DISTRIBUTIVE EXTENSIONS AND QUASI-FRAMAL ALGEBRAS

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1. Introduction. In (**2**; **3**; **4**), A. L. Foster defined Boolean extensions of framal algebras and bounded Boolean extensions of framal-in-the-small algebras. Foster proved that the class of Boolean (of bounded Boolean) extensions of a framal (a framal-in-the-small) algebra A is coextensive up to isomorphism with a certain class of subdirect powers of A, namely, the class of normal (of bounded normal) subdirect powers of A. His proofs apply, however, to considerably more general situations. Indeed, as remarked in (**2**), the construction of Boolean extensions may be carried out for an arbitrary universal algebra with finitary operations; this is done, in fact, in (**4**). Using precisely the same methods of proof as those in (**2**; **3**; **4**), we extend some of Foster's results in two directions:

1. We allow the algebras in question to admit (possibly) infinitary operations.
2. We construct extensions of algebras by using distributive lattices in place of Boolean algebras. This leads us to consider quasi-framal algebras, which include framal algebras in Foster's sense.

We devote this paper to the statement of these generalizations. In order to establish them in a self-contained exposition, we shall reproduce Foster's proofs in full (and, in places, in greater detail and somewhat more precise notations).

In §2, we give certain preliminaries and go to some lengths to clarify the foundations of our subject; here we introduce the concept of functional rank of a species of algebras, on which the later developments lean rather heavily. In §§3–7, we construct certain extensions of an arbitrary universal algebra. The distributive extension defined generalizes directly the Boolean extension in Foster's sense and is, as described by Foster, a kind of pseudo-hypercomplex algebra; while the lattice extension is defined by purely formal analogy with Foster's extension and is related to the distributive extensions in much the same way as the ring of formal power series over a ring R is related to the ring of polynomials over R. In §8, we prove some structure theorems—in terms of subdirect factorizations—and this is accomplished, a little surprisingly, for entirely arbitrary universal algebras, with no particular identities assumed. Special classes of algebras such as quasi-framal algebras are treated in §§9 and 10. For these algebras, stronger structure theorems may be derived. Finally, we consider framal algebras in §11, now, however, in the general setting of quasi-framal algebras.

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2. Some preliminaries. Let $\mathfrak{S}$ be a non-empty set and ν an ordinal. An *$\mathfrak{S}$-function of rank ν* or a *ν-ary operation on $\mathfrak{S}$* is a mapping $f: \mathfrak{S}^\nu \to \mathfrak{S}$. We write rank $f = \nu$. If $(s_\xi)_{\xi<\nu} \in \mathfrak{S}^\nu$, the image $f((s_\xi)_{\xi<\nu})$ is called *the composite of $(s_\xi)_{\xi<\nu}$ under f.* In particular, an $\mathfrak{S}$-function of rank 0 is just the selection of a constant in $\mathfrak{S}$.

Let $\sigma = (\sigma_i)_{i\in I}$ be a family of ordinals. An *algebra of species σ* is an ordered pair $\tilde{A} = (A, (f_i)_{i\in I})$, where A is a non-empty set and each f_i $(i \in I)$ is a σ_i-ary operation on A called a *fundamental operation of the algebra $\tilde{A}$*. We define ord $\tilde{A}$, the *order of $\tilde{A}$*, to be the cardinality of A : ord $\tilde{A}$ = card A. We define a class of A-functions called *homogeneous $\tilde{A}$-functions* or *homogeneous induced operations of $\tilde{A}$* recursively as the smallest class of A-functions satisfying the following two properties:

1. The identity mapping of A is a homogeneous $\tilde{A}$-function of *order* 0.

2. Let $i \in I$ and $(g_\xi)_{\xi<\sigma_i}$ be a (transfinite) sequence of type σ_i of homogeneous $\tilde{A}$-functions, each g_ξ $(\xi < \sigma_i)$ being of rank ν_ξ and order c_ξ (c_ξ being a cardinal). Let c be the least cardinal $> c_\xi$ for every $\xi < \sigma_i$. Denote by $\sum_{\xi<\sigma_i} \nu_\xi$ the ordinal sum of $(\nu)_{\xi<\sigma_i}$. Let

$$\varphi : A^{\Sigma_{\xi<\sigma_i}\nu_\xi} \to \prod_{\xi<\sigma_i} A^{\nu_\xi}$$

be the canonical bijection, and

$$\prod_{\xi<\sigma_i} g_\xi : \prod_{\xi<\sigma_i} A^{\nu_\xi} \to A^{\sigma_i}$$

the induced mapping. Then

$$f_i \circ \left(\prod_{\xi<\sigma_i} g_\xi\right) \circ \varphi : A^{\Sigma_{\xi<\sigma_i}\nu_\xi} \to A$$

is a homogeneous $\tilde{A}$-function of *order* c, and is denoted by $f_i((g_\xi)_{\xi<\sigma_i}$.

In particular, each fundamental operation f_i $(i \in I)$ is a homogeneous $\tilde{A}$-function of order 1. Note that

$$\operatorname{rank}(f_i((g_\xi)_{\xi<\sigma_i})) = \sum_{\xi<\sigma_i} \nu_\xi = \sum_{\xi<\operatorname{rank} f_i} (\operatorname{rank} g_\xi).$$

Let f be a homogeneous $\tilde{A}$-function of rank ν and order c and let μ be an ordinal $\leqslant \nu$. Let $(\mathfrak{S}_\xi)_{\xi<\mu}$ be a partition of the initial segment $\{\xi | \xi < \nu\}$ of ν into mutually disjoint non-empty subsets. For each $\eta < \nu$, let $\xi(\eta)$ be the unique ordinal $<\mu$ such that $\eta \in \mathfrak{S}_{\xi(\eta)}$. Then there is induced a mapping φ: $A^\mu \to A^\nu$ defined by

$$\varphi((a_\xi)_{\xi<\mu}) = (a_{\xi(\eta)})_{\eta<\nu} \qquad \text{for any } (a_\xi)_{\xi<\mu} \in A^\mu.$$

We call the A-function $f \circ \varphi$: $A^\mu \to A$ a *strict $\tilde{A}$-function of order c* or an *induced operation of $\tilde{A}$ of order c.* In particular, any homogeneous $\tilde{A}$-function is a strict $\tilde{A}$-function.

Let f be a strict $\tilde{A}$-function of rank ν and order c and let $\mathfrak{S}$ be a subset of the initial segment $\{\xi|\xi < \nu\}$ of ν. Since $\mathfrak{S}$ has the structure of a well-ordered set, it has ordinal number μ and $\mu \leqslant \nu$. There is a unique order-isomorphism of the initial segment $\{\xi|\xi < \mu\}$ of ν onto $\mathfrak{S}$, and this induces in turn a bijection

$\varphi: A^\mu \to A^{\mathfrak{S}}$. For each $\eta \in \{\xi | \xi < \nu\} - \mathfrak{S}$, let there be given a fixed $c_\eta \in A$. Define the mapping $\psi: A^{\mathfrak{S}} \to A^\nu$ as follows: If $(a_s)_{s\in\mathfrak{S}} \in A^{\mathfrak{S}}$, then

$$\psi((a_s)_{s\in\mathfrak{S}}) = (b_\eta)_{\eta<\nu}$$

where $b_\eta = a_\eta$ if $\eta \in \mathfrak{S}$ and $b_\eta = c_\eta$ otherwise. We call the A-function $f \circ \psi \circ \varphi: A^\mu \to A$ an *$\tilde{A}$-function of order c.* In particular, any strict $\tilde{A}$-function is an $\tilde{A}$-function and any constant $a \in A$ is an $\tilde{A}$-function.

Roughly speaking, a homogeneous $\tilde{A}$-function is an A-function that can be constructed directly via the fundamental operations of $\tilde{A}$; a strict $\tilde{A}$-function is an A-function obtained from a homogeneous $\tilde{A}$-function by allowing repeated arguments; and an $\tilde{A}$-function is an A-function obtained from a strict $\tilde{A}$-function by holding certain arguments constant. $\tilde{A}$-functions are roughly the "polynomial" functions over $\tilde{A}$.

Convention 2.1. We shall always denote formally distinct $\tilde{A}$-functions, i.e., $\tilde{A}$-functions constructed in different ways, by distinct letters. Under this convention, distinct letters may represent the same A-function; and if h denotes an $\tilde{A}$-function, it has a unique order, dependent only on the manner in which h is constructed, which we denote by ord h. More precisely, we should first define a class of symbols generated by the family $(f_i)_{i\in I}$ of fundamental operations and then interpret them as $\tilde{A}$-functions according to the previous definitions. But this would take us too far back into the logical foundations of universal algebra.

Let $\mathfrak{S}$ be a set and $\mathfrak{C}$ a class of $\mathfrak{S}$-functions. We say that $\mathfrak{C}$ is *closed under composition* if it satisfies the condition stated below: Let $f \in \mathfrak{C}$ be of rank α, and $(g_\xi)_{\xi<\alpha}$ a sequence of type α in $\mathfrak{C}$, each g_ξ ($\xi < \alpha$) being of rank ν_ξ. Denote by $\sum_{\xi<\alpha} \nu_\xi$ the ordinal sum of $(\nu_\xi)_{\xi<\alpha}$. Let

$$\varphi: \mathfrak{S}^{\Sigma_{\xi<\alpha}\nu_\xi} \to \prod_{\xi<\alpha} \mathfrak{S}^{\nu_\xi}$$

be the canonical bijection, and

$$\prod_{\xi<\alpha} g_\xi : \prod_{\xi<\alpha} \mathfrak{S}^{\nu_\xi} \to \mathfrak{S}^\alpha$$

the induced mapping. Then

$$h = f \circ \left(\prod_{\xi<\alpha} g_\xi\right) \circ \varphi : \mathfrak{S}^{\Sigma_{\xi<\alpha}\nu_\xi} \to \mathfrak{S}$$

is in $\mathfrak{C}$. h is denoted by $f((g_\xi)_{\xi<\alpha})$; clearly

$$\text{rank } h = \textstyle\sum_{\xi<\text{rank } f}(\text{rank } g_\xi).$$

THEOREM 2.1. *Let $\tilde{A}$ be an algebra. Then the class of homogeneous $\tilde{A}$-functions (of strict $\tilde{A}$-functions, or $\tilde{A}$-functions) is closed under composition.*

Proof. We prove only the (simplest) case of homogeneous $\tilde{A}$-functions. Let $\mathfrak{C}$ be the class of homogeneous $\tilde{A}$-functions. We use the notations in the definition of closure given above, with A in place of $\mathfrak{S}$. Let ord $f = c$. We shall show that $h \in \mathfrak{C}$ by (transfinite) induction on c. If $c = 0$, there is nothing to prove.

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Assume that $c > 0$ and that the assertion is true for homogeneous $\tilde{A}$-functions of order $<c$. Suppose that the species of $\tilde{A}$ is $\sigma = (\sigma_i)_{i\in I}$. By definition, there exist an index $i \in I$, a sequence $(h_\eta)_{\eta<\sigma_i}$ of type σ_i in $\mathfrak{C}$, each h_η $(\eta < \sigma_i)$ being of rank μ_η and order $<c$, satisfying the condition stated below: If

$$\psi: A^{\Sigma_{\eta<\sigma_i}\mu_\eta} \to \prod_{\eta<\sigma_i} A^{\mu_\eta}$$

is the canonical bijection, and

$$\prod_{\eta<\sigma_i} h_\eta: \prod_{\eta<\sigma_i} A^{\mu_\eta} \to A^{\sigma_i}$$

is the induced mapping, then

$$f = f_i \circ \left(\prod_{\eta<\sigma_i} h_\eta\right) \circ \psi,$$

where f_i is the fundamental operation of $\tilde{A}$ of index i. Therefore we have

$$h = f_i \circ \left(\prod_{\eta<\sigma_i} h_\eta\right) \circ \psi \circ \left(\prod_{\xi<\alpha} g_\xi\right) \circ \varphi.$$

Note that $\alpha = \operatorname{rank} f = \sum_{\eta<\sigma_i} \mu_\eta$, so that

$$\sum_{\xi<\alpha} \nu_\xi = \sum_{\xi<\Sigma_{\eta<\sigma_i}\mu_\eta} \nu_\xi = \sum_{\eta<\sigma_i} \left(\sum_{\xi<\mu_\eta} \nu_\xi\right).$$

For each $\eta < \sigma_i$, denote by

$$\Phi_\eta: A^{\Sigma_{\xi<\mu_\eta}\nu_\xi} \to \prod_{\xi<\mu_\eta} A^{\nu_\xi}$$

the canonical bijection; and denote by

$$\Psi: A^{\Sigma_{\eta<\sigma_i}(\Sigma_{\xi<\mu_\eta}\nu_\xi)} \to \prod_{\eta<\sigma_i} A^{\Sigma_{\xi<\mu_\eta}\nu_\xi}$$

also the canonical bijection. Then the induced mapping $\prod_{\eta<\sigma_i} \Phi_\eta$ is the canonical bijection of

$$\prod_{\eta<\sigma_i} A^{\Sigma_{\xi<\mu_\eta}\nu_\xi} \quad \text{onto} \quad \prod_{\eta<\sigma_i}\left(\prod_{\xi<\mu_\eta} A^{\nu_\xi}\right)$$

and hence $\left(\prod_{\eta<\sigma_i} \Phi_\eta\right) \circ \Psi$ is the canonical bijection of

$$A^{\Sigma_{\xi<\alpha}\nu_\xi} = A^{\Sigma_{\eta<\sigma_i}(\Sigma_{\xi<\mu_\eta}\nu_\xi)} \quad \text{onto} \quad \prod_{\eta<\sigma_i}\left(\prod_{\xi<\mu_\eta} A^{\nu_\xi}\right).$$

It follows that

$$\begin{aligned}\psi \circ \left(\prod_{\xi<\alpha} g_\xi\right) \circ \varphi &= \psi \circ \left(\prod_{\xi<\Sigma_{\eta<\sigma_i}\mu_\eta} g_\xi\right) \circ \varphi \\ &= \left(\prod_{\eta<\sigma_i}\left(\prod_{\xi<\mu_\eta} g_\xi\right)\right) \circ \left(\prod_{\eta<\sigma_i} \Phi_\eta\right) \circ \Psi.\end{aligned}$$

Consequently,

$$\begin{aligned}h &= f_i \circ \left(\prod_{\eta<\sigma_i} h_\eta\right) \circ \left(\prod_{\eta<\sigma_i}\left(\prod_{\xi<\mu_\eta} g_\xi\right)\right) \circ \left(\prod_{\eta<\sigma_i} \Phi_\eta\right) \circ \Psi \\ &= f_i \circ \left(\prod_{\eta<\sigma_i}\left(h_\eta \circ \left(\prod_{\xi<\mu_\eta} g_\xi\right) \circ \Phi_\eta\right)\right) \circ \Psi.\end{aligned}$$

Since $\operatorname{ord} h_\eta < c$ for every $\eta < \sigma_i$, the inductive hypothesis implies that

$$h_\eta \circ \left(\prod_{\xi<\mu_\eta} g_\xi\right) \circ \Phi_\eta \in \mathfrak{C}$$

for every $\eta < \sigma_i$. Hence $h \in \mathfrak{C}$, by definition. This completes the proof.

We define *the functional rank of the species* $\sigma = (\sigma_i)_{i \in I}$, denoted by $\mathfrak{F}(\sigma)$, as the least infinite cardinal $>$card ν for any $\tilde{A}$-function of rank ν, $\tilde{A}$ being an algebra of species σ. Given an $\tilde{A}$-function of rank ν, there always exists a homogeneous $\tilde{A}$-function of rank $\geqslant \nu$. Therefore, $\mathfrak{F}(\sigma)$ may be defined solely in terms of homogeneous $\tilde{A}$-functions, $\tilde{A}$ an arbitrary algebra of species σ. In particular, any finitary species has functional rank $\aleph_0$.

A cardinal m is said to be *regular* if the sum of any family of less than m cardinals, each of which is less than m, is less than m (e.g. 0, 1, 2, $\aleph_0$, $\aleph_1$, $\aleph_2, \ldots$ are regular, but $\aleph_\omega$ is not).

THEOREM 2.2. *The functional rank* $\mathfrak{F}(\sigma)$ *of the species* $\sigma = (\sigma_i)_{i \in I}$ *is the least regular infinite cardinal* $>$card σ_i *for every* $i \in I$.

Proof. Denote by $\aleph$ the least regular infinite cardinal $>$card σ_i for every $i \in I$. Let $\tilde{A}$ be an arbitrary algebra of species σ.

Let h be any homogeneous $\tilde{A}$-function of rank α and order c. In order to prove that $\mathfrak{F}(\sigma) \leqslant \aleph$, it suffices to prove that card $\alpha < \aleph$. We proceed by induction on c. If $c = 0$, the assertion is trivial. Assume that $c > 0$ and that card $\nu < \aleph$ for any homogeneous $\tilde{A}$-function of rank ν and order $<c$. By definition, there exist an index $i \in I$, and a sequence $(g_\xi)_{\xi<\sigma_i}$ of homogeneous $\tilde{A}$-functions of order $<c$, each g_ξ $(\xi < \sigma_i)$ being of rank ν_ξ, such that $h = f_i((g_\xi)_{\xi<\sigma_i})$, where f_i is the fundamental operation of $\tilde{A}$ of index i. We have $\alpha = \text{rank } h = \sum_{\xi<\sigma_i} \nu_\xi$ and hence

$$\text{card } \alpha = \textstyle\sum_{\xi<\sigma_i}(\text{card } \nu_\xi).$$

But card $\sigma_i < \aleph$, by definition of $\aleph$; moreover, card $\nu_\xi < \aleph$ for every $\xi < \sigma_i$, by the inductive hypothesis. Therefore card $\alpha < \aleph$, by the regularity of $\aleph$.

To prove that $\aleph \leqslant \mathfrak{F}(\sigma)$, we need only show that $\mathfrak{F}(\sigma)$ is regular. Let $(c_\xi)_{\xi<\nu}$ be a sequence of type ν (ν an ordinal) of cardinals such that $c_\xi < \mathfrak{F}(\sigma)$ for every $\xi < \nu$ and card $\nu < \mathfrak{F}(\sigma)$. By definition of $\mathfrak{F}(\sigma)$, there exist an ordinal $\alpha \geqslant \nu$, an $\tilde{A}$-function f of rank α, and a sequence $(g_\xi)_{\xi<\alpha}$ of type α of $\tilde{A}$-functions, each g_ξ $(\xi < \alpha)$ being of rank μ_ξ, such that $c_\xi \leqslant \text{card } \mu_\xi$ for every $\xi < \nu$. We have

$$\begin{aligned}\textstyle\sum_{\xi<\nu} c_\xi \leqslant \sum_{\xi<\nu} \text{card } \mu_\xi \leqslant \sum_{\xi<\alpha} \text{card } \mu_\xi &= \text{card}(\textstyle\sum_{\xi<\alpha} \mu_\xi) \\ &= \text{card } (\text{rank}(f((g_\xi)_{\xi<\alpha}))).\end{aligned}$$

But $f((g_\xi)_{\xi<\alpha})$ is an $\tilde{A}$-function (Theorem 2.1). Hence

$$\textstyle\sum_{\xi<\nu} c_\xi \leqslant \text{card}(\text{rank}(f((g_\xi)_{\xi<\alpha}))) < \mathfrak{F}(\sigma).$$

This proves that $\mathfrak{F}(\sigma)$ is regular.

Convention 2.2. Given two algebras $\tilde{A} = (A, (f_i)_{i\in I})$ and $\tilde{B} = (B, (g_i)_{i\in I})$ of the same species $\sigma = (\sigma_i)_{i\in I}$, we shall agree to write f_i for g_i for every $i \in I$, provided no confusion can arise. In other words, we shall use the same symbols for the fundamental operations of algebras of species σ uniformly over the class

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of algebras of species σ. Under this convention and convention 2.1, the same letters will also be used to denote corresponding strict $\tilde{A}$-functions and strict $\tilde{B}$-functions, i.e., strict $\tilde{A}$-functions and strict $\tilde{B}$-functions constructed in the same way. We also call f_i $(i \in I)$ a *fundamental operation of the species* σ and a strict $\tilde{A}$-function f an *induced operation of the species* σ.

Let $\mathfrak{S}$ be a set, $\mathfrak{T} \subseteq \mathfrak{S}$, and f an $\mathfrak{S}$-function of rank ν. We say that $\mathfrak{T}$ is *stable* (or *closed*) under f if $f(\mathfrak{T}^\nu) \subseteq \mathfrak{T}$. Assume that this is the case and that $\mathfrak{T} \neq \emptyset$. Then $f \mid \mathfrak{T}^\nu$, the restriction of f to $\mathfrak{T}^\nu$, is a $\mathfrak{T}$-function of rank ν called *the $\mathfrak{T}$-function induced by f*. Clearly, any intersection of subsets of $\mathfrak{S}$ that are stable under f is stable under f (unless, of course, both $\nu = 0$ and the intersection is empty).

Let $\tilde{A} = (A, (f_i)_{i\in I})$ be an algebra of species $\sigma = (\sigma_i)_{i\in I}$, and B a non-empty subset of A that is stable under each f_i $(i \in I)$. Then $\tilde{B} = (B, (f_i \mid B^{\sigma_i})_{i\in I})$ is an algebra of species σ called a *subalgebra of* $\tilde{A}$. By convention 2.2, we write f_i for $f_i \mid B^{\sigma_i}$ $(i \in I)$. If h is a strict $\tilde{A}$-function of rank ν, then $h \mid B^\nu$ is also a strict B-function of rank ν and is precisely the one that is denoted also by h according to Convention 2.2. Let $(\tilde{B}_j)_{j\in J}$ be a family of subalgebras of $\tilde{A}$ such that $\bigcap_{j\in J} B_j \neq \emptyset$. Then

$$(\bigcap_{j\in J} B_j, (f_i \mid \bigcap_{j\in J} B_j)_{i\in I})$$

is a subalgebra of $\tilde{A}$ called *the intersection of* $(\tilde{B}_i)_{i\in I}$ and is denoted by $\bigcap_{j\in J} \tilde{B}_j$. Given a subset $\mathfrak{S}$ of A, if the intersection of all subalgebras containing $\mathfrak{S}$ is non-empty (which is the case if $\mathfrak{S}$ is non-empty), it is called *the subalgebra of $\tilde{A}$ generated by* $\mathfrak{S}$. A subalgebra of $\tilde{A}$ is easily seen to be stable under each induced operation of $\tilde{A}$, and the subalgebra of $\tilde{A}$ generated by $\mathfrak{S}$ consists precisely of the composites of elements of $\mathfrak{S}$ under the induced operations (or the homogeneous induced operations) of $\tilde{A}$.

Certain properties of the functional rank $\mathfrak{F}(\sigma)$ of a species σ are interesting in themselves:

1. *Let $\tilde{A}$ be an algebra of species σ generated by a set of cardinality $<\mathfrak{F}(\sigma)$. Then any set of generators of $\tilde{A}$ includes a set of generators of $\tilde{A}$ of cardinality $<\mathfrak{F}(\sigma)$.*

2. *Let $\tilde{A}$ be an algebra of species σ which has a minimal set of generators of cardinality $\aleph \geqslant \mathfrak{F}(\sigma)$. Then any set of generators of $\tilde{A}$ has cardinality $\geqslant \aleph$. In particular, any two minimal sets of generators of $\tilde{A}$ are equipotent.*

3. *Let $\mathfrak{C}$ be a chain of subalgebras of $\tilde{A}$ which is $\aleph$-complete for every cardinal $\aleph < \mathfrak{F}(\sigma)$. Then $\bigcup_{\tilde{B}\in\mathfrak{C}} \tilde{B}$ is a subalgebra of $\tilde{A}$.*

Let $\tilde{A}, \tilde{B}$ be algebras of the same species σ, and h an induced operation of rank ν of the species σ. A mapping $f: A \to B$ is said to *preserve* h if

$$f(h((a_\xi)_{\xi<\nu})) = h((f(a_\xi))_{\xi<\nu})$$

for any $(a_\xi)_{\xi<\nu} \in A^\nu$. This condition may be stated more formally as follows:

If $f^\nu : A^\nu \to B^\nu$ is the induced mapping, then $h \circ f^\nu = f \circ h$, i.e., the diagram

$$\begin{tikzcd} A^\nu \arrow[r, "f^\nu"] \arrow[d, "h"'] & B^\nu \arrow[d, "h"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is commutative. A *homomorphism of $\tilde{A}$ into $\tilde{B}$* is a mapping $f:A \to B$ which preserves all fundamental operations of the species σ. It is easily seen that a homomorphism preserves all induced operations of the species σ. A surjective (an injective, bijective) homomorphism is called an *epimorphism* (a *monomorphism*, *isomorphism*). A homomorphism (an isomorphism) of an algebra into itself is called an *endomorphism* (an *automorphism*). The class of algebras of a fixed species together with homomorphisms is evidently a bicategory.

Let f, g be $\tilde{A}$-functions of the same rank. Then we speak of $f = g$ as an *$\tilde{A}$-identity*. If both f and g are homogeneous (are strict) $\tilde{A}$-functions, then $f = g$ is called a *homogeneous* (a *strict*) $\tilde{A}$-identity. In case we are dealing with several algebras of the same species, and f, g are strict $\tilde{A}$-functions, we also write $f = g \pmod{\tilde{A}}$ when the identity $f = g$ holds in $\tilde{A}$. Similarly, if $\mathfrak{S}$ is a subset of A, $f = g \pmod{\mathfrak{S}}$ means that $f = g$ holds in $\mathfrak{S}$. Clearly, a strict identity that holds in $\tilde{A}$ holds in any subalgebra or homomorphic image of $\tilde{A}$.

Remark. Let $\tilde{A}$ be an algebra of species σ, and denote by $\mathfrak{F}(\sigma)$ the functional rank of σ. We may define functional completeness as follows: $\tilde{A}$ is *functionally complete* (is *strictly*, *homogeneously*, or *fundamentally functionally complete*) if each A-function of rank $< \mathfrak{F}(\sigma)$ is representable as an $\tilde{A}$-function (a strict, homogeneous, or fundamental $\tilde{A}$-function). For σ finitary, these definitions reduce to the corresponding ones of A. L. Foster **(2)**. Certain results of Foster may be extended. For example, every *strictly functionally complete algebra $\tilde{A}$ is simple and has no subalgebra except $\tilde{A}$ itself.*

3. Scalar subdirect powers of an algebra. Let $\tilde{A}$, $\tilde{B}$ be algebras of the same species, and $f:\tilde{A} \to \tilde{B}$ a monomorphism. Then the pair $(\tilde{B}, f)$ is called an *extension of $\tilde{A}$* and $\tilde{A}$ is said to be *embedded in $\tilde{B}$ via f*; we also say that $\tilde{B}$ is an extension of $\tilde{A}$, f being now understood. $\tilde{A}$ is called *the kernel of the extension* (B, f). $\tilde{A}$ is isomorphic to a subalgebra of $\tilde{B}$, namely, the homomorphic image $f(\tilde{A})$, and we often identify $\tilde{A}$ and $f(\tilde{A})$ via f.

Let $(\tilde{A}_j)_{j \in J}$ be a family of algebras of the same species, and $\tilde{B}$ a subalgebra of the direct product $\prod_{j\in J} \tilde{A}_j$. For each $k \in J$, denote by $\pi_k: \prod_{j \in J} \tilde{A}_j \to \tilde{A}_k$ the projection of index k. We say that $\tilde{B}$ is a *subdirect product of* $(A_j)_{j\in I}$ if $\pi_j(B) = A_j$ for every $j \in J$.

Let $\tilde{A}$ be an algebra, $\mathfrak{S}$ any set, and $\tilde{B}$ a subalgebra of the direct power $\tilde{A}^{\mathfrak{S}}$. Assume that for any $a \in A$ the element $(a_s)_{s \in \mathfrak{S}} \in A^{\mathfrak{S}}$ defined by $a_s = a$ for every $s \in \mathfrak{S}$ belongs to B; in other words, B contains all the constant mappings of $\mathfrak{S}$ into A. Then $\tilde{B}$ is clearly a subdirect power of $\tilde{A}$. We say that $\tilde{B}$ is a *scalar subdirect power of* $\tilde{A}$. If $\mathfrak{S} \neq \emptyset$, the mapping $\varphi: A \to B$, which maps each $a \in A$ upon the element $(a_s)_{s \in \mathfrak{S}} \in B$ with $a_s = a$ for every $a \in A$, is easily seen to be a monomorphism of $\tilde{A}$ into $\tilde{B}$. φ is called the *natural monomorphism of* $\tilde{A}$ *into* $\tilde{B}$. By definition, $(\tilde{B}, \varphi)$ is an extension with kernel $\tilde{A}$.

Now let $\tilde{A}$ be an algebra of *finitary* species, $\mathfrak{S}$ any set, and $\tilde{B}$ a subalgebra of the direct power $\tilde{A}^{\mathfrak{S}}$. Assume that for any $(a_s)_{s \in \mathfrak{S}} \in B$ the set $\{a_s | s \in \mathfrak{S}\}$ is finite (which is always the case if either A or $\mathfrak{S}$ is finite). Then $\tilde{B}$ is called a *bounded subalgebra of* $\tilde{A}^{\mathfrak{S}}$.

4. Lattice extensions of an algebra. Let $\tilde{A}$ be an algebra of species σ, and let $\mathfrak{F}(\sigma)$ denote the functional rank of σ. A lattice $\tilde{L}$ is said to be $\tilde{A}$-*admissible* if it is a lattice with 0 and 1 which satisfies the following two conditions: (1) If ord $\tilde{A} = 1$, $\tilde{L}$ is $\aleph$-complete for every cardinal $\aleph < \mathfrak{F}(\sigma)$. (2) If ord $\tilde{A} > 1$, $\tilde{L}$ is $(\text{ord } A)^{\aleph}$-complete for every cardinal $\aleph < \mathfrak{F}(\sigma)$. Thus, if σ is a finitary species and $\tilde{A}$ is a finite algebra, an $\tilde{A}$-admissible lattice is an arbitrary lattice with 0 and 1.

Let $\mathfrak{S}$, $\mathfrak{T}$ be sets, $x \in \mathfrak{S}^{\mathfrak{T}}$, and $t \in \mathfrak{T}$. Then we shall use $[x]_t$ to denote the co-ordinate of index t of x.

Assume that $\tilde{L}$ is an $\tilde{A}$-admissible lattice. If f is an A-function of rank ν, where card $\nu < \mathfrak{F}(\sigma)$, we define an L^A-function $\bar{f}$ of rank ν as follows:

$$\bar{f}((x_\xi)_{\xi<\nu}) = (\bigcup_{(c_\xi)_{\xi<\nu} \in f^{-1}(a)} (\bigcap_{\xi<\nu} [x_\xi]_{c_\xi}))_{a \in A}, \qquad (x_\xi)_{\xi<\nu} \in (L^A)^\nu.$$

We call $\bar{f}$ *the* L^A-*function induced by* f *with core* $\tilde{L}$. We use here the usual convention that the join (the meet) of the empty family of elements of L is 0 (is 1). We have card $\nu < \mathfrak{F}(\sigma)$ and

$$\text{card } f^{-1}(a) \leqslant \text{card } A^\nu = (\text{ord } \tilde{A})^{\text{card } \nu}.$$

Our definition is therefore valid.

Let $\tilde{A} = (A, (f_i)_{i \in I})$ be an algebra of species σ, and $\tilde{L}$ an $\tilde{A}$-admissible lattice. For each $i \in I$, let $\bar{f}_i$ be the L^A-function induced by f_i with core $\tilde{L}$. Then $(L^A, (\bar{f}_i)_{i \in I})$ is an algebra of species σ called *the lattice extension of* $\tilde{A}$ *with core* $\tilde{L}$. (The extension terminology will be justified later.) According to a previous convention, we shall write f_i for $\bar{f}_i$ $(i \in I)$. Note that L^A also has the structure of a lattice with 0 and 1, namely, as a direct power of $\tilde{L}$.

Assume that $\tilde{L}$ is any lattice with 0 and 1, and that I is any set. As usual, we define the Kronecker delta as the mapping $\delta: I \times I \to L$ defined by $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$ $((i,j) \in I \times I)$.

THEOREM 4.1. *Let* $\tilde{A}$ *be an algebra,* $\tilde{L}$ *an* $\tilde{A}$-*admissible lattice of order* >1, *and* $\tilde{L}^{\tilde{A}}$ *the lattice extension of* $\tilde{A}$ *with core* $\tilde{L}$. *Then the mapping* $\varphi: A \to L^A$ defined by $\varphi(c) = (\delta_{c,a})_{a \in A}$ is a monomorphism of $\tilde{A}$ into $\tilde{L}^{\tilde{A}}$.

Proof. That φ is injective is trivial. Assume that f is a fundamental operation of $\tilde{A}$ of rank α and that $(c_\xi)_{\xi<\alpha} \in A^\alpha$. Take $a \in A$. We have

$$[f((\varphi(c_\xi))_{\xi<\alpha})]_a = \bigcup_{(b_\xi)_{\xi<\alpha} \in f^{-1}(a)} (\bigcap_{\xi<\alpha}[\varphi(c_\xi)]_{b_\xi}).$$

If $a = f((c_\xi)_{\xi<\alpha})$, then $(c_\xi)_{\xi<\alpha} \in f^{-1}(a)$ and $[\varphi(c_\xi)]_{c_\xi} = \delta_{c_\xi, c_\xi} = 1$ for every $\xi < \alpha$. On the other hand, suppose that $a \neq f((c_\xi)_{\xi<\alpha})$. Then $(c_\xi)_{\xi<\alpha} \notin f^{-1}(a)$. Thus, if $(b_\xi)_{\xi<\alpha} \in f^{-1}(a)$, we must have $b_\eta \neq c_\eta$ for some $\eta < \alpha$, so that

$$[\varphi(c_\eta)]_{b_\eta} = \delta_{c_\eta, b_\eta} = 0.$$

In either case, we readily obtain

$$[f((\varphi(c_\xi))_{\xi<\alpha}]_a = [\varphi(f((c_\xi)_{\xi<\alpha}))]_a.$$

It follows that $f((\varphi(c_\xi))_{\xi<\alpha}) = \varphi(f((c_\xi)_{\xi<\alpha}))$. We conclude that φ is a monomorphism.

The notations being as in Theorem 4.1, φ is called *the natural monomorphism of $\tilde{A}$ into $\tilde{L}^{\tilde{A}}$*. This being said, we now call $\tilde{A}$ the kernel of the lattice extension.

5. Distributive extensions of an algebra. Let $\aleph$ be a cardinal. A lattice $\tilde{L}$ is said to be *$\aleph$-distributive* if it satisfies the following condition: Let I be a set of cardinality $\leqslant \aleph$, and for each $i \in I$ let $(x_{i,j_i})_{j_i \in J_i}$ be a family of elements of $\tilde{L}$, J_i being a set of cardinality $\leqslant \aleph$; then

$$\bigcap_{i \in I}(\bigcup_{j_i \in J_i} x_{i,j_i}) = \bigcup_{(j_i)_{i \in I} \in \Pi_{i \in I} J_i}(\bigcap_{i \in I} x_{i,j_i})$$

and dually, whenever both sides are defined.

Let $\tilde{A}$ be an algebra of species σ, and let $\mathfrak{F}(\sigma)$ denote the functional rank of σ. A lattice $\tilde{L}$ is said to be *distributively $\tilde{A}$-admissible* if it is $\tilde{A}$-admissible and $(\text{ord } \tilde{A})^{\aleph}$-distributive for every cardinal $\aleph < \mathfrak{F}(\sigma)$. Thus, if σ is a finitary species and $\tilde{A}$ is a finite algebra, a distributively $\tilde{A}$-admissible lattice is an arbitrary distributive lattice with 0 and 1.

Let $\tilde{A}$ be an algebra of species σ, and $\tilde{L}$ a distributively $\tilde{A}$-admissible lattice. Let $A(L)$ denote the subset of L consisting of all $x \in L^A$ such that (1) $[x]_a \cap [x]_b = 0$ for $a, b \in A$ and $a \neq b$

$$(\text{i.e. } [x]_a \cap [x]_b = [x]_a \cap [x]_b \cap \delta_{a,b} \text{ for any } (a, b) \in A \times A),$$

and (2) $\bigcup_{a \in A} [x]_a = 1$. In other words, $A(L)$ consists of all partitions of $\tilde{L}$ indexed by A. Let f be an A-function of rank ν, and $\bar{f}$ the L^A-function induced by f with core $\tilde{L}$ (here rank $\nu < \mathfrak{F}(\sigma)$). We shall prove that $A(L)$ is stable under f. Let $(x_\xi)_{\xi<\nu} \in A(L)^\nu$. We must show that

$$\bar{f}((x_\xi)_{\xi<\nu}) \in A(L).$$

The proof is divided into two parts:

1. Let $a, b \in A$, $a \neq b$. Take $(c_\xi)_{\xi<\nu} \in f^{-1}(a)$ and $(d_\xi)_{\xi<\nu} \in f^{-1}(b)$. Then $f((c_\xi)_{\xi<\nu}) = a \neq b = f((d_\xi)_{\xi<\nu})$. Hence there exists $\eta < \nu$ such that $c_\eta \neq d_\eta$ so that $[x_\eta]_{c_\eta} \cap [x_\eta]_{d_\eta} = 0$ and thus

$$\bigcap_{\xi<\nu}([x_\xi]_{c_\xi} \cap [x_\xi]_{d_\xi}) = 0.$$

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Using the assumed distributivity of $\tilde{L}$, we obtain

$$\begin{aligned} 0 &= \bigcup_{((c_\xi)_{\xi<\nu},(d_\xi)_{\xi<\nu}) \in f^{-1}(a)\times f^{-1}(b)} (\bigcap_{\xi<\nu}([x_\xi]_{c_\xi} \cap [x_\xi]_{d_\xi})) \\ &= (\bigcup_{(c_\xi)_{\xi<\nu}\in f^{-1}(a)} (\bigcap_{\xi<\nu}[x_\xi]_{c_\xi})) \cap (\bigcup_{(d_\xi)_{\xi<\nu}\in f^{-1}(b)} (\bigcap_{\xi<\nu}[x_\xi]_{d_\xi})) \\ &= [\bar{f}((x_\xi)_{\xi<\nu})]_a \cap [\bar{f}((x_\xi)_{\xi<\nu})]_b. \end{aligned}$$

2. Let $a \in A$. Define $(a_\xi)_{\xi<\nu} \in A^\nu$ by $a_\xi = a$ for every $\xi < \nu$ and $\bar{a}$ by $\bar{a} = f((a_\xi)_{\xi<\nu})$. Then $(a_\xi)_{\xi<\nu} \in f^{-1}(\bar{a})$. Hence

$$\bigcap_{\xi<\nu}[x_\xi]_a = \bigcap_{\xi<\nu}[x_\xi]_{a_\xi} \subseteq \bigcup_{(c_\xi)_{\xi<\nu}\in f^{-1}(\bar{a})} (\bigcap_{\xi<\nu}[x_\xi]_{c_\xi}) = [\bar{f}((x_\xi)_{\xi<\nu})]_{\bar{a}}.$$

Using the assumed distributivity of $\tilde{L}$, we get

$$1 = \bigcap_{\xi<\nu}(\bigcap_{a\in A}[x_\xi]_a) = \bigcup_{a\in A}(\bigcap_{\xi<\nu}[x_\xi]_a) \subseteq \bigcup_{a\in A}[\bar{f}((x_\xi)_{\xi<\nu})]_{\bar{a}}.$$

Here we also made use of (1). Since $\{\bar{a} \mid a \in A\} \subseteq A$, we immediately obtain $\bigcup_{a\in A}[\bar{f}((x_\xi)_{\xi<\nu})]_a = 1$. This completes the proof of $\bar{f}((x_\xi)_{\xi<\nu}) \in A(L)$.

Since $A(L)$ is stable under $\bar{f}$, the $A(L)$-function $\bar{f} \mid A(L)^\nu$ induced by f is defined. We call $\bar{f} \mid A(L)^\nu$ *the $A(L)$-function induced by f with core $\tilde{L}$.*

Let L^A denote the lattice extension of $\tilde{A}$ with core $\tilde{L}$, $\tilde{L}$ being distributively $\tilde{A}$-admissible. As a consequence of the result above, $A(L)$ is stable under the fundamental operations of $\tilde{L}^{\tilde{A}}$ and hence inherits the structure of a subalgebra of $\tilde{L}^{\tilde{A}}$; we call this subalgebra *the distributive extension of $\tilde{A}$ with core $\tilde{L}$.* We immediately obtain

THEOREM 5.1. *Let $\tilde{A}$ be an algebra, $\tilde{L}$ a distributively $\tilde{A}$-admissible lattice of order >1, and $\tilde{A}(\tilde{L})$ the distributive extension of $\tilde{A}$ with core $\tilde{L}$. Then the mapping φ: $A \to A(L)$ defined by $\varphi(c) = (\delta_{c,a})_{a\in A}$* is a *monomorphism of $\tilde{A}$ into $\tilde{A}(\tilde{L})$.*

The notations being as in Theorem 5.1, φ is called *the natural monomorphism of $\tilde{A}$ into $\tilde{A}(\tilde{L})$.* This being said, we now call $\tilde{A}$ the kernel of the distributive extension.

Remark. The assumptions being as above, note that there is also a rather "natural" subalgebra of $\tilde{L}^{\tilde{A}}$ which contains the algebra $\tilde{A}(\tilde{L})$ as a subalgebra, namely, the subalgebra determined by the set of elements $x \in L^A$ such that $[x]_a \cap [x]_b = 0$ for $a, b \in A$ and $a \neq b$.

In the particular case when we are dealing with Boolean algebras, a distributive extension is also called a *Boolean extension.*

6. Bounded lattice extensions of an algebra of finitary species. Let $\tilde{A}$ be an algebra of finitary species and $\tilde{L}$ any lattice with 0 and 1. Let $\mathfrak{B}(L^A)$ be the set of all $x \in L^A$ such that $[x]_a = 0$ for almost all $a \in A$ (i.e. $[x]_a \neq 0$ for all but a finite number of indices $a \in A$). If f is an A-function of finite rank n, we define a $\mathfrak{B}(L^A)$-function $\bar{f}$ of rank n as follows:

$$\bar{f}((x_j)_{j<n}) = (\bigcup_{(c_j)_{j<\eta}\in f^{-1}(a)} (\bigcap_{j<n}[x_j]_{c_j}))_{a\in A}, \qquad (x_j)_{j<n} \in \mathfrak{B}(L^A)^n.$$

We need to show that $\bar{f}$ is well-defined. By definition, for each $j < n$, there exists a finite subset $B_j \subseteq A$ such that $[x_j]_a = 0$ for all $a \in A - B_j$. Let $C = \bigcup_{j<n} B_j$. Then C is also a finite subset of A and $[x_j]_a = 0$ for any $j < n$ and $a \in A - C$. For any $(c_j)_{j<n} \in A^n - C^n$, we have $[x_j]_{c_j} = 0$ for any $j < n$ and hence $\bigcap_{j<n}[x_j]_{c_j} = 0$; consequently, for the join operation involved in the definition of $\bar{f}$, we may replace the index set $f^{-1}(a)$ by $f^{-1}(a) \cap C^n$. Since $f^{-1}(a) \cap C^n \subseteq C^n$ and both n and C are finite, the co-ordinate $[\bar{f}((x_j)_{j<n})]_a$ is well-defined for any $a \in A$. Moreover, $f(C^n)$ is also finite; and if $a \in A - f(C^n)$, we have $[\bar{f}((x_j)_{j<n})]_a = 0$. Therefore $\bar{f}((x_j)_{j<n}) \in \mathfrak{B}(L^A)$. This shows that $\bar{f}$ is a well-defined $\mathfrak{B}(L^A)$-function. We call $\bar{f}$ the *$\mathfrak{B}(L^A)$-function induced by f with core $\tilde{L}$.*

Let $\tilde{A} = (A, (f_i)_{i \in I})$ be an algebra of a finitary species σ, and $\tilde{L}$ any lattice with 0 and 1. For each $i \in I$, let $\bar{f}_i$ be the $\mathfrak{B}(L^A)$-function induced by f_i with core $\tilde{L}$. Then $(\mathfrak{B}(L^A), (\bar{f}_i)_{i \in I})$ is an algebra of species σ called *the bounded lattice extension of $\tilde{A}$ with core $\tilde{L}$.* (The extension terminology will be justified later.) According to a previous convention, we shall write f_i for $\bar{f}_i$ $(i \in I)$.

As in Section 4, we readily obtain

THEOREM 6.1. *Let $\tilde{A}$ be an algebra of a finitary species, $\tilde{L}$ a lattice with 0 and 1 of order >1, and $\mathfrak{B}(\tilde{L}^{\tilde{A}})$ the bounded lattice extension of $\tilde{A}$ with core $\tilde{L}$. Then the mapping $\varphi : A \to \mathfrak{B}(L^A)$ defined by $\varphi(c) = (\delta_{c,a})_{a \in A}$ is a monomorphism of $\tilde{A}$ into $\mathfrak{B}(\tilde{L}^{\tilde{A}})$.*

φ is called *the natural nonomorphism of $\tilde{A}$ into $\mathfrak{B}(\tilde{L}^A)$.* This being said, we now call $\tilde{A}$ the kernel of the bounded lattice extension.

THEOREM 6.2. *Let $\tilde{A}$ be an algebra of a finitary species, and $\tilde{L}$ an $\tilde{A}$-admissible lattice. Then the bounded lattice extension of $\tilde{A}$ with core $\tilde{L}$ is a subalgebra of the lattice extension of $\tilde{A}$ with core $\tilde{L}$.*

In particular, if $\tilde{A}$ is a finite algebra, the two extensions coincide.

7. Bounded distributive extensions of an algebra of finitary species. Let $\tilde{A}$ be an algebra of finitary species, and $\tilde{L}$ a distributive lattice with 0 and 1. As in §6, we define $\mathfrak{B}(L^A)$ as the set of all $x \in L^A$ such that $[x]_a = 0$ for almost all $a \in A$. We define $\mathfrak{B}(A(L))$ as the subset of $\mathfrak{B}(L^A)$ consisting of all $x \in \mathfrak{B}(L^A)$ such that (1) $[x]_a \cap [x]_b = 0$ for $a, b \in A$ and $a \neq b$, and (2) $\bigcup_{a \in A}[x]_a = 1$. In other words, $\mathfrak{B}(A(L))$ consists of all finite "partitions" of $\tilde{L}$ indexed by A. Let f be an A-function of finite rank n, and $\bar{f}$ the $\mathfrak{B}(L^A)$-function induced by f with core $\tilde{L}$. Then as in §5, we can prove that $\mathfrak{B}(A(L))$ is stable under $\bar{f}$; hence the $\mathfrak{B}(A(L))$-function $\bar{f} \mid \mathfrak{B}(A(L))^n$ induced by $\bar{f}$ is defined. We call $\bar{f} \mid \mathfrak{B}(A(L))^n$ the $\mathfrak{B}(A(L))$-function induced by f with core $\tilde{L}$.

Let $\mathfrak{B}(\tilde{L}^{\tilde{A}})$ denote the bounded lattice extension of $\tilde{A}$ with core $\tilde{L}$, $\tilde{L}$ being a distributive lattice with 0 and 1. As a consequence of the result above, $\mathfrak{B}(A(L))$ is stable under the fundamental operations of $\mathfrak{B}(\tilde{L}^{\tilde{A}})$ and hence

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inherits the structure of a subalgebra of $\mathfrak{B}(\tilde{L}^{\tilde{A}})$; we call this subalgebra *the bounded distributive extension of $\tilde{A}$ with core $\tilde{L}$.*

THEOREM 7.1. *Let $\tilde{A}$ be an algebra of a finitary species, $\tilde{L}$ a distributive lattice with 0 and 1 of order >1, and $\mathfrak{B}(\tilde{A}(\tilde{L}))$ the bounded distributive extension of $\tilde{A}$ with core $\tilde{L}$. Then the mapping $\varphi : A \to \mathfrak{B}(A(L))$ defined by $\varphi(c) = (\delta_{c,a})_{a\in A}$ is a monomorphism of $\tilde{A}$ into $\mathfrak{B}(\tilde{A}(\tilde{L}))$.*

φ is called *the natural monomorphism of $\tilde{A}$ into $\mathfrak{B}(\tilde{A}(\tilde{L}))$.* This being said, we call $\tilde{A}$ the kernel of the bounded distributive extension.

THEOREM 7.2. *Let $\tilde{A}$ be an algebra of a finitary species, and $\tilde{L}$ a distributively $\tilde{A}$-admissible lattice. Then the bounded distributive extension of $\tilde{A}$ with core $\tilde{L}$ is a subalgebra of the distributive extension of $\tilde{A}$ with core $\tilde{L}$.*

In particular, if $\tilde{A}$ is a finite algebra, the two extensions coincide.

A bounded distributive extension whose core is a Boolean algebra is also called a *bounded Boolean extension.*

8. The structure of distributive extensions: subdirect factorizations:

THEOREM 8.1. *Let $\tilde{A}$ be an algebra, $\tilde{L}$ a distributively $\tilde{A}$-admissible and atomic Boolean algebra, and $\tilde{A}(\tilde{L})$ the Boolean extension of $\tilde{A}$ with core $\tilde{L}$. Then $\tilde{A}(\tilde{L})$ is isomorphic to a scalar subdirect power of $\tilde{A}$. Moreover, if $\tilde{L}$ is complete, $\tilde{A}(\tilde{L})$ is isomorphic to a direct power of $\tilde{A}$.*

Proof. Let $\mathfrak{S}$ be the set of atoms of $\tilde{L}$. Fix $s \in \mathfrak{S}$. If $x \in A(L)$, there exists an index $c \in A$ such that $s \subseteq [x]_c$, because $s \subseteq 1 = \bigcup_{a\in A}[x]_a$; moreover, c is unique, because $[x]_a \cap [x]_b = 0$ for $a, b \in A$ and $a \neq b$. Denoting c by $f_s(x)$, we obtain a mapping $f_s : A(L) \to A$. We have $s \subseteq [x]_{f_s(x)}$ for every $x \in A(L)$. We shall now prove that f_s is a homomorphism of $\tilde{A}(\tilde{L})$ into $\tilde{A}$. Let h be a fundamental operation of rank ν, and let $(x_\xi)_{\xi<\nu} \in A(L)$. We are to prove that

$$f_s(h((x_\xi)_{\xi<\nu})) = h((f_s(x_\xi))_{\xi<\nu}).$$

For this purpose, it is sufficient to show that $s \subseteq [h((x_\xi)_{\xi<\nu})]_b$ where

$$b = h((f_s(x_\xi))_{\xi<\nu}).$$

By definition, $s \subseteq [x_\xi]_{f_s(x_\xi)}$ for every $\xi < \nu$ so that $s \subseteq \bigcap_{\xi<\nu}[x_\xi]_{f_s(x_\xi)}$. But $(f_s(x_\xi))_{\xi<\nu} \in h^{-1}(b)$. Hence

$$s \subseteq \bigcap_{\xi<\nu}[x_\xi]_{f_s(x_\xi)} \subseteq \bigcup_{(c_\xi)_{\xi<\nu}\in h^{-1}(b)} \left(\bigcap_{\xi<\nu}[x_\xi]_{c_\xi}\right) = [h((x_\xi)_{\xi<\nu})]_b.$$

This proves our claim.

For each $t \in \mathfrak{S}$, let $\pi_t : A^{\mathfrak{S}} \to A$ be the projection of index t. Denote by $\tilde{A}^{\mathfrak{S}}$ the direct power of $\tilde{A}$ with exponent $\mathfrak{S}$. Then there exists a unique homomorphism $f : \tilde{A}(\tilde{L}) \to \tilde{A}^{\mathfrak{S}}$ such that $\pi_s \circ f = f_s$ for every $s \in \mathfrak{S}$. In order to show that f is injective, it suffices to prove that the family $(f_s)_{s\in\mathfrak{S}}$ of mappings of

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$A(L)$ into A *distinguishes points* in the sense that if $x, y \in A(L)$ and $x \neq y$, then $f_t(x) \neq f_t(y)$ for some $t \in \mathfrak{S}$. Assume then that $x, y \in A(L)$ and $x \neq y$. Then there exists $a \in A$ such that $[x]_a \neq [y]_a$. For this $a \in A$, we must have $[x]_a \cap [y]_a^* \neq 0$ or $[y]_a \cap [x]_a^* \neq 0$, where * denotes complimentation in the Boolean algebra $\tilde{L}$. We may assume that $[x]_a \cap [y]_a^* \neq 0$. By atomicity of $\tilde{L}$, there exists an atom $t \subseteq [x]_a \cap [y]_a^*$ so that $t \subseteq [x]_a$ but $t \subseteq [y]_a$ does not hold. Hence $f_t(x) = a \neq f_t(y)$. We conclude that f is a monomorphism.

If $c \in A$, then f maps $(\delta_{c,a})_{a \in A} \in A(L)$ upon the element $(c_s)_{s \in \mathfrak{S}} \in f(A(L))$ defined by $c_s = c$ for every $s \in \mathfrak{S}$. Hence the homomorphic image $f(\tilde{A}(\tilde{L}))$ must be a scalar subdirect power of $\tilde{A}$. This proves the first assertion of the theorem.

Assume now that $\tilde{L}$ is complete. We must prove that f is surjective. Let $(a_s)_{s \in \mathfrak{S}} \in A^{\mathfrak{S}}$. For each $a \in A$, define $\mathfrak{T}_a = \{s \in \mathfrak{S} \mid a_s = a\}$ and define $x = (\bigcup_{s \in \mathfrak{T}_a} s)_{a \in A}$. By the completeness of $\tilde{L}$, $\bigcup_{s \in \mathfrak{T}_a} s$ is defined for every $a \in A$, so that $x \in L^A$. The proof that $x \in A(L)$ is in two parts:

1. Let $a, b \in A$ and $a \neq b$. If $s \in \mathfrak{T}_a$ and $t \in \mathfrak{T}_b$, then s and t are distinct atoms and hence $s \cap t = 0$. Therefore

$$[x]_a \cap [x]_b = \left(\bigcup_{s \in \mathfrak{T}_a} s\right) \cap \left(\bigcup_{t \in \mathfrak{T}_b} t\right) = \bigcup_{(s,t) \in \mathfrak{T}_a \times \mathfrak{T}_b} (s \cap t) = 0.$$

(The distributivity used here is valid in any complete Boolean algebra **(1)**.)

2. Since $\bigcup_{a \in A} \mathfrak{T}_a = \mathfrak{S}$, where $\bigcup$ here denotes set-theoretic union, by the completeness of $\tilde{L}$, we have

$$\bigcup_{a \in A} [x]_a = \bigcup_{a \in A} \left(\bigcup_{s \in \mathfrak{T}_a} s\right) = \bigcup_{s \, s \in \cup_{a \in A} \mathfrak{T}_a} s = \bigcup_{s \in \mathfrak{S}} s.$$

But $\bigcup_{s \in \mathfrak{S}} s = 1$, by the atomicity of $\tilde{L}$. Thus, $\bigcup_{a \in A}[x]_a = 1$. We conclude that $x \in A(L)$. We have

$$\pi_s(f(x)) = (\pi_s \circ f)(x) = f_s(x) = f_s\left(\left(\bigcup_{t \in \mathfrak{T}_a} t\right)_{a \in A}\right) = a_s \text{ for every } s \in \mathfrak{S}.$$

Therefore $f(x) = (a_s)_{s \in \mathfrak{S}}$. This completes the proof of Theorem 8.1.

A Boolean algebra is said to be *completely distributive* if it is complete and $\aleph$-distributive for every cardinal $\aleph$. A result of Tarski shows that a completely distributive Boolean algebra is isomorphic to the field of all subsets of some set **(1)**. Therefore a completely distributive Boolean algebra is also atomic.

THEOREM 8.2. *Let $\tilde{A}$ be an algebra. Then the class of Boolean extensions of $\tilde{A}$ with completely distributive cores is co-extensive up to isomorphism with the class of direct powers of $\tilde{A}$.*

Proof. Let $\tilde{L}$ be a completely distributive Boolean algebra. Then $\tilde{L}$ is distributively $\tilde{A}$-admissible, complete, and atomic. Therefore the Boolean extension of $\tilde{A}$ with core $\tilde{L}$ is isomorphic to a direct power of $\tilde{A}$, by Theorem 8.1.

On the other hand, let $\mathfrak{S}$ be any set, and $\tilde{A}^{\mathfrak{S}}$ the direct power of $\tilde{A}$ with exponent $\mathfrak{S}$. Let $\mathfrak{T} = \{\{s\} \mid s \in \mathfrak{S}\}$ (i.e. $\mathfrak{T}$ is the set of singleton subsets of $\mathfrak{S}$).

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Then the direct powers $\tilde{A}^{\mathfrak{S}}$ and $\tilde{A}^{\mathfrak{T}}$ are isomorphic. Let $\tilde{L}$ be the field of all subsets of $\mathfrak{S}$. Then $\tilde{L}$ is a completely distributive Boolean algebra with $\tilde{\mathfrak{T}}$ as its set of atoms. As in the proof of Theorem 8.1, we see that the Boolean extension of $\tilde{A}$ with core $\tilde{L}$ is isomorphic to $\tilde{A}^{\mathfrak{T}}$ and hence also to $\tilde{A}^{\mathfrak{S}}$.

THEOREM 8.3. *Let $\tilde{A}$ be an algebra of finitary species, and $\tilde{L}$ a distributive lattice with* 0 *and* 1. *Then the bounded distributive extension of $\tilde{A}$ with core $\tilde{L}$ is isomorphic with a bounded scalar subdirect power of $\tilde{A}$.*

Proof. $\tilde{L}$ can be represented isomorphically as a ring of subsets of a set in such a way that 0 (that 1) of $\tilde{L}$ corresponds to the empty set $\emptyset$ (to the entire set) **(1)**. Considering the field of all subsets of this set, we see that $\tilde{L}$ is embedded in a completely distributive Boolean algebra $\tilde{M}$ in such a way that the 0 (the 1) of $\tilde{L}$ corresponds to the 0 (the 1) of $\tilde{M}$. We may assume that $\tilde{M}$ contains $\tilde{L}$ as a sublattice. Let $\mathfrak{S}$ be the set of atoms of $\tilde{M}$. Now we may proceed in the same way as in the proof of Theorem 8.1.

9. Quasi-framal algebras. A *quasi-frame* is an algebra $\tilde{Q} = (Q, (0, 1, +, \times))$ of species (0, 0, 2, 2) satisfying the following identities:

$$0 + q = q + 0 = q, \quad 1 \times q = q \times 1 = q, \quad 0 \times q = q \times 0 = 0 \qquad (q \in Q).$$

A subalgebra of a quasi-frame $\tilde{Q}$ is again a quasi-frame called a *sub-quasi-frame* of Q. Note that the set $\{0, 1\}$ determines a sub-quasi-frame of $\tilde{Q}$, in fact, the unique one generated by $\emptyset$; and if $0 \neq 1$, this quasi-frame may be considered a chain of order 2 (i.e. isomorphic to the ordinal 2) by regarding $+$ (by regarding $\times$) as join (as meet).

Let $\tilde{A}$ be an algebra. Assume that there exist $\tilde{A}$-functions (strict $\tilde{A}$-functions, homogeneous $\tilde{A}$-functions, or fundamental operations of $\tilde{A}$) 0, 1, $+$, $\times$ such that $\tilde{Q} = (A, (0, 1, +, \times)$ is a quasi-frame. Then $\tilde{Q}$ is called *a quasi-frame* (*a strict, homogeneous,* or *fundamental quasi-frame*) *for* $\tilde{A}$, and $\tilde{A}$ is said to be *quasi-framal* (to be *strictly, homogeneously,* or *fundamentally quasi-framal*).

Thus, any lattice with 0 and 1 and any (not necessarily associative) linear algebra with identity over an associative ring with identity are quasi-framal algebras; in particular, any (not necessarily associative)ring with identity is a quasi-framal algebra.

Let $\tilde{A}$ be a quasi-framal algebra with $\tilde{Q} = (A, (0, 1, +, \times))$ as a quasi-frame for $\tilde{A}$, and $\mathfrak{S}$ any set. For each $a \in A$, the mapping $\rho_a: A^{\mathfrak{S}} \to A^{\mathfrak{S}}$ defined by $\rho_a(a_s)_{s \in \mathfrak{S}}) = (\delta_{a_s, a})_{s \in \mathfrak{S}}$ is called *the projector of $\tilde{A}^{\mathfrak{S}}$ of index a relative to the quasi-frame $\tilde{Q}$.* Let $\tilde{B}$ be a subalgebra of $\tilde{A}$ which is a scalar subdirect power of $\tilde{A}$. If $\bigcup_{a \in A} \rho_a(B) \subseteq B$, then $\tilde{B}$ is said to be a *normal subdirect power of $\tilde{A}$ relative to $\tilde{Q}$.*

The notations being as above, assume that $\tilde{B}$ is, in fact, a normal subdirect power of $\tilde{A}$ relative to the quasi-frame $\tilde{Q}$. Let L be the set of all elements $(a_s)_{s \in \mathfrak{S}} \in B$ such that $a_s = 0$ or 1 for every $s \in \mathfrak{S}$. Since $\tilde{B}$ is a scalar subdirect power of $\tilde{A}$, B must contain the element in A each of whose co-ordinates is 0 (is 1). Thus, $L \neq \emptyset$. It is easy to see that L determines a sub-quasi frame $\tilde{L}$

of the quasi-frame $\tilde{Q}^{\mathfrak{S}}$ for $\tilde{A}^{\mathfrak{S}}$. It is further obvious that $\tilde{L}$ is isomorphic to a subdirect power of the ordinal 2 and is hence a distributive lattice **(1)**. $\tilde{L}$ is in fact a distributive lattice with 0 and 1, whose 0 (whose 1) is the element each of whose co-ordinates is $0 \in A$ (is $1 \in A$). We call *$\tilde{L}$ the core of $\tilde{B}$ induced by $\tilde{Q}$.*

If $\tilde{B}$ is a normal subdirect power of $\tilde{A}$ relative to any quasi-frame (to any strict quasi-frame, homogeneous quasi-frame, or fundamental quasi-frame) for $\tilde{A}$, then $\tilde{B}$ is said to be a *normal* (a *strictly*, *homogeneously*, or *fundamentally normal*) *subdirect power of* $\tilde{A}$.

THEOREM 9.1. *Let $\tilde{A}$ be a quasi-framal (a strictly, homogeneously, or fundamentally quasi-framal) algebra, $\tilde{L}$ a distributively $\tilde{A}$-admissible and atomic Boolean algebra, and $\tilde{A}(\tilde{L})$ the Boolean extension of $\tilde{A}$ with core $\tilde{L}$. Then $\tilde{A}(\tilde{L})$ is isomorphic to a normal (a strictly, homogeneously, or fundamentally normal) subdirect power of $\tilde{A}$.*

Proof. We use the notations of the proof of Theorem 8.1. We already know that $f(\tilde{A}(\tilde{L}))$ is a scalar subdirect power of $\tilde{A}$. Let $\tilde{Q} = (A, (0, 1, +, \times))$ be any quasi-frame (any strict quasi-frame, homogeneous quasi-frame, or fundamental quasi-frame) for A, and let $c \in A$. Denote by $\rho_c : A^{\mathfrak{S}} \to A^{\mathfrak{S}}$ the projector of index c relative to $\tilde{Q}$. Let $x \in A(L)$. We need only prove that $\rho_c(f(x)) \in f(A(L))$. We have

$$\rho_c(f(x)) = \rho_c((f_s(x))_{s\in\mathfrak{S}}) = (\delta_{f_s(x).c})_{s\in\mathfrak{S}}.$$

Let the element $y \in A(L)$ be defined as follows: $[y]_0 = \bigcup_{a\in A-\{c\}}[x]_a$, $[y]_1 = [x]_c$, and $y_a = 0$ for $a \in A$, $a \neq 0, 1$. Now $f_s(y) = 1$ if and only if $s \subseteq [y]_1 = [x]_c$ and the latter holds if and only if $f_s(x) = c$; furthermore, $f_s(y) = 0$ if $f_s(x) \neq c$. Hence $f_s(y) = \delta_{f_s(x),c}$ and so

$$\rho_c(f(x)) = (\delta_{f_s(x),c})_{s\in\mathfrak{S}} = (f_s(y))_{s\in\mathfrak{S}} = f(y) \in f(A(L)).$$

This proves Theorem 9.1.

Theorem 9.1 is a sharper version of Theorem 8.1 for quasi-framal algebras. In the same way, we can prove a sharper version of Theorem 8.3 for quasi-framal algebras of finitary species. We shall, however, give a stronger result in §10.

10. Quasi-framal-in-the-small algebras of finitary species. Let $\tilde{A}$ be an algebra of finitary species, and let f be an A-function. Assume that for any *finite* subset $\mathfrak{S}$ of A, there exists an $\tilde{A}$-function (a strict $\tilde{A}$-function, homogeneous $\tilde{A}$-function, or fundamental operation of $\tilde{A}$) f_s such that $f = f_{\mathfrak{S}}$ (mod $\mathfrak{S}$. Then we say that f is *$\tilde{A}$-representable in the small* (*is strictly $\tilde{A}$-representable in the small, homogeneously $\tilde{A}$-representable in the small*, or *fundamentally $\tilde{A}$-representable in the small*).

Let $\tilde{A}$ be an algebra of finitary species, and $\tilde{Q} = (A, (0, 1, +, \times))$ a quasi-frame with the same underlying set A as $\tilde{A}$. Assume that 0, 1, +, $\times$ are all

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$\tilde{A}$-representable in the small. Then we call $\tilde{Q}$ a *small quasi-frame for* $\tilde{A}$ and we say that $\tilde{A}$ is *quasi-framal-in-the-small.* Related notions such as *small strict quasi-frame for* $\tilde{A}$, *strictly quasi-framal-in-the-small algebra*, etc. are defined in the same way.

Again, *projectors relative to small quasi-frames, normal-in-the-small subdirect powers*, etc. are defined in the same way as in §9.

Obviously any quasi-framal algebra of finitary species is also quasi-framal-in-the-small. On the other hand, any finite quasi-framal-in-the-small algebra is quasi-framal.

THEOREM 10.1. *Let* $\tilde{A}$ *be a quasi-framal-in-the-small (a strictly, homogeneously, or fundamentally quasi-framal-in-the-small) algebra of finitary species. Then the class of bounded distributive extensions of* $\tilde{A}$ *is coextensive up to isomorphism with the class of bounded normal-in-the-small (of strictly, homogeneously, or fundamentally normal-in-the-small) subdirect power of* $\tilde{A}$.

Proof. Using Theorem 8.3 and a proof analogous to that of Theorem 9.1, we readily see that any bounded distributive extension of $\tilde{A}$ is isomorphic to a bounded normal-in-the small subdirect power of $\tilde{A}$.

On the other hand, let $\tilde{B}$ be a bounded normal-in-the-small subdirect power of $\tilde{A}$; we may assume that $\tilde{B}$ is a subalgebra of the direct power $\tilde{A}^{\mathfrak{S}}$. Let $\tilde{Q}$ be a small quasi-frame for $\tilde{A}$, and $\tilde{L}$ the core of $\tilde{B}$ induced by $\tilde{Q}$. For each $a \in A$, denote by $\rho_a : A^{\mathfrak{S}} \to A^{\mathfrak{S}}$ the projector of index a; we note that each ρ_a $(a \in A)$ maps $\tilde{B}$ into $\tilde{L}$. Define the mapping $f: B \to \mathfrak{B}(A(L))$ by $f(b) = (\rho_a(b))_{a\in A}$. Now any $b \in B$ is of the form $(b_s)_{s\in\mathfrak{S}}$ where $b_s \in A$ for every $s \in \mathfrak{S}$, so that

$$f(b) = f((b_s)_{s\in\mathfrak{S}}) = (\rho_a((b_s)_{s\in\mathfrak{S}}))_{a\in A} = ((\delta_{b_s,a})_{s\in\mathfrak{S}})_{a\in A}.$$

In order to show that f is well-defined, we must prove that $f(b) \in \mathfrak{B}(A(L))$. The proof of this falls into three parts:

1. Since $\tilde{B}$ is a bounded subalgebra of $\tilde{A}^{\mathfrak{S}}$, the set $\{b_s | s \in \mathfrak{S}\}$ is finite and so for all but finitely many $a \in A$ we have $b_s \neq a$ for every $s \in \mathfrak{S}$. Hence $\rho_a(b) = (\delta_{b_s,a})_{s\in\mathfrak{S}} = 0$ (= the least element of L) for almost all $a \in A$.

2. Let $a, a' \in A$ and $a \neq a'$. Then

$$\rho_a(b) \times \rho_{a'}(b) = (\delta_{b_s,a})_{s\in\mathfrak{S}} \times (\delta_{b_s,a'})_{s\in\mathfrak{S}} = (\delta_{b_s,a} \times \delta_{b_s,a'})_{s\in\mathfrak{S}} = (\delta_{a,a'})_{s\in\mathfrak{S}} = 0$$

(= the least element of $\tilde{L}$). Here $\times = \cap =$ meet operation in the lattice $\tilde{L}$.

3. $\sum_{a\in A} \rho_a(b) = \sum_{a\in A} (\delta_{b_s,a})_{s\in\mathfrak{S}}$. Since $\delta_{b_s,a} = 0$ for every $s \in \mathfrak{S}$, for almost all $a \in A$, the last expression is equal to $(\sum_{a\in A} \delta_{b_s,a})_{s\in\mathfrak{S}} = 1$ (= the greatest element of $\tilde{L}$). Here $\sum = \cup =$ join operation in the lattice $\tilde{L}$.

Thus, f is well-defined.

As in the proof of (**2**, Theorem 17, Part II), we can prove that f is an isomorphism.

11. Framal algebras and framal-in-the-small algebras of finitary species. We shall now show that quasi-framal algebras generalize *framal* *algebras* in the sense of A. L. Foster (**2**; **3**; **4**) by giving a definition of *frames* in terms of quasi-frames which is essentially equivalent to that given by Foster.

A *frame* is an algebra $\tilde{F} = (F, (0, 1, {}^{\cup}, {}^{\cap}, +, \times))$ of species (0, 0, 1, 1, 2, 2) such that $(F, (0, 1, +, \times))$ is a quasi-frame and the following identities are satisfied:

$$0^{\cup} = 0^{\cap} = 1, \qquad 1^{\cup} = 1^{\cap} = 0;$$
$$g^{\cup\cap} = g^{\cap\cup} = g \qquad (g \in F);$$
$$g + r = (g^{\cup} \times r^{\cup})^{\cap} \qquad (g, r \in F).$$

The notions of *framal-algebras*, *framal-in-the-small algebras* of finitary species, etc. are defined in the same way as in §9.

Combining Theorem 10.1 and a result of Foster (**4**, Theorem **11.1**), we immediately get

THEOREM 11.1. *Let $\tilde{A}$ be a framal-in-the-small (a strictly, homogeneously, or fundamentally framal-in-the-small) algebra of finitary species. Then the following classes of algebras are coextensive up to isomorphism:*

(1) *the class of bounded distributive extensions of $\tilde{A}$,*

(2) *the class of bounded Boolean extensions of $\tilde{A}$,*

(3) *the class of bounded normal-in-the-small (of strictly, homogeneously, or fundamentally normal-in-the-small) subdirect powers of $\tilde{A}$.*

It should be observed here that "normal-in-the-small" in our sense is identical with "normal" in Foster's sense (**4**).

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