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ON MOROZOV'S DISCREPANCY PRINCIPLE FOR NONLINEAR ILL-POSED EQUATIONS

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Abstract

Morozov's discrepancy principle is one of the simplest and most widely used parameter choice strategies in the context of regularization of ill-posed operator equations. Although many authors have considered this principle under general source conditions for linear ill-posed problems, such study for nonlinear problems is restricted to only a few papers. The aim of this paper is to apply Morozov's discrepancy principle for Tikhonov regularization of nonlinear ill-posed problems under general source conditions.

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1. Introduction

Many of the inverse problems that appear in science and engineering can be modelled as ill-posed equations of the form

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to Y$ is an operator (not necessarily linear) between Hilbert spaces X and Y (see, for example, [1]). We assume that the operator F and the data y are such that the above equation has a solution $\widehat{x}$ which minimizes the function $x \mapsto \|x - \overline{x}\|$, where $\overline{x}$ is an initial guess at the unknown solution. As the data available is often noisy, say y^δ in place of y with

$$\|y - y^\delta\| \leq \delta \tag{1.2}$$

for known noise level $\delta > 0$, the ill-posedness of the equation demands application of certain regularization methods for obtaining stable approximate solutions. Tikhonov regularization is one of the most widely used such methods. In Tikhonov regularization, one looks for a minimizer x_α^δ of the function

$$x \mapsto J_\alpha(x, y^\delta) := \|F(x) - y^\delta\|^2 + \alpha\|x - \overline{x}\|^2,$$

where $\alpha > 0$ is a parameter to be chosen appropriately.

It is known (see [1]) that if F is weakly closed, then for every $(y, \overline{x}) \in Y \times X$ and for every $\alpha > 0$, there exists $x \in D(F)$ such that

$$J_\alpha(x, y) = \inf_{u \in D(F)} J_\alpha(u, y).$$

Suppose that $D(F)$ contains a neighbourhood of $\widehat{x}$, and F is Fréchet differentiable at $\widehat{x}$. It is known that if $\widehat{x}$ belongs to a source set of the form

$$M_{\nu,\rho} := \{(A^*A)^\nu v : \|v\| \le \rho\}$$

for some $\rho > 0$ and $0 < \nu \le 1$, with $A := F'(\widehat{x})$, the Fréchet derivative of F at $\widehat{x}$, then by choosing α either a priori as $\alpha := c_0\delta^{2/(2\nu+1)}$ or by a Morozov-type discrepancy principle in which α satisfies the inequality

$$c_1\delta \le \|F(x_\alpha^\delta) - y^\delta\| \le c_2\delta \tag{1.3}$$

with $c_2 \ge c_1 \ge 1$, then we obtain the 'optimal estimate'

$$\|\widehat{x} - x_\alpha^\delta\| = O(\delta^{2\nu/(2\nu+1)}).$$

But there are many examples of inverse problems where the requirement that $\widehat{x}$ belongs to $M_{\nu,\rho}$ becomes too restrictive (see, for example, [5, 6, 14]). A general form of source set which was found convenient and useful in many linear ill-posed problems is

$$M_{\varphi,\rho} := \{\sqrt{\varphi(A^*A)}\, v : \|v\| \le \rho\},$$

where φ is a suitable function which is general enough to include many of the standard source conditions, including the *Hölder type* in which $\varphi(\lambda) := \lambda^\nu$ for $0 < \nu \le 1$, and the *logarithmic type* in which $\varphi(\lambda) := [\ln(1/\lambda)]^{-p}$ for $p > 0$.

The general source condition in the context of linear ill-posed problems has been considered extensively in recent years (see, for example, [10–14, 17]). Extensions of such source conditions to nonlinear problems have also been considered (see, for example, [7, 8]). However, the general source condition combined with Morozov's discrepancy principle, one of the most widely used parameter choice strategies in the context of linear ill-posed equations, does not seem to have been studied for nonlinear problems.

The aim of this short paper is to obtain an order optimal estimate for the error involved in nonlinear ill-posed equations under a general source condition, along the lines of the analysis in [13], by choosing the regularization parameter α according to the Morozov-type discrepancy principle (1.3).

We may observe that if A is injective and φ is an *index function*, that is, if φ is a positive monotonically increasing, continuous function defined on a suitable interval $[0, a]$ satisfying $\lim_{\lambda\to 0}\varphi(\lambda) = 0$, then the source set $M_{\varphi,\rho}$ defined above can be thought of as a closed ball of radius ρ in a Hilbert space X_φ which is the completion of the subspace

$$D_\varphi := \{x \in X : \exists v_x \in X \text{ with } x = \sqrt{\varphi(A^*A)}\, v_x\}$$

with respect to the norm

$$\|x\|_\varphi := \|v_x\|, \quad x \in D_\varphi.$$

The *variable Hilbert scale* $\{X_\varphi : \varphi \text{ an index function}\}$ was considered first by Hegland (see [3, 4]) as a generalization of the concept of *Hilbert scale*.

In this short paper we consider such general source condition and use Morozov's discrepancy principle (1.3) as the parameter choice strategy for choosing the regularization parameter α.

2. The main result

The following main theorem of this paper extends the known result (see [13]) for linear ill-posed problems to the nonlinear case. We shall make use of the following assumptions.

ASSUMPTION 2.1. (a) The exact data y belongs to $R(F)$, the range of F, and $\widehat{x} \in D(F)$ is such that $F(\widehat{x}) = y$.

(b) The operator F is Fréchet differentiable at $\widehat{x}$ and there exists $\eta > 0$ such that

$$\|F'(\widehat{x})(\widehat{x} - x)\| \le \eta \|F(\widehat{x}) - F(x)\| \quad \forall x \in B_r(\widehat{x}); \quad r \ge \|\widehat{x} - \overline{x}\|.$$

(c) There exist c_1, c_2 with $c_2 \ge c_1 \ge 1$ such that for every $\delta > 0$, there exists $\alpha := \alpha(\delta, y^\delta)$ satisfying

$$c_1\delta \le \|F(x_\alpha^\delta) - y^\delta\| \le c_2\delta.$$

(d) There exists $\rho > 0$ such that

$$\widehat{x} - \overline{x} \in \{\sqrt{\varphi(A^*A)}\, v : \|v\| \le \rho\},$$

where $A := F'(\widehat{x})$ and $\varphi : [0, a] \to [0, \infty)$ with $a \ge \|A\|^2$ is a monotonically increasing, continuous and concave function satisfying $\lim_{\lambda \to 0} \varphi(\lambda) = 0$.

We now state and prove the main theorem of this paper.

THEOREM 2.2. *Under Assumption 2.1,*

$$\|\widehat{x} - x_\alpha^\delta\| \le c_0 \sqrt{\psi^{-1}(\delta^2/\rho^2)},$$

where $c_0 := \max\{2, \eta(1 + c_2)\}$ *and* $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ *for* $\lambda \in [0, a]$.

PROOF. By the definition of x_α^δ and by the fact that $y = F(\widehat{x})$, we have

$$\begin{aligned}
\delta^2 + \alpha\|x_\alpha^\delta - \overline{x}\|^2 &\le c_1\delta^2 + \alpha\|x_\alpha^\delta - \overline{x}\|^2 \\
&\le \|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - \overline{x}\|^2 \\
&\le \|F(\widehat{x}) - y^\delta\|^2 + \alpha\|\widehat{x} - \overline{x}\|^2 \\
&\le \delta^2 + \alpha\|\widehat{x} - \overline{x}\|^2.
\end{aligned}$$

Hence, $\|x_\alpha^\delta - \bar{x}\|^2 \le \|\widehat{x} - \bar{x}\|^2$. Thus, by Assumption 2.1(d),

$$\begin{aligned}\|x_\alpha^\delta - \widehat{x}\|^2 &\le \|\widehat{x} - \bar{x}\|^2 + \|x_\alpha^\delta - \widehat{x}\|^2 - \|x_\alpha^\delta - \bar{x}\|^2 \\ &= 2\langle x_\alpha^\delta - \widehat{x},\ \widehat{x} - \bar{x}\rangle \\ &= 2\langle x_\alpha^\delta - \widehat{x},\ \sqrt{\varphi(A^*A)}\, v\rangle \\ &= 2\langle \sqrt{\varphi(A^*A)}(x_\alpha^\delta - \widehat{x}),\ v\rangle \\ &\le 2\rho\|\sqrt{\varphi(A^*A)}\,(x_\alpha^\delta - \widehat{x})\| \\ &\le c_0\rho\|\sqrt{\varphi(A^*A)}\,(x_\alpha^\delta - \widehat{x})\|.\end{aligned}$$

Therefore,

$$\frac{\|x_\alpha^\delta - \widehat{x}\|^2}{c_0^2\rho^2} \le \frac{\|\sqrt{\varphi(A^*A)}(x_\alpha^\delta - \widehat{x})\|^2}{\|x_\alpha^\delta - \widehat{x}\|^2}.$$

Using the spectral representation of the self-adjoint operator A^*A (see [9]),

$$\frac{\|\sqrt{\varphi(A^*A)}x\|^2}{\|x\|^2} = \frac{\int_0^a \varphi(\lambda)\, d\langle E_\lambda x, x\rangle}{\int_0^a d\langle E_\lambda x, x\rangle}$$

and hence, by Jensens's inequality (see [16]),

$$\varphi^{-1}\left(\frac{\|\sqrt{\varphi(A^*A)}x\|^2}{\|x\|^2}\right) \le \frac{\int_0^a \lambda\, d\langle E_\lambda x, x\rangle}{\int_0^a d\langle E_\lambda x, x\rangle} = \frac{\|Ax\|^2}{\|x\|^2}.$$

Thus,

$$\varphi^{-1}\left(\frac{\|x_\alpha^\delta - \widehat{x}\|^2}{c_0^2\rho^2}\right) \le \varphi^{-1}\left(\frac{\|\sqrt{\varphi(A^*A)}(x_\alpha^\delta - \widehat{x})\|^2}{\|x_\alpha^\delta - \widehat{x}\|^2}\right) \le \frac{\|A(x_\alpha^\delta - \widehat{x})\|^2}{\|x_\alpha^\delta - \widehat{x}\|^2}.$$

By (a) and (b) in Assumption 2.1,

$$\|A(x_\alpha^\delta - \widehat{x})\| \le \eta\|y - F(x_\alpha^\delta)\| \le \eta(\|y - y^\delta\| + \|y^\delta - F(x_\alpha^\delta)\|) = \eta(\delta + c_2\delta) = c_0\delta.$$

Thus,

$$\varphi^{-1}\left(\frac{\|x_\alpha^\delta - \widehat{x}\|^2}{c_0^2\rho^2}\right) \le \frac{\|A(x_\alpha^\delta - \widehat{x})\|^2}{\|x_\alpha^\delta - \widehat{x}\|^2} \le \frac{c_0^2\delta^2}{\|x_\alpha^\delta - \widehat{x}\|^2}$$

so that

$$\psi\left(\frac{\|x_\alpha^\delta - \widehat{x}\|^2}{c_0^2\rho^2}\right) \le \frac{\delta^2}{\rho^2}.$$

Equivalently, $\|x_\alpha^\delta - \widehat{x}\|^2 \le c_0\rho\sqrt{\psi^{-1}(\delta^2/\rho^2)}$. This completes the proof. □

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REMARK 2.3.

(i) Well-known special cases for φ in Assumption 2.1(d) are $\varphi(\lambda) := \lambda^\nu$ for $0 < \nu \le 1$ and $\varphi(\lambda) := [\ln(1/\lambda)]^{-p}$ for $p > 0$; the first case corresponds to the so-called mildly ill-posed problems, and second to the exponentially ill-posed problems (see, for example [5, 6]).

(ii) It is known that many of the ill-posed problems that occur in applications satisfy a condition of the form

$$\|F(x) - F(\widehat{x}) - F'(\widehat{x})(\widehat{x} - x)\| \le \eta_0 \|F(\widehat{x}) - F(x)\|$$

for all $x \in B_r(\widehat{x})$ for some $\eta_0 > 0$. This is known in the literature as the *η-condition* ([2]). Clearly, condition (b) in Assumption 2.1 is equivalent to the above *η-condition*.

(iii) Ramlau [15] has given sufficient conditions under which the regularization parameter α satisfying

$$c_1\delta \le \|F(x_\alpha^\delta) - y^\delta\| \le c_2\delta$$

as in Assumption 2.1(d) exists.

3. Conclusion

We have applied a Morozov-type discrepancy principle for choosing the regularization parameter in Tikhonov regularization of a nonlinear ill-posed equation. Under suitable assumptions on the nonlinear operator, it is shown that the procedure yields order optimal error estimates under a general source condition.

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