In this note we study the convergence rate problem for the sequences $\Pr\{|S_n - a_n| > \varepsilon s_n \sqrt{n \log n}\}$,

$$\Pr\left\{\max_{1 \leqq k \leqq n} \left|\frac{S_k}{s_n \sqrt{n \log n}} - b_k\right| > \varepsilon\right\} \text{ and } \Pr\left\{\sup_{k \geqq n} \left|\frac{S_k}{s_k \sqrt{k \log k}} - c_k\right| > \varepsilon\right\}$$

where a_k, b_k, c_k are appropriate centering constants. The corresponding problem for the special case of identically distributed summands has

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recently been treated by Davis in [2] where he considers only the first and the third of above sequences.

In Theorems A and B in section 2 we assume that (1) and (2) hold and that the sequence of normed sums $n^{-\frac{1}{2}}s_n^{-1}S_n$ converges in law to the normal distribution so that, in particular, (3) holds. $L(\cdot)$ is a nonnegative, nondecreasing and continuous function of slow variation [3].

2. Results

THEOREM A. *For $t \geqq 0$ the following statements are equivalent:*

(a) $n^t L(n) \Pr\{|S_n| > \varepsilon s_n \sqrt{n \log n}\} \to 0$ *for all* $\varepsilon > 0$.

(b) $n^t L(n) \Pr\{\max_{1\leqq k\leqq n} |S_k| > \varepsilon s_n \sqrt{n \log n}\} \to 0$ *for all* $\varepsilon > 0$.

If $t > 0$, the above statements are equivalent to

(c) $n^t L(n) \Pr\left\{\sup_{k\geqq n} \left|\frac{S_k}{s_k\sqrt{k \log k}}\right| > \varepsilon\right\} \to 0$ *for all* $\varepsilon > 0$.

THEOREM B.

(a) *For $t \geqq 0$,* $\sum_{n=1}^{\infty} n^{t-1} L(n) \Pr\{|S_n| > \varepsilon s_n \sqrt{n \log n}\} < \infty$ *for all* $\varepsilon > 0$ *if, and only if*

$$\sum_{n=1}^{\infty} n^{t-1} L(n) \Pr\left\{\max_{1\leqq k\leqq n} \left|\frac{S_k}{s_n\sqrt{n \log n}} - \operatorname{med}\left(\frac{S_k - S_n}{s_n\sqrt{n \log n}}\right)\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$.

(b) *For $t > 0$,*

$$\sum_{n=1}^{\infty} n^{t-1} L(n) \Pr\left\{\left|\frac{S_n}{s_n\sqrt{n \log n}} - \operatorname{med}\left(\frac{S_n}{s_n\sqrt{n \log n}}\right)\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$ if, and only if

$$\sum_{n=1}^{\infty} n^{t-1} L(n) \Pr\left\{\sup_{k\geqq n} \left|\frac{S_k}{s_k\sqrt{k \log k}} - \operatorname{med}\left(\frac{S_k}{s_k\sqrt{k \log k}}\right)\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$.

(c) *For $t \geqq 1$ the following statements are equivalent.*

(c_1) $\sum_{n=1}^{\infty} n^{t-1} L(n) \Pr\{|S_n| > \varepsilon s_n \sqrt{n \log n}\} < \infty$ *for all* $\varepsilon > 0$.

(c_2) $\sum_{n=1}^{\infty} n^{t-1} L(n) \Pr\{\max_{1\leqq k\leqq n} |S_k| > s_n \sqrt{n \log n}\} < \infty$ *for all* $\varepsilon > 0$.

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$$(c_3)\ \sum_{n=1}^{\infty} n^{t-1}L(n)\Pr\left\{\sup_{k\geqq n}\left|\frac{S_k}{s_k\sqrt{k\log k}}\right| > \varepsilon\right\} < \infty \qquad \textit{for all } \varepsilon > 0.$$

THEOREM C. *For* $t \geqq 1$,

$$\sum_{n=1}^{\infty} n^{t-1}(\log n)^t \Pr\{|S_n| > \varepsilon\sqrt{n\log n}\} < \infty$$

for all $\varepsilon > 0$ *implies* $E|X_k|^{2t} < \infty$ *for all* k.

REMARK 1. The $L(n) = \log n$ case of part (b) of Theorem B has been obtained by Davis [2] in the special case of identically distributed summands.

PROOFS. The (a), (b) equivalence part of Theorem A and part (a) of Theorem B follows from the inequalities

$$(4)\quad \Pr\{|S_n| > \varepsilon s_n\sqrt{n\log n}\}$$
$$\leqq \Pr\left\{\max_{1\leqq k\leqq n}\left|\frac{S_k}{s_n\sqrt{n\log n}} - \operatorname{med}\left(\frac{S_k-S_n}{s_n\sqrt{n\log n}}\right)\right| > \varepsilon\right\}$$
$$\leqq 2\Pr\{|S_n| > \varepsilon s_n\sqrt{n\log n}\}.$$

The first of these inequalities is trivial while the second follows from Lévy's inequality [5, 247].

The (c_1), (c_2) equivalence part of Theorem B follows since

$$\sum_{n=1}^{\infty} n^{t-1}L(n)\Pr\{|S_n| > \varepsilon s_n\sqrt{n\log n}\} < \infty$$

for all $\varepsilon > 0$ implies

$$\operatorname{med}\left(\frac{S_k-S_n}{s_n\sqrt{n\log n}}\right) \to 0 \text{ as } n\to\infty.$$

For (a), (c) equivalence part of Theorem A and part (b) of Theorem B the proof can be constructed on the lines of [4] and we do not intend to repeat the computations.

The (c_1), (c_3) equivalence in Theorem B follows similarly using once again the fact that for $t \geqq 1$

$$\sum_{n=1}^{\infty} n^{t-1}L(n)\Pr\{|S_n| > \varepsilon s_n\sqrt{n\log n}\} < \infty$$

for all $\varepsilon > 0$ implies

$$\operatorname{med}\left(\frac{S_k-S_n}{s_n\sqrt{n\log n}}\right) \to 0 \text{ as } n\to\infty.$$

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In the case of Theorem C we use the methods of Baum, Katz and Read [1] and Lemma 1 of Davis [2]. We omit the details.

REMARK 2. In Theorems A and B we may replace $L(n)$ by an arbitrary non-negative, non-decreasing function of n.

REMARK 3. The result of Theorem C cannot be improved. This follows trivially by considering the sequences for which $X_k = 0$, $k = 2, 3, \cdots$ and $E|X_1|^{2t} < \infty$ but $E|X_1|^{2t+\delta} = \infty$ for $\delta > 0$.

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