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MAILLET'S PROPERTY AND MAHLER'S CONJECTURE ON LIOUVILLE NUMBERS FAIL FOR MATRICES

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Abstract

In the early 1900s, Maillet [*Introduction a la theorie des nombres transcendants et des proprietes arithmetiques des fonctions* (Gauthier–Villars, Paris, 1906)] proved that the image of any Liouville number under a rational function with rational coefficients is again a Liouville number. The analogous result for quadratic Liouville matrices in higher dimensions turns out to fail. In fact, using a result by Kleinbock and Margulis ['Flows on homogeneous spaces and Diophantine approximation on manifolds', *Ann. of Math. (2)* **148**(1) (1998), 339–360], we show that among analytic matrix functions in dimension $n \geq 2$, Maillet's invariance property is only true for Möbius transformations with special coefficients. This implies that the analogue in higher dimensions of an open question of Mahler on the existence of transcendental entire functions with Maillet's property has a negative answer. However, extending a topological argument of Erdős ['Representations of real numbers as sums and products of Liouville numbers', *Michigan Math. J.* **9** (1962), 59–60], we prove that for any injective continuous self-mapping on the space of rectangular matrices, many Liouville matrices are mapped to Liouville matrices. Dropping injectivity, we consider setups similar to Alniaçik and Saias ['Une remarque sur les G_δ-denses', *Arch. Math. (Basel)* **62**(5) (1994), 425–426], and show that the situation depends on the matrix dimensions m, n. Finally, we discuss extensions of a related result by Burger ['Diophantine inequalities and irrationality measures for certain transcendental numbers', *Indian J. Pure Appl. Math.* **32** (2001), 1591–1599] to quadratic matrices. We state several open problems along the way.

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1. Introduction: Maillet's property and Mahler's problem

A Liouville number is an irrational real number x for which $|x - p/q| < q^{-N}$ has a rational solution p/q for any N. Denote by $\mathscr{L} = \mathscr{L}_{1,1}$ the set of Liouville numbers. It is well known that $\mathscr{L}$ is comeagre, equivalently a dense G_δ set, and of Hausdorff dimension 0; see [20]. Maillet [12] proved that if f is a nonconstant rational function with rational coefficients, then $f(a) \in \mathscr{L}$ for any $a \in \mathscr{L}$, or equivalently, $f(\mathscr{L}) \subseteq \mathscr{L}$.

We say a real function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, with I a nonempty open interval, has the Maillet property if $f(I \cap \mathscr{L}) \subseteq \mathscr{L}$ or equivalently, $f^{-1}(\mathscr{L}) = f^{-1}(\mathscr{L}) \cap \mathscr{L} = \mathscr{L} \cap I$. An open question by Mahler [11] asks the following in the classical setup $I = \mathbb{R}$.

PROBLEM 1.1. Do there exist transcendental entire functions having Maillet's property?

Recall that a function f is called transcendental if $P(z, f(z)) \neq 0$ for any nontrivial bivariate polynomial $P(X, Y) = \sum c_{i,j} X^i Y^j$ with complex coefficients $c_{i,j}$, otherwise f is called algebraic. Some advances to Mahler's question by providing entire transcendental functions f mapping large subclasses of Liouville numbers into $\mathscr{L}$ (or even itself) were made in [13, 16]; see also [14]. Conversely, claims mildly indicating towards a negative answer of Problem 1.1 in the context of [16, Corollary 2.2] were obtained in [9, 10, 15, 17]. In addition to Maillet's result, it is known that any 'reasonable' function enjoys the weaker property that, while not all, many Liouville numbers are mapped to Liouville numbers. Indeed, as shown in [2] for any continuous function f that is nowhere constant as above, the set $f^{-1}(\mathscr{L}) \cap \mathscr{L}$ is a dense G_δ subset of Liouville numbers on I. In fact, we can intersect over countably many preimages under such functions f_k at once. See also the preceding papers [6, 22, 24].

In this paper, we want to discuss analogous claims for Liouville matrices to be defined. Let $\|.\|$ denote the supremum norm on a Euclidean space of any dimension.

DEFINITION 1.2. We call a real $m \times n$ matrix A a Liouville matrix if

$$A \cdot \mathbf{q} - \mathbf{p} \neq \mathbf{0}, \quad (\mathbf{q}, \mathbf{p}) \in \mathbb{Z}^{n+m} \setminus \{\mathbf{0}\}, \tag{1-1}$$

and $\|A\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|^{-N}$ has a solution in integer vectors $\mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ for any N. We denote by $\mathscr{L}_{m,n}$ the set of $m \times n$ Liouville matrices.

This agrees with the definition of Liouville numbers if $m = n = 1$, so $\mathscr{L} = \mathscr{L}_{1,1}$. Studying small values of $\|A\mathbf{q} - \mathbf{p}\|$ is an intensely studied topic in Diophantine approximation, for example, [18], so the definition appears very natural. Property (1-1) means that the sequence of best approximating integer vectors associated to A does not terminate (called good matrices in [18]), so we consider it more natural than the less restrictive condition $A \notin \mathbb{Q}^{m\times n}$; see however Theorem 3.9 on the latter. Since approximation becomes easier the more free variables we have and the fewer equations we need to satisfy, the following observations are obvious but may be helpful for the reader.

PROPOSITION 1.3. *If $A \in \mathscr{L}_{m,n}$, then any line $(a_{j,1}, \dots, a_{j,n})$ of A either has $\mathbb{Z}$-linearly dependent coordinates together with $\{1\}$ or lies in $\mathscr{L}_{1,n}$. If some column of A lies in $\mathscr{L}_{m,1}$ and A satisfies (1-1), then $A \in \mathscr{L}_{m,n}$.*

In general, $\mathscr{L}_{n,m} \neq \mathscr{L}_{m,n}^t$, superscript t denoting the transpose; moreover, $A \in \mathscr{L}_{n,m}$ by no means implies that its entries are Liouville numbers, nor is the converse true.

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2. Mahler's question has a negative answer for matrices

We focus on $m = n$ in this section. Given $I \ni 0$ an open interval and any analytic function $f : I \to \mathbb{R}$, we extend f to (a nonempty open connected subset of) the ring of $n \times n$ matrices via the same local power series expansion. More precisely, we know that $f(z) = \sum c_j z^j$ converges absolutely in some subinterval $(-r, r) \subseteq I$, $r > 0$, and for $A \in \mathbb{R}^{n\times n}$, we denote by $\mathfrak{f}$ the extension of f to the matrix ring via

$$\mathfrak{f}(A) = \sum_{j=0}^{\infty} c_j A^j, \tag{2-1}$$

which converges absolutely as soon as A has operator norm less than r. Denoting by I_n the identity matrix, this setup appears natural and contains, for example, any rational function $(c_0 I_n + c_1 A + \cdots + c_u A^u) \cdot (d_0 I_n + \cdots + d_v A^v)^{-1}$, $d_0 \neq 0$, the matrix exponential function $e^A = \sum_{j\geq 0} A^j/j!$ and its inverse $\log(I_n + A) = \sum_{j\geq 1} (-1)^{j+1} A^j/j$.

The following is not hard to see and we provide a sketch of the proof in Section 8.

PROPOSITION 2.1. *If $A \in \mathscr{L}_{n,n}$, and $R_1, R_2, S, T \in \mathbb{Q}^{n\times n}$ and R_i are invertible, then*

$$R_1 A R_2 + S, \quad (R_1 A R_2 + T)^{-1} + S \tag{2-2}$$

again belong to $\mathscr{L}_{n,n}$ (if defined in the latter case).

REMARK 2.2. The regularity of R_i is only needed to avoid rational matrices in (2-2) which do not satisfy (1-1), especially if A is not invertible. A refined claim on invariance of the irrationality exponent defined below can be obtained. Moreover, some claims can be generalized to rectangular matrices.

Let us extend naturally Maillet's property for real f to given $n \geq 1$ via imposing $\mathfrak{f}(\mathscr{L}_{n,n}) \subseteq \mathscr{L}_{n,n}$ for the induced $\mathfrak{f}$ from (2-1). Taking in (2-2) diagonal matrices

$$R_1 = rI_n,\ (r \neq 0), \quad R_2 = I_n, \quad S = sI_n, \quad T = tI_n,$$

Maillet's property holds for any $n \geq 1$ and the two types of algebraic functions

$$f(z) = rz + s, \quad f(z) = s + (rz + t)^{-1} \tag{2-3}$$

with $0 \neq r, s, t \in \mathbb{Q}$. Equivalently, Maillet's property holds for Möbius maps f with rational coefficients

$$f(z) = \tau_{a,b,c,d}(z) := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Q},\ ad - bc \neq 0. \tag{2-4}$$

However, we show that for $n \geq 2$, Maillet's property fails for any other analytic function not of the form (2-3) with real parameters. Thereby, we get an almost comprehensive description of analytic functions with Maillet's property for $n \geq 2$, leaving only a small gap of the case (2-3) (or (2-4)) with nonrational constants. In fact, we show a refined claim that needs some more preparation. For general rectangular

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matrices $A \in \mathbb{R}^{m\times n}$, let $\omega^{m\times n}(A)$ be the irrationality exponent of A, defined as the supremum of w such that

$$\|A\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|^{-w}$$

for infinitely many integer vector pairs $\mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n$. We notice that

$$\omega^{m\times n}(A) \geq \frac{n}{m}$$

for any real A by a well-known variant Dirichlet's theorem, in particular, for quadratic matrices, the lower bound is 1. Let us just write $\omega(A)$ when the dimensions are clear. A real $m \times n$ matrix A is a Liouville matrix if and only if (1-1) holds and $\omega(A) = \infty$. In the following, we always naturally identify

$$\mathbb{R}^{m\times n} \cong \mathbb{R}^{mn},$$

where we can assume that the lines of the matrix are put one by one into a vector (we can choose any coordinate ordering; however, we must stick to it as $\mathscr{L}_{m,n}$ is not invariant under entry bijections as soon as $\min\{m, n\} \geq 2$). This induces a topology and a Lebesgue measure (generally Hausdorff measures) on the matrix set. Our main result is the following theorem.

THEOREM 2.3. *Let $n \geq 2$ be an integer and $I \subseteq \mathbb{R}$ an open interval containing* 0*. Let $f_k : I \to \mathbb{R}$ be any sequence of real analytic functions not of the form (2-3) for real numbers r, s, t (possibly* 0*) and $\mathfrak{f}_k$ their extensions as in (2-1). Then, there exists a set $\Omega \subseteq \mathscr{L}_{n,n} \subseteq \mathbb{R}^{n^2}$ of Hausdorff dimension $\dim_H(\Omega) = (n-2)^2 + 1$ so that for any $A \in \Omega$ and $k \geq 1$, $\mathfrak{f}_k(A)$ is defined but $\mathfrak{f}_k(A) \notin \mathscr{L}_{n,n}$, and thus, $\bigcup_{k\geq 1} \mathfrak{f}_k^{-1}(\mathscr{L}_{n,n}) \cap \Omega = \emptyset$. In short,*

$$\dim_H\Big(\bigcap_{k\geq 1} \mathfrak{f}_k^{-1}(\mathscr{L}_{n,n}^c) \cap \mathscr{L}_{n,n}\Big) \geq (n-2)^2 + 1.$$

In fact, for any $A \in \Omega$,

$$\omega(\mathfrak{f}_k(A)) \leq 2, \quad k \geq 1.$$

If $f_k(0) = 0$ for some k, then equality $\omega(\mathfrak{f}_k(A)) = 2$ can be obtained.

REMARK 2.4. When $m = n$, we may additionally impose for Liouville matrices the constraint of being transcendental, that is, imposing $P(A) \neq \mathbf{0}$ for any nonzero $P \in \mathbb{Z}[X]$. This would exclude especially nilpotent matrices (it is not hard to construct nilpotent Liouville matrices in our setting for $n \geq 2$). Our results remain valid in this setting as well with minor modifications in some proofs. Another reasonable restriction would be to only consider invertible matrices.

REMARK 2.5. Note that we exclude all real r, s, t, not only rationals. We believe that when choosing any r, s, t within the uncountable subset of $\mathscr{L}_{1,1}$ of so-called strong Liouville numbers, the functions $\mathfrak{f}$ derived from f in (2-3) satisfy $\mathfrak{f}(\mathscr{L}_{n,n}) \subseteq \mathscr{L}_{n,n}$. By a result of Petruska [21], this is true for $n = 1$; indeed if c_0, c_1 are strong Liouville numbers, then, for example, $f(a) = c_0 + c_1 a$ is a Liouville number

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for any Liouville number a. In fact, a weaker property of c_i being so-called semistrong Liouville numbers [1] suffices. The general case $n \geq 1$ seems to admit a similar proof of $\mathfrak{f}(A) = c_0 I_n + c_1 A \in \mathscr{L}_{n,n}$ for any $A \in \mathscr{L}_{n,n}$.

REMARK 2.6. We believe that $\omega(\mathfrak{f}_k(A)) = 1$ can be reached as well in the framework of the theorem. It may be possible to obtain a stronger result, that the $f_k(A)$ are badly approximable, that is, $\|\mathfrak{f}_k(A)\mathbf{q} - \mathbf{p}\| \geq d_k \|\mathbf{q}\|^{-1}$ for any integer vectors $\mathbf{p}, \mathbf{q} \neq \mathbf{0}$ and absolute $d_k > 0$ (maybe even with uniform $d_k = d$). However, even considering only one function, this would require new ideas in the proof.

There is no reason to believe that the stated lower bound on $\dim_H(\Omega)$ is sharp; see also Section 9 below. However, topologically, Ω is small. Indeed, it follows from Theorem 3.3 below that even for only one function $f = f_1$, it must be meagre. For our proof of Theorem 2.3, it is convenient to use a deep result by Kleinbock and Margulis [8]. However, weaker partial claims can be obtained with elementary methods.

Since the functions in (2-3) (or (2-4)) are algebraic, Theorem 2.3 also shows that Mahler's Problem 1.1 has a negative answer in higher dimensions.

COROLLARY 2.7. *Let $n \geq 2$. For any transcendental entire function f, its extension $\mathfrak{f}$ to the $n \times n$ matrix ring via (2-1) does not have the Maillet property.*

It is however not clear if Corollary 2.7 can be regarded a strong indication for a negative answer in the case where $n = 1$ (Mahler's Problem 1.1) as well. Indeed, the following remarks illustrate that the situation over the matrix ring is different from the scalar one.

REMARK 2.8. The proof of Theorem 2.3 shows that we can choose the matrix in the corollary to be an ultra-Liouville matrix defined analogously to [13] for $n = 1$, so the main claim from [13] of the invariance of this set under certain entire transcendental maps fails for $n \geq 2$ as well. The result [16, Theorem 4.3], stating that the parametrized subclasses of Liouville numbers from [16, Definition 3.1] are mapped to Liouville numbers for certain transcendental functions f, fails for $n \geq 2$ as well by similar arguments.

REMARK 2.9. In [16, Corollary 2.2], it is shown that if an entire function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(\mathbb{Q}) \subseteq \mathbb{Q}$ with denominators of order $\mathrm{denom}(f(p/q)) \ll q^N$ for some absolute N, then f has Maillet's property. This property fails over the matrix ring. Indeed, for any polynomial $f \in \mathbb{Z}[X]$, the according properties $\mathfrak{f}(\mathbb{Q}^{n\times n}) \subseteq \mathbb{Q}^{n\times n}$ and $\mathrm{denom}(\mathfrak{f}(A/q)) \ll q^N$ for $A \in \mathbb{Z}^{n\times n}$ hold (with $N = \deg f$), but f does not have Maillet's property by Theorem 2.3 if $\deg f \geq 2$. The reason is basically that A being Liouville according to Definition 1.2 does not imply that A is well approximable by rational matrices B/q, $B \in \mathbb{Z}^{n\times n}$, $q \in \mathbb{N}$, as a function of q with respect to supremum norm on $\mathbb{R}^{n\times n}$; see also Proposition 1.3.

The claim of Corollary 2.7 holds for any transcendental analytic function defined on any open neighbourhood of 0. Since the setting above is almost completely solved, we ask the following variants of Mahler's question for wider classes of functions.

PROBLEM 2.10. Characterize all functions $f : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ with the Maillet property that are (a) continuous and (b) analytic in the sense that each of the mn output entry functions is a power series in the mn entries of A.

For $m = n = 1$, part (b) just reduces to Problem 1.1. Any piecewise defined continuous function that is locally a rational function function with rational coefficients is an example for part (a). Possibly, the set for part (a) is too large to allow a natural classification.

3. Continuous images of Liouville matrices

3.1. A converse property for analytic functions and a conjecture. By contrast to the results in Section 2, while not all, there is still a large subset of quadratic Liouville matrices of any dimension that are mapped to Liouville matrices under any given nonconstant analytic function f (more precisely, by its extension $\mathfrak{f}$). Indeed, via a diagonalization argument and a result from [2] for $n = 1$ (see Section 1), we obtain the following theorem.

THEOREM 3.1. *Let $n \geq 2$ be an integer and $I \subseteq \mathbb{R}$ an open interval containing 0. Let $f_k : I \to \mathbb{R}$ be any sequence of nonconstant real analytic functions on I and $\mathfrak{f}_k$ their extensions via (2-1). Then, there is a set $\Omega \subseteq \mathscr{L}_{n,n} \subseteq \mathbb{R}^{n^2}$ of Hausdorff dimension $(n-1)^2$ so that for any $A \in \Omega$ and any $k \geq 1$, we have $\mathfrak{f}_k(A) \in \mathscr{L}_{n,n}$. In other words, the set*

$$\bigcap_{k\geq 1} \mathfrak{f}_k^{-1}(\mathscr{L}_{n,n}) \cap \mathscr{L}_{n,n} \tag{3-1}$$

has Hausdorff dimension at least $(n-1)^2$; in particular, it is not empty.

While the main idea of the proof is simple, the condition (1-1) causes some technicality, so we move it to Section 6. The lower bound on the Hausdorff dimension is presumably again not optimal, possibly it is the same value $n(n-1)$ as for the full set $\mathscr{L}_{n,n}$; see for example [3] for a considerably stronger result. The set (3-1) is also large in a topological sense; see Section 3.2 below.

The combination of Theorems 2.3 and 3.1 suggests the following conjecture, in the spirit of the open problem closing Burger's paper [4] or [16, Theorem 6.2].

CONJECTURE 3.2. *Let n, I be as above. Suppose $f_k : I \mapsto \mathbb{R}$ and $g_k : I \mapsto \mathbb{R}$, $k \geq 1$, are sequences of analytic functions, with the properties that:*

(i) *for any $k \geq 1$, the functions g_k are not of the form (2-4) for any $a, b, c, d \in \mathbb{R}$;*
(ii) *for $\tau_{a,b,c,d}$ as in (2-4) with any $a, b, c, d \in \mathbb{R}$, we have the nonidentity of functions on I*

$$f_{k_1}(z) \neq \tau_{a,b,c,d}(g_{k_2})(z), \qquad k_1, k_2 \in \mathbb{N}.$$

Then, there exist $A \in \mathscr{L}_{n,n}$ such that $\mathfrak{f}_k(A) \in \mathscr{L}_{n,n}$ and $\mathfrak{g}_k(A) \notin \mathscr{L}_{n,n}$ for any $k \geq 1$, with definitions as in (2-1). Equivalently,

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$$\bigcap_{k\geq 1}(\mathfrak{f}_k^{-1}(\mathscr{L}_{n,n})\cap \mathfrak{g}_k^{-1}(\mathscr{L}_{n,n}^c))\cap \mathscr{L}_{n,n}\neq \emptyset.$$

The assumptions are natural in view of our results and cannot be relaxed apart from possibly restricting a, b, c, d to subsets of $\mathbb{R}$. We point out that if the conjecture holds for some $n = \ell$, then it does also for any $n \geq \ell$. This can be shown by considering $A = \mathrm{diag}(A_\ell, B)$ with $A_\ell \in \mathscr{L}_{\ell,\ell}$ any such matrix and B any $(n-\ell)\times(n-\ell)$ matrix so that $\mathfrak{g}_\ell(B) \notin \mathscr{L}_{n-\ell,n-\ell}$ for every $k \geq 1$. Such matrices B are easily seen to exist by metrical means; see the proof of Theorem 2.3 below for more details. However, even for just one pair of functions $f = f_1, g = g_1$ satisfying the hypotheses (i), (ii) with $k = k_1 = k_2 = 1$, the claim of Conjecture 3.2 is far from obvious.

When $n = 1$, for a similar claim, we would certainly need to exclude more relations in view of Maillet's result, and a complete description requires a comprehensive understanding of Mahler's problem in the original formulation.

3.2. One-to-one continuous maps. We now study the images of Liouville matrices under functions $f_k : U \subseteq \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$. Following the short topological argument in [24], a similar result to Theorem 3.1 can be obtained for injective continuous maps.

Theorem 3.3. *Let m, n be positive integers. Let $U \subseteq \mathbb{R}^{m\times n}$ be a nonempty open set and $f_k : U \to \mathbb{R}^{m\times n}$ a sequence of injective, continuous functions. Then, there exists a dense G_δ subset $\Omega \subseteq U \cap \mathscr{L}_{m,n}$ within U such that $f_k(A) \in \mathscr{L}_{m,n}$ for any $A \in \Omega$ and any $k \geq 1$. In other words,*

$$\bigcap_{k\geq 1} f_k^{-1}(\mathscr{L}_{m,n})\cap \mathscr{L}_{m,n}$$

is a dense G_δ subset of U.

In the proof, we use the famous result of Brouwer that injective continuous self-maps on an Euclidean space are open onto their images and thus locally induce homeomorphisms. A variant of Theorem 3.3 assuming instead the weaker property of the f_k being open can be formulated, leading to a slightly more general claim.

Proof. As indicated, we use an analogous argument as in [24]. First, notice that, similar to $n = 1$, for any integer $h \geq 1$,

$$Y_h := \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\|\mathbf{q}\|\geq 2}\ \bigcup_{\mathbf{p}\in\mathbb{Z}^m} Z_h(\mathbf{p},\mathbf{q}),\qquad Z_h(\mathbf{p},\mathbf{q}) := \{A\in\mathbb{R}^{m\times n} : 0 < \|A\mathbf{q}-\mathbf{p}\| < \|\mathbf{q}\|^{-h}\}$$

is an open dense set in $\mathbb{R}^{mn}$. Indeed, it is open by continuity of the maps $\varphi_{\mathbf{p},\mathbf{q}} : A \mapsto A\mathbf{q}-\mathbf{p}$, and it is dense since for any $A \in \mathbb{Q}^{m\times n}$, there are obviously $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ so that $A\mathbf{q}-\mathbf{p} = \mathbf{0}$ and again by continuity of $\varphi_{\mathbf{q},\mathbf{p}} \not\equiv \mathbf{0}$. So, the intersection of Y_h over $h \geq 1$ is a dense G_δ set. Moreover, if $(L_h)_{h\geq 1}$ denotes the countable collection of rational hyperplanes in $\mathbb{R}^{mn}$, then $\bigcap_{h\geq 1} L_h^c$ is obviously a dense G_δ set as well. Hence,

$$\mathscr{L}_{m,n} \supseteq \bigcap_{h\geq 1} L_h^c \cap Y_h$$

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is again dense G_δ (there is equality in the inclusion if we restrict to a subset of L_h inducing $\mathbb{Z}$-dependent columns), so $\mathscr{L}_{m,n} \cap U$ is dense G_δ within U.

The remaining, short argument based on Baire's theorem is precisely as for $m = n = 1$ in [24]. Since the f_k are injective by Brouwer's result, we have that the images $U_k := f_k(U)$ are open sets, and hence we can find a dense G_δ-subset of $\mathscr{L}_{m,n}$ in each U_k, say $\emptyset \neq \Lambda_k := U_k \cap \mathscr{L}_{m,n}$. However, since the f_k induce homeomorphisms, this means their preimages $Z_k = f_k^{-1}(\Lambda_k)$ are again dense G_δ-sets (in U). Hence, the countable intersection $\Omega := \bigcap_{k\geq 1} Z_k \cap \mathscr{L}_{m,n}$ is a dense G_δ subset of U as well. Any matrix in this set Ω has the claimed property. □

The same topological result can be obtained in the setting of Theorem 3.1 as well, meaning (3-1) is again a dense G_δ set for $\mathfrak{f}_k$ derived from nonconstant scalar analytic f_k via (2-1). However, as it cannot be directly deduced from Theorem 3.3 and the complete proof we found is slightly technical, we prefer to omit it here. (For instance, a technical problem is that the derivatives f_k' may vanish within I, and then f_k and $\mathfrak{f}_k$ are not locally injective everywhere. This is related to Lemma 5.1 and its proof below.) However, it is not clear to us if a positive Hausdorff dimension result can be obtained in the context of Theorem 3.3.

A special case is the following generalization of a result by Erdős [6] for $m = n = 1$.

COROLLARY 3.4. *Any $A \in \mathbb{R}^{m\times n}$ can be written $A = B + C$ with $B, C \in \mathscr{L}_{m,n}$.*

PROOF. Apply Theorem 3.3 with $f(X) = A - X$. □

We should notice that the ideas in all papers [2, 24, 25] as well as our proof of Theorem 3.3 above originate in this work of Erdős. See also [22]. By Theorem 3.3, we can further directly extend several results from [25] to the matrix setting; we only state the analogue of [25, Corollary 7].

COROLLARY 3.5. *Let $U \subseteq \mathbb{R}^{m\times n}$ be a nonempty open connected set and $\varphi : U \to U$ be an injective, continuous self-map. Then, there exists a dense G_δ set of $A \in \mathscr{L}_{m,n} \cap U$ so that the orbit $\varphi^k(A) = \varphi \circ \varphi \cdots \circ \varphi(A)$, $k \geq 1$, consists only of elements in $\mathscr{L}_{m,n} \cap U$.*

In the original formulation in [25], it is assumed that φ is bijective; however, surjectivity is not needed.

PROOF. Apply Theorem 3.3 to $f_k = \varphi^k$, which are defined and easily seen to inherit the properties of being continuous and injective from φ. □

3.3. On relaxing conditions of Theorem 3.3. For $m = n = 1$, the assumption of injectivity in Theorem 3.3 can be relaxed. As stated before, indeed, it was shown in [2] that we only need f_k to be nowhere constant on an interval I, meaning not constant on any nonempty open subinterval of I, for the implication of Theorem 3.3. The latter indeed defines a strictly larger set of functions. In higher dimensions, *a priori* the most natural way to extend the concept of nowhere constant seems to be the following.

DEFINITION 3.6. Let m_1, m_2, n_1, n_2 be positive integers and $U \subseteq \mathbb{R}^{m_1\times n_1} \cong \mathbb{R}^{m_1 n_1}$ be an open, nonempty set. We call a matrix function $f : U \to \mathbb{R}^{m_2\times n_2}$ *nowhere constant* if it is not constant on any nonempty open subset of U.

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When $n \geq 2$, it is easy to see that the analogue of [2] fails. Consider the function that maps $A = (a_{i,j})_{1\leq i\leq m, 1\leq j\leq n} \in U$ to the $m \times n$ matrix $f(A) = B = (b_{i,j})_{1\leq i\leq m, 1\leq j\leq n}$ with each entry $b_{i,j} = a_{1,1}$. Indeed, this function is continuous and nowhere constant but any point in the image satisfies a fixed linear dependence of columns relation over $\mathbb{Z}$, and hence $f(U)$ has empty intersection in a trivial way with $\mathscr{L}_{m,n}$ as (1-1) fails for any $B \in f(U)$. So, it seems reasonable to also consider the following additional property.

DEFINITION 3.7. A function $f : U \subseteq \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ satisfies the *LIC* property (linearly independent columns) if for every set $\Omega \subseteq U$ with nonempty interior, there are no fixed $\mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $B\mathbf{q} - \mathbf{p} = \mathbf{0}$ for any $B \in f(\Omega)$.

The LIC property is easily seen to be independent from the condition of being nowhere constant. However, even assuming both the nowhere constant and LIC properties is not enough to guarantee the conclusion of Theorem 3.3 in general.

THEOREM 3.8. *Let m, n be positive integers and $U \subseteq \mathbb{R}^{m\times n}$ be an open, nonempty set.*

(i) *If $m = n = 1$, then for every nowhere constant continuous function $f : U \to \mathbb{R}$, the set $f^{-1}(\mathscr{L}_{1,1}) \cap \mathscr{L}_{1,1}$ is dense G_δ in U; in particular, nonempty.*
(ii) *If $(m, n) \neq (1, 1)$, then there exists a nowhere constant continuous function $f : U \to \mathbb{R}^{m\times n}$ such that $f(A) \notin \mathscr{L}_{m,n}$ for any matrix $A \in U$, so $f^{-1}(\mathscr{L}_{m,n}) = \emptyset$.*
(iii) *If $m \geq 2$, there exists a nowhere constant continuous function with the LIC property $f : V \to \mathbb{R}^{m\times n}$ such that $f(A) \notin \mathscr{L}_{m,n}$ for any matrix $A \in U$, so $f^{-1}(\mathscr{L}_{m,n}) = \emptyset$.*

PROOF. Claim (i) is just [2]; if $n \geq 2$, claim (ii) has already been observed above. Note that claim (iii) also contains the remaining case $n = 1, m \geq 2$ of claim (ii). So it remains to prove claim (iii). Let $m \geq 2$ and consider a function f that acts as the identity on the first line $\mathbf{a}_1 = (a_{1,1}, \ldots, a_{1,n})$ of a matrix $A = (a_{i,j}) \in U$ and is constant on the remaining lines. Hereby, we choose the constant image vectors $\mathbf{b}_2, \ldots, \mathbf{b}_m \in \mathbb{R}^n \setminus (\mathscr{L}_{1,n} \cup \Pi(L_j))$, where the L_j are the countable collection of all rational hyperplanes of $\mathbb{R}^{n+1}$ and $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the restriction by chopping off the last coordinate. Such $\mathbf{b}_i$ clearly exist: in fact, this set has full n-dimensional Lebesgue measure as $\mathscr{L}_{m,n}$ and $\Pi(L_j)$ are all nullsets. Then, f is clearly continuous and nowhere constant, and satisfies the LIC property since $\mathbf{b}_2 \notin \cup\Pi(L_j)$ implies (1-1) for any matrix $B \in f(U)$. However, since the second line $\mathbf{b}_2 \notin \mathscr{L}_{1,n}$, we conclude $f(A) \notin \mathscr{L}_{m,n}$ by Proposition 1.3; indeed, $\infty > \omega^{2\times n}(\mathbf{b}_2) \geq \omega^{m\times n}(f(A))$. □

Observe that claims (ii), (iii) apply to all matrices A, not only Liouville matrices. The main reason for the failure in claims (ii), (iii) for $m \geq 2$ is that in higher dimensions, the property of being nowhere constant for a function is insufficient to guarantee that its image contains an open set, which was used in the argument for $m = n = 1$ in [2].

There is a gap in Theorem 3.8 for $m = 1, n \geq 2$ and nowhere constant, continuous functions with the LIC property. The converse result of the next theorem illustrates why this case is more complicated. Define $\mathscr{L}^*_{m,n} \supseteq \mathscr{L}_{m,n}$ to be the set of irrational

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real matrices $A \notin \mathbb{Q}^{m\times n}$ such that $\|A\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|^{-N}$ has a solution in integer vectors $\mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ for any N. As indicated in Section 1, for $n = 1$, we have $\mathscr{L}^*_{m,1} = \mathscr{L}_{m,1}$, but for $n \geq 2$, the inclusion is proper.

THEOREM 3.9. *Let $m = 1$ and $U \subseteq \mathbb{R}^{1\times n}$ be an open, nonempty set. For any sequence of nowhere constant continuous functions $f_k : U \to \mathbb{R}^{1\times n}$, there exists a dense G_δ set $\Lambda \subseteq \mathscr{L}_{1,n} \cap U$ such that $f_k(A) \in \mathscr{L}^*_{1,n}$ for any $A \in \Lambda$ and every $k \geq 1$. In other words,*

$$\bigcap_{k\geq 1} f_k^{-1}(\mathscr{L}^*_{1,n}) \cap \mathscr{L}_{1,n}$$

is a dense G_δ subset of U.

We believe the claim remains true for $\mathscr{L}_{1,n}$ in place of $\mathscr{L}^*_{1,n}$ throughout, but a proof would be desirable. However, for $m \geq 2$, the analogue of Theorem 3.8(iii) holds as well for $\mathscr{L}^*_{m,n}$ by the same proof, which is formally a stronger claim.

PROOF. We can assume $n \geq 2$ as the case $m = n = 1$ reduces to [2]. First, note that by Proposition 1.3, case $m = 1$, a line vector belongs to $\mathscr{L}^*_{1,n}$ (but not necessarily $\mathscr{L}_{1,n}$) as soon as some entry is a Liouville number. Hence, for any nonempty open interval I, if $L_I = \mathscr{L}_{1,1} \cap I \subseteq I$ denotes the dense G_δ set of Liouville numbers in I, the cylinder sets

$$Z_j = \mathbb{R}^{j-1} \times L_I \times \mathbb{R}^{n-j}, \quad 1 \leq j \leq n,$$

consist only of elements of $\mathscr{L}^*_{1,n}$, so

$$Z_j \subseteq \mathscr{L}^*_{1,n}.$$

Take any nonempty open box $\mathcal{B}$ in $U \subseteq \mathbb{R}^n$. First, consider just one function $f = f_1$. Then, since $f = (f^1, \ldots, f^n)$ is nowhere constant, some coordinate function f^j is not constant on $\mathcal{B}$. So, since connectedness is preserved under continuous maps, its image $f^j(\mathcal{B}) \subseteq \mathbb{R}$ contains a nonempty open interval I. However, this means $f(\mathcal{B})$ has nonempty intersection with the cylinder set Z_j above, or equivalently, $f^{-1}(Z_j) \cap \mathcal{B} \subseteq f^{-1}(\mathscr{L}^*_{1,n}) \cap \mathcal{B}$ is nonempty. Thus, as $\mathcal{B}$ was arbitrary in U, the set $f^{-1}(\mathscr{L}^*_{1,n})$ is dense in U. Moreover, as f is continuous and $\mathscr{L}^*_{1,n} \supseteq \mathscr{L}_{1,n}$ is dense G_δ (see the proof of Theorem 3.3), $f^{-1}(\mathscr{L}^*_{1,n})$ is a G_δ set. Hence, it is a dense G_δ set. Now, we apply this to $f = f_k$ for all $k \geq 1$ simultaneously, and again, since $\mathscr{L}_{1,n}$ is dense G_δ as well, we see that the set

$$\Lambda := \bigcap_{k\geq 1} f_k^{-1}(\mathscr{L}^*_{1,n}) \cap \mathscr{L}_{1,n}$$

is a dense G_δ set in $U \subseteq \mathbb{R}^n$ as well, with the property of the theorem that $f_k(A) \in \mathscr{L}^*_{1,n}$ for any $A \in \Lambda \subseteq \mathscr{L}_{1,n}$ and every $k \geq 1$. □

Even if assuming the LIC property, the counterexamples in Theorem 3.8(ii), (iii) are still slightly artificial as the image has certain constant entry functions. To avoid this, we propose an alternative to our definition of a nowhere constant function. Since,

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very similarly to Proposition 2.1, the Liouville property of $A \in \mathbb{R}^{m\times n}$ is preserved under actions

$$A \mapsto R_1 A R_2 + T, \quad R_1 \in \mathbb{Q}^{m\times m}, R_2 \in \mathbb{Q}^{n\times n}, T \in \mathbb{Q}^{m\times n} \tag{3-2}$$

with invertible matrices R_j, the following strengthening seems natural.

DEFINITION 3.10. We refer to $A, B \in \mathbb{R}^{m\times n}$ as $\mathscr{L}$-equivalent and write $A \sim B$ if B arises from A via (3-2). We write $[A]_\sim$ for the class of A. We say functions $f, g : U \subseteq \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ are $\mathscr{L}$-equivalent and write $f \sim g$ if $g(A) = R_1 f(A) R_2 + T$ for fixed R_j, T as in (3-2) and any $A \in U$, and again write $[f]_\sim$ for the class of f. We finally call f *strongly nowhere constant* if for every $g \in [f]_\sim$, every scalar entry function $g^{i,j} : U \to \mathbb{R}$, $1 \le i \le m, 1 \le j \le n$, is nowhere constant in the sense of Definition 3.6.

Plainly, we can restrict to $g \in [f]_\sim$ derived via $T = \mathbf{0}$ for testing the property strongly nowhere constant. Thus, the property means that the $\mathbb{Z}$-span of the entries $f^{i,j}(A)$ of $f(A)$ together with the $\mathbf{1}$ function can only generate the $\mathbf{0}$ function in the trivial way. For strongly nowhere constant functions, we prove the following result on column vectors.

THEOREM 3.11. *Let $m \ge 2$, $n = 1$. Then, there exists a strongly nowhere constant continuous function $f : \mathbb{R}^{m\times 1} \to \mathbb{R}^{m\times 1}$ with the LIC property and such that $f(A) \notin \mathscr{L}_{m,1}$ for any $A \in \mathbb{R}^{m\times 1}$, in other words, $f^{-1}(\mathscr{L}_{m,1}) = \emptyset$.*

This is stronger than Theorem 3.8(ii), (iii), for the special case $m \ge 2, n = 1$. The principal idea of the proof is to consider a function whose image is contained in an algebraic variety without rational points, which by a result in [23] means they contain no Liouville (column) vectors.

PROOF. Write $A = (a_1, \dots, a_m)^t$ and define the coordinate functions of $f = (f^1, \dots, f^m)^t$ by

$$f^j(a_1, \dots, a_m) = a_j, \ (1 \le j \le m-1), \quad f^m(a_1, \dots, a_m) = \sqrt[3]{a_{m-1}^2 - N},$$

where $N \in \mathbb{Z}$ is so that $Y^2 = X^3 + N$ has no rational solution (see [19] for existence). It is not hard to see that this induces a continuous, strongly nowhere constant function on $\mathbb{R}^m$ with the LIC property. Consider the projection $\Pi : \mathbb{R}^{m\times 1} \to \mathbb{R}^{2\times 1}$ onto the last two coordinates, so that $\Pi(f(\mathbb{R}^{m\times 1}))$ equals the variety $X_m^3 + N = X_{m-1}^2$ defined over $\mathbb{Q}[X_{m-1}, X_m]$, without rational points. As there is no rational point in $\Pi(f(\mathbb{R}^{m\times 1})) \subseteq \mathbb{R}^{2\times 1}$, from [23, Theorem 2.1], we see that any such projected vector $\mathbf{a} \in \Pi(f(\mathbb{R}^{m\times 1})) \subseteq \mathbb{R}^2$ has irrationality exponent $\omega^{2\times 1}(\mathbf{a}) \le 2$, and hence $\mathbf{a} \notin \mathscr{L}_{2,1}$. However, as clearly the irrationality exponent of a column vector cannot decrease under the projection Π (see Proposition 1.3), *a fortiori*, any $\mathbf{b} \in \Pi^{-1}(\mathbf{a})$ has exponent at most $\omega^{m\times 1}(\mathbf{b}) \le \omega^{2\times 1}(\mathbf{a}) \le 2$ as well, and thus $\mathbf{b} \notin \mathscr{L}_{m,1}$. So, since $\Pi^{-1}(\Pi(f(\mathbb{R}^{m\times 1}))) \supseteq f(\mathbb{R}^{m\times 1})$ and $\mathbf{a} \in \Pi(f(\mathbb{R}^{m\times 1}))$ was arbitrary, we have $f(\mathbb{R}^{m\times 1}) \cap \mathscr{L}_{m,1} = \emptyset$, which is equivalent to the claim. □

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REMARK 3.12. The function $f^m(a_1, \dots, a_m) = \sqrt{3 - a_{m-1}^2}$ is an example of lower degree due to $X^2 + Y^2 = 3$ having no rational points again; however, it is not globally defined.

For $m = 1$, a converse result in the form of an analogue of Theorem 3.9 clearly holds for strongly nowhere constant functions *a fortiori*, and presumably also for $\mathscr{L}_{1,n}$ (without the 'star'). This leaves the nonvector cases $\min\{m, n\} > 1$ open where the situation seems unclear.

PROBLEM 3.13. Determine all pairs m, n for which the analogue of Theorem 3.11 holds.

Above, we have only considered self-mappings. We want to finish this section with an open problem for a function $f : \mathbb{R} \to \mathbb{R}^{n\times n}$.

PROBLEM 3.14. Given $A \in \mathbb{R}^{n\times n}$, what can be said about the set

$$\mathcal{X}_A = \{\lambda \in \mathbb{R} : A - \lambda I_n \in \mathscr{L}_{n,n}\}?$$

Is it always nonempty/uncountable/dense G_δ?

It is not hard to see by topological arguments similar to Theorem 3.3 that for any A, all rows of $A - \lambda I_n$ can be made Liouville vectors in $\mathbb{R}^{1\times n}$ simultaneously for many (dense G_δ set of) λ; however, this is in general insufficient for $A - \lambda I_n$ being in $\mathscr{L}_{n,n}$. This argument works if A is diagonalizable via a rational base change matrix (in particular, if A is diagonal), via Proposition 2.1. However, it is easy to construct A such that for no λ, is any column of $A - \lambda I_n$ a Liouville vector in $\mathbb{R}^{m\times 1}$; however, this is not necessary for $A - \lambda I_n$ being in $\mathscr{L}_{n,n}$. If $\mathcal{X}_A$ is nonempty, it is dense in $\mathbb{R}$ by its invariance under rational translations via Proposition 2.1 again. In general, we do not know what to conjecture, but remark that Problem 3.14 can be naturally generalized to $\mathcal{X}_{A,B} = \{\lambda \in \mathbb{R} : A - \lambda B \in \mathscr{L}_{m,n}\}$ for $A, B \in \mathbb{R}^{m\times n}$.

4. On a property of Burger

Let us return to quadratic matrices. As noticed in Section 3.2, it was shown by Erdős [6] and is a special case of Corollary 3.4 that any real number can be written as the sum of two Liouville numbers. Burger [4] noticed that Erdős' proof can be adapted to show that any transcendental real number can be written as a sum of two algebraically independent Liouville numbers. The converse is easy to prove, giving a characterization of transcendental real numbers. See also [25, Proposition 3] for a generalization. Let us consider the problem in the matrix setting. Naturally, we call a quadratic real matrix algebraic (over $\mathbb{Z}$) if $P(A) = \mathbf{0}$ for some nonzero polynomial $P \in \mathbb{Z}[X]$, otherwise we call it transcendental. Similarly, let us call two real $n \times n$ matrices B, C algebraically dependent (over $\mathbb{Z}$) if there exists nonzero $P \in \mathbb{Z}[X, Y]$ so that $P(B, C) = \mathbf{0}$, otherwise they are algebraically independent. Note hereby that bivariate polynomials over a noncommutative matrix ring have a more complicated

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form as in the commutative case $n = 1$, for example, $5X^2Y^3X - 5X^3Y^3$ is not the 0 polynomial when $n \geq 2$. We study Burger's property in this setting.

PROBLEM 4.1. For $n \geq 2$, is it true that $A \in \mathbb{R}^{n\times n}$ is transcendental if and only if it has a representation as a sum of two algebraically independent Liouville matrices?

For $n = 2$, Problem 4.1 has a negative answer in a trivial sense, as it turns out that we have the following proposition.

PROPOSITION 4.2. *For $n \geq 2$, there are no algebraically independent matrix pairs at all.*

PROOF. For $n = 2$, this is due to Hall's identity

$$X(YZ - ZY)^2 = (YZ - ZY)^2 X$$

for any integer (or real) 2×2 matrices X, Y, Z. Indeed, it suffices, for example, to let $X = Y = A, Z = B$ to see that any two 2×2 matrices A, B are algebraically dependent according to our definition above. Similar examples can be found for general $n \geq 2$, as the matrix rings are all so-called polynomial identity rings, by the Amitsur–Levitzki theorem. We refer to [5]. □

It appears that the ordinary concept of algebraic independence is too strong over our matrix rings. However, if we modify our definitions of transcendence and algebraic independence, Burger's problem becomes more interesting. We propose to work with the following concepts.

DEFINITION 4.3.

(i) Call $A \in \mathbb{R}^{n\times n}$ *weakly algebraic* if there exists an integer $\ell \geq 0$ and a polynomial $P \in \mathbb{Z}[X_0, \dots, X_\ell]$ such that

$$P(A, B_1, \dots, B_\ell) = \mathbf{0}$$

for any $B_1, \dots, B_\ell \in \mathbb{R}^{n\times n}$, but

$$P(C_0, \dots, C_\ell) \neq \mathbf{0}$$

for some $C_0, \dots, C_\ell \in \mathbb{R}^{n\times n}$.

(ii) Call $A, B \in \mathbb{R}^{n\times n}$ *weakly algebraically independent* if $P(A, B) = \mathbf{0}$ for $P \in \mathbb{Z}[X, Y]$ implies $P(C_0, C_1) = \mathbf{0}$ for all matrices $C_0, C_1 \in \mathbb{R}^{n\times n}$.

We only want the polynomials as functions to act nontrivially in definitions (i), (ii), so essentially, we factor out the nontrivial polynomial relations in $\mathbb{Z}[X, Y]$ over the matrix ring. Algebraic implies weakly algebraic as we may let $\ell = 0$, and the ring $\mathbb{Z}[X]$ in a single real $n \times n$, $n \geq 2$, matrix variable is not a polynomial identity ring (it is a principal ideal domain). The implication follows also from Theorem 4.5 below. The concepts algebraic and weakly algebraic may however be equivalent; this question forms part of Problem 4.6 below. Moreover, it is immediate that weakly algebraically independent implies algebraically independent in the classical sense. For $n = 1$, the

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respective concepts coincide, if we assume that we are working over a commutative ring (that is, identifying AB with BA). However, we have the following.

PROPOSITION 4.4. *For $n \geq 2$, weakly algebraically independent matrix pairs exist.*

Thus, comparing with Proposition 4.2, weakly algebraically independent is a strictly weaker concept than algebraically independent.

PROOF. Fix for now $P \in \mathbb{Z}[X, Y]$ that does not induce the $\mathbf{0}$ function. Then, as any entry $P_{i,j}(A, B)$, $1 \leq i, j \leq n$, of $P(A, B)$ is a scalar-valued multivariate polynomial in the entries of $A = (a_{i,j})$ and $B = (b_{i,j})$, and some $P_{i_0,j_0}(A, B)$ is not identical to 0, we find many (a full Lebesgue measure set in $\mathbb{R}^{2n^2}$) real matrix pairs A, B so that $P_{i_0,j_0}(A, B) \neq 0$ and thus also $P(A, B) \neq \mathbf{0}$. Since $\mathbb{Z}[X, Y]$ is only countable, we are still left with a full measure set that avoids all algebraic varieties. □

Alternatively, the claim follows from the next Theorem 4.5. It generalizes Burger's result, with similar underlying proof ideas. However, we use a topological argument rather than a cardinality consideration as in [4], and a few more twists.

THEOREM 4.5. *Let $n \geq 1$ and $A \in \mathbb{R}^{n\times n}$. Then:*

(i) *if $A = B + C$ holds for some weakly algebraically independent $B, C \in \mathbb{R}^{n\times n}$, then A is transcendental;*
(ii) *if A is not weakly algebraic (in particular, A is transcendental), then there exist weakly algebraically independent $B, C \in \mathscr{L}_{n,n}$ with $B + C = A$.*

In fact, in claim (ii), we only need to assume that A is not weakly algebraic for $\ell = 1$, which is a stronger claim. As stated above, for $n = 1$, we get a new proof of Burger's result avoiding Bezout's theorem. However, for $n \geq 2$, claims (i), (ii) do not give rise to any logical equivalence. Consider the claims: (I) A is transcendental; (II) A is not weakly algebraic; (III) A can be written $A = B + C$ with B, C weakly algebraically independent. Then, Theorem 4.5 is equivalent to $(II) \Longrightarrow (III) \Longrightarrow (I)$. Indeed, the following remains open.

PROBLEM 4.6. When $n \geq 2$, is it actually true that $(I) \Longleftrightarrow (III)$ or even $(I) \Longleftrightarrow (II) \Longleftrightarrow (III)$?

Clearly if (and only if) weakly algebraic is actually the same as algebraic, then we have the full equivalence. Possibly, generalizations of Theorem 4.5 as in [25, Proposition 3] to more general expressions (polynomials) in B, C in place of the plain sum $B + C$ hold; however, our proofs of neither claim (i) nor claim (ii) extend in an obvious way. Moreover, there may be alternative natural variants of Burger's problem for matrices to that discussed above worth studying. We stop our investigation here.

5. Proof of Theorem 2.3

To get the full metrical statement, we use the following structural lemma on extensions $\mathfrak{f}$. Possibly, the result is known, but we found no reference.

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LEMMA 5.1. *Let $f : I \to \mathbb{R}$ with $I \subseteq \mathbb{R}$ open containing 0 be a nonconstant analytic function. Then, the extension $\mathfrak{f} : U \subseteq \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ via (2-1) is locally bi-Lipschitz and open onto its image on a set $\mathcal{Y} \subseteq U$ of full n^2-dimensional Lebesgue measure.*

PROOF. Since f is nonconstant analytic, f' is analytic and vanishes at most on a discrete countable set (complex included) R. So, by the Lagrange inversion theorem, outside this set R, the function f is invertible with its inverse locally being an analytic function.

Let $\mathcal{Y} \subseteq \mathbb{R}^{n^2}$ be the set of real $n \times n$ matrices avoiding the discrete set R of eigenvalues and also avoiding multiple (possibly complex) eigenvalues. Then, $\mathcal{Y}$ has full n^2-dimensional Lebesgue measure by the sigma-additivity of measures and if we prescribe a fixed eigenvalue λ, this defines an algebraic equation via $\det(A - \lambda I) = 0$; a similar argument applies to double roots using the discriminant of the characteristic polynomial.

We claim that the restriction of $\mathfrak{f}$ to $\mathcal{Y}$ is bijective and locally bi-Lipschitz onto its image. Let $A \in \mathcal{Y}$ and write $A = PJP^{-1}$ with J the diagonal Jordan matrix of A and P its matrix of eigenvectors. Note that $\mathfrak{f}(A) = P \cdot \mathfrak{f}(J) \cdot P^{-1}$ since f is analytic. We first show local injectivity. Let $U' \ni A$ be an open neighbourhood of A within $\mathcal{Y}$, so that all its matrices, like A, avoid eigenvalues in R and repeated eigenvalues. Clearly, such a U' exists by continuity (in other words, $\mathcal{Y}$ is open). Assume $\mathfrak{f}(A) = \mathfrak{f}(B)$ for some $B \in U'$. Then, $\mathfrak{f}(B)$ has the same eigenvalues as $\mathfrak{f}(A)$ and the same eigenvectors (we may scale them to length one without loss of generality). However, since f is locally injective and $\mathfrak{f}(J) = \mathrm{diag}(f(J_i))$ if $J = \mathrm{diag}(J_i)$, this means A and B have the same Jordan matrix J as well provided U' was chosen small enough. However, then they must have the same eigenvector matrices (that is, P) as well, as otherwise, we would have $\mathfrak{f}(A) \neq \mathfrak{f}(B)$. Hence, $A = B$ is necessary, showing local injectivity. From Brouwer's result, it follows immediately that $\mathfrak{f}$ restricted to $\mathcal{Y}$ is an open map, and thus, in particular, has a continuous inverse.

Clearly, $\mathfrak{f}$ is locally Lipschitz. The Lipschitz property of the inverse is not immediate. Recall $U' \subseteq U$ as above is an open neighbourhood of A, where $\mathfrak{f}$ is injective. If $B \in U'$ and $C = \mathfrak{f}(B)$ lies in the image of $\mathfrak{f}$, then if we write the Jordan form of B as $B = \tilde{P}\tilde{J}\tilde{P}^{-1}$, then $C = \tilde{P} \cdot \mathrm{diag}(f(\tilde{J}_i)) \cdot \tilde{P}^{-1}$ by analyticity of f. Moreover, the inverse is $\mathfrak{f}^{-1}(C) = \tilde{P} \cdot \mathrm{diag}(f^{-1}(\tilde{J}_j)) \cdot \tilde{P}^{-1}$, with the same $\tilde{P} = \tilde{P}(B)$. Since $\tilde{P}, \tilde{J}$ depend locally Lipschitz continuously on $B \in U'$ and, thus, also on $C \in \mathfrak{f}(U')$ (for this step, we use that $\mathfrak{f}$ is Lipschitz as well so we may multiply their Lipschitz constants), and f^{-1} is real analytic and thus Lipschitz. Putting all this together, indeed $\mathfrak{f}^{-1}$ is Lipschitz on the open set $\mathfrak{f}(U')$ as well. □

First, let $n = 2$. The principal idea is to consider matrices of the form

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

where $b \in \mathscr{L}_{1,1}$ is any fixed Liouville number and a is to be chosen later. Then, $A \in \mathscr{L}_{2,2}$ for any a not linearly dependent with $\{1, b\}$ over $\mathbb{Z}$, since for any p, q

$|qb - p| = \|A \cdot (0, q)^t - (0, p)^t\|$. First, consider just one function $f(z) = f_1(z) = \sum_{j\geq 0} c_j z^j$. We find many a such that $\mathfrak{f}(A) = \sum c_j A^j$ is not Liouville. Note first that

$$A^j = \begin{pmatrix} a^j & ba^{j-1} \\ 0 & 0 \end{pmatrix}, \quad j \geq 1$$

so

$$\mathfrak{f}(A) = \begin{pmatrix} r(a) & s(a) \\ 0 & c_0 \end{pmatrix},$$

where

$$r(z) = \sum_{j\geq 0} c_j z^j = f(z), \quad s(z) = \sum_{j\geq 0} bc_{j+1} z^j.$$

Hence,

$$r(z) - c_0 = \frac{1}{b} \cdot s(z)z. \tag{5-1}$$

Then, for $\mathbf{v}_a := (r(a), s(a)) \in \mathbb{R}^{1\times 2}$, Proposition 1.3 implies $\omega^{2\times 2}(\mathfrak{f}(A)) \leq \omega^{1\times 2}(\mathbf{v}_a)$, with equality if $c_0 = 0$ (as the additional Diophantine condition on $|q_2 c_0 - p_2|$ holds for $p_2 = 0$ if $c_0 = 0$). So, it suffices to show $\omega^{1\times 2}(\mathbf{v}_a) = 2$ for many a.

Now, $C := \{(r(a), s(a)) : a \in \mathbb{R}\}$ defines an analytic curve in $\mathbb{R}^2$. We use a deep result from [8] to conclude. Let us call a planar curve locally parametrized by $(x, f(x))$ nondegenerate if the critical points of $f''(x)$, that is, where it vanishes or does not exist, occur only within a set x of Lebesgue measure 0.

THEOREM 5.2 (Kleinbock, Margulis). *Let C be a nondegenerate planar curve given by parametrization $(x(t), y(t))$, $t \in I$. Then, for almost all $t \in I$ with respect to Lebesgue measure, we have $\omega^{1\times 2}(x(t), y(t)) = 2$.*

So, if C above is nondegenerate, we know that for almost all a, the point $\mathbf{v}_a$ indeed has Diophantine exponent 2. By omitting additionally the countable set of a where (1-1) fails, that is, excluding elements $\mathbb{Q}$-linearly dependent with $\{b, 1\}$, to make A a Liouville matrix, we are still left with a full measure set, so we are done. (In fact, such a do not exist as $\mathbb{Q}$-linear dependence directly implies $\omega^{1\times 2}(\mathbf{v}_a) = \infty > 2$.)

So, suppose conversely that C is degenerate. Since C is defined via analytic entry functions, the zeros of the according second derivative form a countable discrete set unless the second derivative is constantly zero, so the second derivative must vanish everywhere. However, this requires

$$\frac{d^2 s(z)}{d^2 r(z)} = \frac{d(s_z/r_z)}{dr_z} = \frac{\frac{d(s_z/r_z)}{dz}}{\frac{dr}{dz}} = \frac{s_{zz} r_z - s_z r_{zz}}{r_z^3} = 0$$

identically or is identically undefined. If $s_z \equiv 0$ by (5-1), this yields $f(z) = r(z) = c_0 + c_1 z$ for some real numbers c_0, c_1, among the functions (2-3) excluded in the theorem.

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So, assume $s_z \not\equiv 0$. Then, $r_z/s_z \equiv d$ is constant. We may assume $d \neq 0$ as $d = 0$ leads to $f(z) = r(z) = c_0$ constant, again excluded. Then, again by (5-1), the condition becomes

$$\frac{1}{b} \cdot \frac{s_z(z)z + s(z)}{s_z} = \frac{1}{b}(z + s/s_z) \equiv d$$

or $z + s/s_z \equiv g$ for a new constant $g = bd$. This is further equivalent to

$$(\log s)_z = \frac{s_z}{s} \equiv -\frac{1}{z-g}$$

or $\log s = -\log|z-g| + h$ for some h, and finally $s(z) = H/(z-g)$ for $H = e^h$. Hence,

$$f(z) = r(z) = \frac{1}{b} \cdot s(z)z + c_0 = c_0 + J\frac{z}{z-g} = c_0 + J + \frac{Jg}{z-g} = \tilde{c}_0 + \frac{\tilde{c}_1}{z-g}$$

for some real numbers $J = H/b$ and $\tilde{c}_i = Jg$, again of the form (2-3) excluded in the theorem. The argument for a single function is complete.

We can apply this argument for any $f = f_k$. Since we get a full measure set of $a \in \mathbb{R}$ for any $k \geq 1$, their countable intersection again has full measure. So $n = 2$ is done.

For larger n, let us for simplicity denote simultaneously $\mathfrak{f}_k$ derived from f_k via (2-1) for matrices in arbitrary dimension (we use dimension 2, $n-2$ and n). Then, we can just take a matrix consisting of two diagonal blocks

$$A = \mathrm{diag}(A_2, B)$$

with $A_2 \in \mathscr{L}_{2,2}$ as above and B any real $(n-2)\times(n-2)$ matrix so that all $\mathfrak{f}_k(B)$, $k \geq 1$, have irrationality exponents $\omega^{(n-2)\times(n-2)}(\mathfrak{f}_k(B)) = 1$. For any k, it is easily seen that such B form a full Lebesgue measure set in $\mathbb{R}^{(n-2)^2}$. Indeed, this follows from the locally bi-Lipschitz property of the analytic maps $B \to \mathfrak{f}_k(B)$ obtained in Lemma 5.1 and a standard Khintchine-type result that the set of matrices

$$\{C \in \mathbb{R}^{(n-2)\times(n-2)} : \omega^{(n-2)\times(n-2)}(C) = 1\}$$

has full Lebesgue measure in $\mathbb{R}^{(n-2)^2}$. So, the same holds for the infinite intersection over $k \geq 1$ as requested. It is easily checked that any resulting $A \in \mathscr{L}_{n,n}$ since $A_2 \in \mathscr{L}_{2,2}$ and the system is decoupled. However, for $n \geq 2, k \geq 1$,

$$\omega^{n\times n}(\mathfrak{f}_k(A)) = \max\{\omega^{2\times 2}(\mathfrak{f}_k(A_2)), \omega^{(n-2)\times(n-2)}(\mathfrak{f}_k(B))\} \leq \max\{2, 1\} = 2,$$

with equality in the inequality if $f_k(0) = 0$, where for the first identity, again we use that the system is decoupled (see also [7, Lemma 9.1]). So, $\mathfrak{f}_k(A) \notin \mathscr{L}_{n,n}$.

Finally, as the set of suitable $a \in \mathbb{R}$ and $B \in \mathbb{R}^{(n-2)\times(n-2)}$ have full Lebesgue measure in the respective Euclidean spaces, the metrical claim follows from a standard estimate on the Hausdorff dimension Cartesian products $\dim_H(X \times Y) \geq \dim_H(X) + \dim_H(Y)$ for measurable X, Y; see Tricot [26].

REMARK 5.3. By choosing b an ultra-Liouville number as defined in [14], the arising matrix A has an analogous property. This justifies Remark 2.8.

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REMARK 5.4. Alternatively, for $n \geq 3$, we can consider $A = \mathrm{diag}(A_2, A_2, \ldots, A_2)$ for n even and $A = \mathrm{diag}(A_2, A_2, \ldots, A_2, \{\ell\})$ for n odd, where $\ell \in \mathbb{R}$ is a number all of whose evaluations $f_k(\ell)$ have exponent $\omega^{1\times 1}(f_k(\ell)) = 1$ (which again exists by the same metrical argument as in the proof above). However, this gives a weaker metrical bound $\lceil n/2 \rceil$.

6. Proof of Theorem 3.1

In view of condition (1-1), we need the following technical lemma. As for Lemma 5.1, possibly stronger claims are known, but we have found no reference, so we prove it directly using the metrical sparsity of zeros of multivariate real power series and the aforementioned Lemma 5.1.

LEMMA 6.1. *Let $n \geq 1$ be an integer, $I \ni 0$ be an open interval and $f : I \to \mathbb{R}$ be any nonconstant analytic map. Let $\mathcal{S} \subseteq \mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ be a proper affine subspace. Then, for $\mathfrak{f} : U \subseteq \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ the extension of f via (2-1), the preimage $\mathfrak{f}^{-1}(\mathcal{S})$ has n^2-dimensional Lebesgue measure zero.*

PROOF. The image $\mathfrak{f}(A) = \sum c_j A^j$ of some $A = (a_{i,j})$ has each entry a scalar power series in its n^2 scalar entries, say $P_{i,j}(a_{1,1}, \ldots, a_{n,n}) \in \mathbb{R}[[X_1, \ldots, X_{n^2}]]$ for $1 \leq i, j \leq n$. So, if $\mathfrak{f}(A)$ lies in a proper affine subspace $\mathcal{S}$ with equation $d_0 + d_{1,1}x_1 + \cdots + d_{n,n}x_{n^2} = 0$, $d_0, d_{i,j} \in \mathbb{R}$ not all 0, it satisfies some fixed scalar power series equation

$$Q(a_{1,1}, \ldots, a_{n,n}) = \sum_{1\leq i,j\leq n} d_{i,j} P_{i,j}(a_{1,1}, \ldots, a_{n,n}) = 0$$

for $Q(a_{1,1}, \ldots, a_{n,n}) \in \mathbb{R}[[X_1, \ldots, X_{n^2}]]$ a power series in n^2 variables $a_{1,1}, \ldots, a_{n,n}$. If $Q \not\equiv 0$ does not vanish identically, it is well known and, for example, a consequence of the Lebesgue density theorem that only a set of Lebesgue measure zero in $\mathbb{R}^{n^2}$ can satisfy such an identity, and we are done. So, assume $Q \equiv 0$. This means every $A \in \mathbb{R}^{n\times n}$ satisfies $\mathfrak{f}(A) \in \mathcal{S}$, so $\mathfrak{f}(U) \subseteq \mathcal{S}$. However, this forces f to be constant by Lemma 5.1, as otherwise, if f is not constant then the image $f(U)$ contains an open set, which is a contradiction. □

For $n = 1$, the claim follows directly from [2], noticing that any nonconstant analytic function on an interval I is nowhere constant on I by identity theorem (see also Section 1). For $n > 1$, we reduce it to this case. Consider $A = \mathrm{diag}(a, B)$ for $a \in \mathscr{L}_{1,1}$ any Liouville number as above, that is, such that $f_k(a) \in \mathscr{L}_{1,1}$ as well, and B for the moment an arbitrary real $(n-1)\times(n-1)$ matrix. Since the f_k are analytic,

$$\mathfrak{f}_k(A) = \mathrm{diag}(f_k(a), \mathfrak{f}_k(B)),$$

where by abuse of notation, we keep the notation $\mathfrak{f}_k$ for the maps $\mathfrak{f}_k : \tilde{U} \subseteq \mathbb{R}^{(n-1)\times(n-1)} \to \mathbb{R}^{(n-1)\times(n-1)}$ defined in the same way as in (2-1). Hence, the system of inequalities induced by $\mathfrak{f}_k(A)\mathbf{q} - \mathbf{p}$ decouples. So, $|q_1 f_k(a) - p_1| < q_1^{-N}$ for some p_1, q_1, N implies $\mathbf{p} = (p_1, 0, \ldots, 0)^t$ and $\mathbf{q} = (q_1, 0, \ldots, 0)^t$ induce equally good approximations $\|\mathfrak{f}_k(A)\mathbf{q} - \mathbf{p}\| = |q_1 f_k(a) - p_1|$ and $\|\mathbf{q}\| = |q_1|$, so $\|\mathfrak{f}_k(A)\mathbf{q} - \mathbf{p}\| < \|\mathbf{q}\|^{-N}$. By irrationality of a, clearly $\mathfrak{f}_k(B)$ and $\mathfrak{f}_k(A)$, $k \geq 0$, satisfying (1-1) is equivalent, with $\mathfrak{f}_0(C) := C$ the identity map. So, it suffices to show that a full measure set of B also gives rise to $\mathfrak{f}_k(B)$, $k \geq 0$, satisfying (1-1).

Choose B with the property that for any $C_k := \mathfrak{f}_k(B)$, $k \geq 0$, there is no nontrivial relation $C_k\mathbf{q}_k - \mathbf{p}_k = \mathbf{0}$ with $(\mathbf{p}_k, \mathbf{q}_k) \in \mathbb{Z}^m \times \mathbb{Z}^n$. We show that this is possible. Indeed, any such relation restricts C_k to a proper affine rational subspace $\mathcal{S}_k$ of $\mathbb{R}^{(n-1)^2}$, so the exceptional $B = \mathfrak{f}_k^{-1}(C_k) \subseteq \mathfrak{f}_k^{-1}(\mathcal{S}_k)$ lie in the preimage of this subspace, and thus, by Lemma 6.1 applied for dimension $n-1$ and $\mathcal{S} = \mathcal{S}_k$, form a set of Lebesgue measure zero. Since there are only countably many f_k and countably many relations (affine rational subspaces), a full measure set in the complement remains to choose B from those giving rise to A as in the theorem.

7. Proof of Theorem 4.5

For claim (i), assume $A \in \mathbb{R}^{n\times n}$ is algebraic over $\mathbb{Z}[X]$. Then, $P(A) = \mathbf{0}$ for some nonzero $P \in \mathbb{Z}[X]$. Assume $A = B + C$ for some $n \times n$ matrices B, C. Then, also $P(B + C) = \mathbf{0}$. However, $P(X + Y)$ can be expanded into a bivariate (noncommutative) polynomial $Q(X, Y) \in \mathbb{Z}[X, Y]$, so that, in particular, $P(A) = P(B + C) = Q(B, C)$. Now, Q does not induce the zero function as otherwise, putting $C = \mathbf{0}$, we would get that $Q(B, \mathbf{0}) = P(B) = \mathbf{0}$ vanishes for all B, but since the polynomial ring in one variable is not a polynomial identity ring, this implies $P(X) \equiv \mathbf{0}$, which is against our assumption. This means B, C are not weakly algebraically independent. Taking the contrapositive, if $A = B + C$ holds for some weakly algebraically independent B, C, then A is transcendental.

For claim (ii), given any real $n \times n$ matrix A, by Theorem 3.3, there is a dense G_δ set $\mathscr{H}_A \subseteq \mathscr{L}_{n,n}$ consisting of real $n \times n$ Liouville matrices B so that $A - B \in \mathscr{L}_{n,n}$ as well. We need to show that if A is not weakly algebraic, then for some B as above, the matrices $B, A - B$ are weakly algebraically independent over $\mathbb{Z}[X, Y]$.

For given A and $P \in \mathbb{Z}[X, Y]$ not inducing the zero function over the matrix ring, denote by $\mathscr{G}_{P,A}$ the set of $B \in \mathbb{R}^{n\times n}$ with $P(B, A - B) = \mathbf{0}$. Then, $\mathscr{G}_{P,A}$ is closed in $\mathbb{R}^{n\times n}$ by continuity. Assume $\mathscr{G}_{P,A}$ has empty interior for all P. Then, by countability of $\mathbb{Z}[X, Y]$, the complement of the union $\cup\mathscr{G}_{P,A}$, taken over all $P \in \mathbb{Z}[X, Y]$ not inducing $\mathbf{0}$, is a dense G_δ set. Hence, as $\mathscr{H}_A$ is also dense G_δ, the set $\mathcal{T}_A := (\cup\mathscr{G}_{P,A})^c \cap \mathscr{H}_A$ is again dense G_δ, in particular, nonempty. Then, any $B \in \mathcal{T}_A$ is suitable.

So assume otherwise for some nonzero $P \in \mathbb{Z}[X, Y]$ not inducing the $\mathbf{0}$ function, the set $\mathscr{G}_{P,A}$ has nonempty interior. Since every entry of $P(B, A - B) \in \mathbb{R}^{n\times n}$ is a multivariate scalar polynomial in the n^2 entries of B, it is clear that then $\mathscr{G}_{P,A} = \mathbb{R}^{n\times n}$ is the entire matrix set. However, this means that there is a relation $P(B, A - B) = \mathbf{0}$ for some $P \in \mathbb{Z}[X, Y]$ not inducing the $\mathbf{0}$ function, and every B. However, we can expand $P(Y, X - Y)$ into a bivariate polynomial in (noncommutative) standard form $R(X, Y)$, $R \in \mathbb{Z}[X, Y]$, so that, in particular, $P(B, A - B) = R(A, B)$ for any B. On the one hand, $R(A, B) = P(B, A - B) = \mathbf{0}$ for all B. On the other hand, R does not induce

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the zero function either. Indeed, if so, this would mean $R(C_0, C_1) = P(C_1, C_0 - C_1) = \mathbf{0}$ for all $C_0, C_1 \in \mathbb{R}^{n\times n}$, implying further that for any $D_0, D_1 \in \mathbb{R}^{n\times n}$, if we let $C_1 = D_0$, $C_0 = D_0 + D_1$, we get $P(D_0, D_1) = P(C_1, C_0 - C_1) = R(C_0, C_1) = \mathbf{0}$, which contradicts the assumption that P does not induce the $\mathbf{0}$ function. Combining these properties, we see that A is weakly algebraic (for $\ell = 1$), which is against our assumption. This argument shows that if A is not weakly algebraic, then there exist weakly algebraically independent $B, C \in \mathscr{L}_{n,n}$ with $B + C = A$.

8. Proof of Proposition 2.1 (Sketch)

We show that the Liouville property is invariant under the maps $A \to RA$, $A \to AR$ for regular $R \in \mathbb{Q}^{n\times n}$, $A \to A + T$ for $T \in \mathbb{Q}^{n\times n}$ and $A \to A^{-1}$ for regular A. Then, clearly it is preserved for the matrices in the proposition that arise from composition of these operations. For $A \to RA$, it suffices to modify good approximation pairs $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ with respect to A via $(\mathbf{p}', \mathbf{q}') := (MN \cdot R\mathbf{p}, MN \cdot \mathbf{q})$, for $M, N \in \mathbb{Z}$ the common denominators of the rational entries of R, S. Similarly, for $A \to AR$, we take $(\mathbf{p}', \mathbf{q}') := (M' \cdot \mathbf{p}, M' \cdot R^{-1}\mathbf{q})$ for $M' \in \mathbb{Z}$ the common denominator of the rational matrix R^{-1}. For $A \to A + T$ with a rational matrix T, write $T = T'/N'$ with an integer matrix T' and an integer N'. Then, we take $(\mathbf{p}', \mathbf{q}') := (N' \cdot T'\mathbf{p} + N'\mathbf{p}, N' \cdot \mathbf{q})$. For $A \to A^{-1}$, we take the reversed vector $(\mathbf{p}', \mathbf{q}') := (\mathbf{q}, \mathbf{p})$. Moreover, using R_i are invertible, it is clear that (1-1) is preserved for the obtained matrices. We leave the details to the reader.

9. Final remarks on Theorem 2.3

We believe the metrical bound in Theorem 2.3 can be improved at least to $(n-2)^2 + 2$. As the full set $\mathscr{L}_{n,n}$ has Hausdorff dimension $n(n-1)$; see [3] for a considerably more general claim, in particular, this would be optimal for $n = 2$. We sketch the proof of a special case. Let us restrict to finitely many polynomials $f_k(z) = c_{0,k} + \cdots + c_{J,k}z^J$, $1 \le k \le K$, of degrees $J = J(k)$ at least two. For $n = 2$, to obtain this result, we consider the larger class of matrices

$$A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix},$$

where again $b \in \mathscr{L}_{1,1}$ is fixed but a, c are real parameters. Again, $A \in \mathscr{L}_{2,2}$ is easily seen as soon as a avoids some countable set. The powers of such matrices have the form

$$A^j = \begin{pmatrix} v_j & w_j \\ * & * \end{pmatrix}, \quad j \ge 1,$$

with v_j, w_j polynomials in a, c satisfying the recursions

$$v_j = ar_{j-1} + bcv_{j-2}, \quad w_j = bv_{j-1}.$$

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Now it can be shown with the inverse function theorem in place of the deep result from [8] that on a joint nonempty open set $V \subseteq \mathbb{R}^2$, the maps $\theta_k : V \to \mathbb{R}^2$, $1 \le k \le K$, defined by

$$\theta_k(a, c) = \Big(\sum_{j=0}^{J} c_{j,k} v_j, \sum_{j=0}^{J} c_{j,k} w_j \Big) \in \mathbb{R}^2$$

with image the first line of $\mathfrak{f}_k(A)$, induce local diffeomorphisms onto their open images $\theta_k(V) =: V_k$ in $\mathbb{R}^2$. By a Khintchine-type result, the images contain large (full measure within the total image V_k) subsets $\mathscr{S}_k \subseteq V_k \subseteq \mathbb{R}^2$ of line vectors $\mathbf{z}_k$ with exponents $\omega^{1\times 2}(\mathbf{z}_k) = 2$. Intersecting finitely many preimages $\mathscr{S} := \cap \theta_k^{-1}(\mathscr{S}_k)$ over $1 \le k \le K$, by the locally Lipschitz property of θ_k^{-1}, we still get a full measure set $\mathscr{S} \subseteq V \subseteq \mathbb{R}^2$ within V. For any pair $(a, c) \in \mathscr{S}$ with corresponding matrix A, as $\mathbf{z}_k$ is the first row of $\mathfrak{f}_k(A)$, by Proposition 1.3, we have $\mathfrak{f}_k(A) \notin \mathscr{L}_{2,2}$, in fact,

$$\omega^{2\times 2}(\mathfrak{f}_k(A)) \le \omega^{1\times 2}(\mathbf{z}_k) = 2, \quad 1 \le k \le K.$$

This finishes the proof for $n = 2$. The extension to larger n works by considering diagonal blocks analogously to the proof of Theorem 2.3. This method also allows us to show that regular matrices A satisfy this relaxed version of Theorem 2.3.

Presumably, the case of countably many analytic functions not of the form (2-4), as in Theorem 2.3, instead of finitely many polynomials, can be treated with some refined argument. The finiteness is only used in the above argument to guarantee a joint open set for all functions f_k simultaneously on which to apply the inverse function theorem; as for transitioning to (almost) arbitrary analytic functions, it may be intricate to verify the regularity hypothesis of the inverse function theorem in this general setup. It may happen that some more analytic functions have to be excluded to achieve this.

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