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THE ASCENDING AND DESCENDING VARIETAL CHAINS OF A VARIETY

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1. Introduction. Let V be a variety (equational class) of algebras. For $n \geqq 0$, V_n is the variety generated by the V-free algebra on n free generators while V^n is the variety of all algebras satisfying each identity of V which has no more than n variables. (Equivalently, V^n is the class of all algebras, $\mathfrak{A}$, such that every n-generated subalgebra of $\mathfrak{A}$ is in V.) Note that unless nullary operation symbols are specified by the similarity type of V, V_0 is the variety of all one element algebras while V^0 is the variety of all algebras. Clearly $V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V \subseteq \ldots \subseteq V^2 \subseteq V^1 \subseteq V^0$; V is generated by $\bigcup_{i=0}^{\infty} V_i$ and $V = \bigcap_{i=0}^{\infty} V^i$. The chain $V_0 \subseteq V_1 \subseteq \ldots$ is called the *ascending varietal chain* of V while $V^0 \supseteq V^1 \supseteq V^2 \supseteq \ldots$ is called the *descending varietal chain* of V. These chains of varieties were first introduced in the study of varieties of groups; see [**5**]. A natural question to ask is what patterns of proper inclusion are possible in these chains. This problem was pointed out to us by G. Gratzer and N. Gupta. After we proved Theorem 2, N. Gupta and F. Levin modified our argument to show that for groups any pattern with finitely many proper inclusions is possible for the descending varietal chain (with the necessary restriction, of course, that $V^0 \neq V^1 \neq V^2 \neq V^3$). For a discussion of the ascending and descending varietal chains of some particular varieties of groups see [**2**] and [**3**] while for a discussion of the ascending varietal chain of a particular variety of commutative Moufang loops see [**1**].

THEOREM 1. *For any set S of non-negative integers there is a variety V with finitely many operations such that for every $n \geqq 0$ $V_n = V_{n+1}$ if and only if $n \in S$.*

THEOREM 2. *For any set S of integers greater than 2 there is a variety V of semigroups such that:*

(i) *For any $n \geqq 0$ $V^n = V^{n+1}$ if and only if $n \in S$.*

(ii) *V is finitely based if and only if S is cofinite.*

(iii) *The equational theory of V is recursive if and only if S is recursive.*

2. The ascending varietal chain of a variety. The similarity type used to establish Theorem 1 is $\langle 0, 0, 2, 2, 3, 3, 2, 1\rangle$ and the corresponding algebras will be denoted $\mathfrak{A} = \langle A; a, b, \wedge, \vee, F, G, H, f\rangle$.

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For $n \geqq 0$ let $A_n = \{0, 1, \ldots, n+1\}$ and define

$$\mathfrak{A}_n = \langle A_n; 0, n+1, \wedge, \vee, F_n, G_n, H_n, f_n \rangle$$

as follows:

$$x \wedge y = \min\{x, y\}; x \vee y = \max\{x, y\}; f_n(x) = 0;$$

$$F_n(x, y, z) = \begin{cases} y \text{ if } x \neq n+1 \text{ and } z = y+1, \\ n+1 \text{ otherwise;} \end{cases}$$

$$G_n(x, y, z) = \begin{cases} z \text{ if } x \neq 0 \text{ and } z = y+1, \\ 0 \text{ otherwise;} \end{cases}$$

$$H_n(x, y) = \begin{cases} n+1 \text{ if } x = 0 \text{ and } y = n+1, \\ 0 \text{ otherwise.} \end{cases}$$

Next we define the following polynomials:

$$P_0(v_0, v_1) = F(a, v_0, v_1); Q_0(v_0, v_1) = G(b, v_0, v_1);$$

$$P_{k+1}(v_0, \ldots, v_{k+2}) = F(P_k(v_1, \ldots, v_{k+2}), v_0, v_1);$$

$$Q_{k+1}(v_0, \ldots, v_{k+2}) = G(Q_k(v_0, \ldots, v_{k+1}), v_{k+1}, v_{k+2});$$

$$R_k(v_0, \ldots, v_{k+1}) = H(P_k(v_0, \ldots, v_{k+1}), Q_k(v_0, \ldots, v_{k+1})).$$

Claim 1. $R_k^{\mathfrak{A}_n}(x_0, \ldots, x_{k+1}) = 0$ *except when* $k = n$ *and* $x_i = i$ *for* $0 \leqq i \leqq k+1$.

Proof. In A_n let $x \prec y$ mean that $y = x + 1$. An easy induction on k shows that

$$P_k^{\mathfrak{A}_n}(x_0, \ldots, x_{k+1}) = \begin{cases} x_0 \text{ if } x_0 \prec x_1 \prec \ldots \prec x_{k+1}, \\ n+1 \text{ otherwise.} \end{cases}$$

Similarly,

$$Q_k^{\mathfrak{A}_n}(x_0, \ldots, x_{k+1}) = \begin{cases} x_{k+1} \text{ if } x_0 \prec x_1 \prec \ldots \prec x_{k+1}, \\ 0 \text{ otherwise.} \end{cases}$$

Consequently, the claim follows from the definition of H_n.

Notice that the following equation (1_k) holds in all $\mathfrak{A}_n$ for $n \neq k$, that it fails in $\mathfrak{A}_k$ but that it holds in every proper subalgebra of $\mathfrak{A}_k$:

(1_k) $\quad R_k(v_0, \ldots, v_{k+1}) = a.$

Let S be the set of integers mentioned in Theorem 1; let $\mathscr{A} = \{\mathfrak{A}_n | n - 1 \notin S\}$ and let $V' = HSP(\mathscr{A})$. For any class $\mathscr{K}$ let $S_n(\mathscr{K})$ be all algebras in $S(\mathscr{K})$ generated by n or fewer elements. Since V' is a variety of lattices with certain additional operations and hence congruence distributive, we have $V' = P_S HSP_P(\mathscr{A})$ (see [**4**]). Consequently,

$$S_n(V') \subseteq P_S HS_n P_P(\mathscr{A}) = P_S HS_n P_P S_n(\mathscr{A}).$$

Note that every subset of A_m containing 0 and $n + 1$ is a subuniverse of $\mathfrak{A}_m$ so that no member of $S_n(\mathscr{A})$ has more than $n + 2$ elements. Hence $S_n(\mathscr{A})$ has only finitely many members. Thus $S_n(V') \subseteq P_S HS_n(\mathscr{A}) = V_n'$. Therefore the identities which hold in V_n' are exactly the identities that hold in

$S_n(\mathscr{A})$. From this we infer that (1_k) holds in V_n' if and only if $\mathfrak{A}_k \notin S_n(\mathscr{A})$. But $\mathfrak{A}_k \notin S_n(\mathscr{A})$ if and only if $k > n$ or $k - 1 \in S$. Thus every identity of the form (1_k) that holds in V_n' will hold in V_{n+1}' except that if $n \notin S$ then (1_{n+1}) holds in V_n' but not in V_{n+1}'. Consequently the identities (1_k) distinguish between V_n' and V_{n+1}' just in the case that $n \notin S$.

However, there may be other identities that distinguish between V_n' and V_{n+1}' even when $n \in S$; thus we cannot claim that V' is the variety called for by Theorem 1. Let V'' be the variety of algebras of the same type as V' such that for any $\mathfrak{A} \in V''$, $\langle A; a, b, \wedge, \vee \rangle$ is a lattice with zero a and unit b, and such that (1_k) holds in V'' for every $k \geqq 1$. Let $V = V' \vee V''$. Since V has distributive congruences, every subdirectly irreducible member of V belongs to either V' or V'' and so $V_n = V_n' \vee V_n''$.

Claim 2. $V_0'' = V''$.

Proof. Let $\mathfrak{F}$ be the free algebra in V'' with free generators $x_0, x_1, \ldots$ and let $p = q$ be an identity not true in V''; we need only show that $p = q$ is not true in V_0''. By assumption, $p^{\mathfrak{F}}(x_0, \ldots) \neq q^{\mathfrak{F}}(x_0, \ldots)$. Since only finitely many elements of $\mathfrak{F}$ are used in building up $p^{\mathfrak{F}}(x_0, \ldots)$ and $q^{\mathfrak{F}}(x_0, \ldots)$ from $x_0, \ldots$, we can choose $n > 0$ such that $f^m(a), f^m(x_0), f^m(x_1), \ldots$ are not used in forming $p^{\mathfrak{F}}(x_0, \ldots)$ or $q^{\mathfrak{F}}(x_0, \ldots)$ for any $m \geqq n - 1$. Notice that $f(a)$, $f^2(a), \ldots, f(x_0), \ldots, f(x_1), \ldots$ are all distinct. Since the defining identities of V'' contain no occurrence of f, V'' has the following property: If $\mathfrak{A} \in V''$ and f is arbitrarily redefined on A to yield $\mathfrak{A}'$ then $\mathfrak{A}' \in V''$. Thus we redefine f in $\mathfrak{F}$ to get $\mathfrak{F}'$ as follows: $x_0 = f^{\mathfrak{F}'}(f^{n-1}(a))$, $x_{i+1} = f^{\mathfrak{F}'}(f^{n-1}(x_i))$ for $i \geqq 0$ and otherwise $f^{\mathfrak{F}'} = f^{\mathfrak{F}}$. Thus $\mathfrak{F}' \in V''$. But we now have $x_i = (f^{\mathfrak{F}'})^{n(i+1)}(a)$ in $\mathfrak{F}'$ and $p^{\mathfrak{F}'}(x_0, \ldots) \neq q^{\mathfrak{F}'}(x_0, \ldots)$. Hence

$$p^{\mathfrak{F}'}((f^{\mathfrak{F}'})^n(a), \ldots) \neq q^{\mathfrak{F}'}((f^{\mathfrak{F}'})_1(a), \ldots).$$

Let $\mathfrak{F}''$ be the constant subalgebra of $\mathfrak{F}'$. Since $\mathfrak{F}' \in V''$, $\mathfrak{F}'' \in V_0''$. On the other hand, $p^{\mathfrak{F}''}((f^{\mathfrak{F}''})^n(a), \ldots) \neq q^{\mathfrak{F}''}((f^{\mathfrak{F}''})^n(a), \ldots)$ and so $p = q$ does not hold in V_0''. Thus $V_0'' = V''$ as claimed.

Claim 3. *For every* $n \geqq 0$, $V_n = V_{n+1}$ *if and only if* $n \in S$.

Proof. By Claim 2, $V_n'' = V''$ and so $V_n = V_n' \vee V''$. If $n \notin S$ then (1_{n+1}) holds in V_n' but not in V_{n+1}'; since (1_{n+1}) holds in V'' we have that (1_{n+1}) holds in V_n but not in V_{n+1} so $V_n \neq V_{n+1}$. Conversely, if $n \in S$ then every member of $S_{n+1}(\mathscr{A})$ belongs to either $S_n(\mathscr{A})$ or V''; this is so because $\mathfrak{A}_{n+1} \notin \mathscr{A}$ and every proper subalgebra of every $\mathfrak{A}_m$ belongs to V''. Thus $V_{n+1}' \subseteq V_n' \vee V''$ so $V_{n+1} = V_{n+1}' \vee V'' \subseteq V_n' \vee V'' = V_n$ and so $V_{n+1} = V_n$.

COROLLARY. *If S is cofinite and* $0 \notin S$ *then there is a finitely based V satisfying* $V_n = V_{n+1}$ *if and only if* $n \in S$ *for all* $n \geqq 0$.

Proof. Define V' as above and note that $\mathscr{A}$ is finite. Let $\mathscr{B} = S(\mathscr{A}) - \mathscr{A}$. It is easily seen that each $\mathfrak{A}_n$ is simple; thus every subdirectly irreducible

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member of V' belongs to either $\mathscr{A}$ or $H(\mathscr{B})$. Consequently, for any $n \in \omega$, $V_n' \neq V_{n+1}'$ if and only if $n \notin S$ or $V_n' \cap H(\mathscr{B}) \neq V_{n+1}' \cap H(\mathscr{B})$.

Let $\mathfrak{B}$ be a direct product of all algebras in $\mathscr{B}$ and let $\mathfrak{C} = \mathfrak{B} \times \mathfrak{B}$. The diagonal subalgebra of $\mathfrak{C}$ is isomorphic to $\mathfrak{B}$ and so has all the members of $\mathscr{B}$ as homomorphic images. Redefine f on $\mathfrak{C}$ to get $\mathfrak{C}'$ by having f cyclically permute the non-diagonal elements among themselves and map the diagonal elements as in $\mathfrak{C}$. It is easily seen that $\mathfrak{C}'$ is generated by any non-diagonal element (since for any $b \in \mathfrak{B}$, $(b, b) = (0, b) \vee (b, 0)$) and that $\mathfrak{B}$ is still isomorphic to the diagonal subalgebra of $\mathfrak{C}'$. Thus let V'' be generated by $\mathfrak{C}'$; since $\mathfrak{C}'$ is 1-generated, $V'' = V_1''$. Finally let $V = V' \vee V''$; as above, $V_n = V_{n+1}$ if and only if $n \in S$. Since V is generated by finitely many finite algebras (i.e., $\mathscr{A} \cup \{\mathfrak{C}'\}$) and is congruence distibutive, it is finitely based by a result of Kirby Baker.

Note that $V_0'' \neq V''$. This is because a 0-generated algebra has no proper subalgebras. In an earlier version of this paper, Theorem 1 was proved using countably many operations. In the case when S is cofinite this earlier construction could be trivially modified to yield the corollary without the restriction on 0. Unfortunately the number of operations needed, although finite, depends on the number of positive integers not in S.

Problem 1. What patterns of proper inclusion are possible in the ascending chain of a variety of groupoids?

3. The descending varietal chain of a variety. If $\Gamma \cup \{\varphi = \psi\}$ is a set of identities then $\varphi = \psi$ is *derivable* from Γ (in symbols, $\Gamma \vdash \varphi = \psi$), just in case there is a finite sequence, $\theta_0, \ldots, \theta_n$, of terms such that φ is θ_0, ψ is θ_n and, for every $i < n$, θ_{i+1} can be obtained from θ_i by replacing some occurrence of one side of a substitution instance of an identity in Γ by the other side. Such a sequence of terms is called a *derivation*. $[\Gamma]$ denotes the set of all identities derivable from Γ, i.e. the equational theory based on Γ, while $[\Gamma]_n$ denotes the set of all identities derivable from all substitution instances in at most n variables of identities in Γ, for each natural number n. If W is the variety of all algebras satisfying Γ then $[\Gamma]$ is the set of identities holding in W and, for each n, $[\Gamma]_n$ is the set of identities holding in W^n.

Consider the following set, Δ, of groupoid identities:

(1) $x(yz) = (xy)z$

(2) $x(xx) = y(z(zz))$

(3) $x(xx) = x(y(xx))$

(4) $x(xx) = x(x(yx))$

(5) $x(xx) = x(x(xy))$

(6) $x(xx) = x(y(x(wx)))$.

Any model of Δ is a semigroup with zero in which the cube of every element is zero. A term (word), φ, will be called *trivial* if some variable occurs at least

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three times in φ; an identity, $\varphi = \psi$, is trivial if both φ and ψ are trivial. Every trivial identity is derivable from Δ.

Since the associative law is present in every equational theory mentioned in this section, parentheses will usually be omitted.

For $n > 2$, let ϵ_n be the following $n + 1$ variable identity:

$$x_0x_1 \ldots x_nx_n \ldots x_1x_0 = x_n \ldots x_1x_0x_0x_1 \ldots x_n.$$

Claim 4. *Let M be any set of natural numbers greater than* 2, *φ be any non-trivial term of length k, and ψ be any term. If $\Delta \cup \{\epsilon_n : n \in M\} \vdash \varphi = \psi$ then*:

(i) ψ is non-trivial.

(ii) Each variable occurs exactly the same number of times in φ as in ψ.

(iii) $\{x(yz) = (xy)z\} \cup \{\epsilon_n : n \in M \text{ and } 2n + 2 \leqq k\} \vdash \varphi = \psi$.

Proof. Let A be the set of natural numbers which are either 0 or never divisible by a perfect cube. Define

$$a \otimes b = \begin{cases} ab \text{ if } ab \in A, \\ 0 \text{ otherwise,} \end{cases}$$

for every $a, b \in A$. Let $\mathfrak{A} = \langle A, \otimes \rangle$. $\mathfrak{A}$ is a model of $\Delta \cup \{\epsilon_n : n \in M\}$. In $\mathfrak{A}$, every trivial term is interpreted by some function with range $\{0\}$ while every non-trivial term is interpreted as a function with an infinite range. Consequently, ψ must be non-trivial and (i) is established. By considering substitution instances of $\varphi = \psi$, (ii) follows from (i). Likewise, (iii) follows from (i) and (ii) by the definition of derivation. Thus the proof is complete.

Claim 5. *Let $n > 2$. Then ϵ_n is not derivable from $\Delta \cup \{\epsilon_k : k \neq n$ and $k > 2\}$.*

Proof. Since ϵ_n is non-trivial, it is only necessary to establish that ϵ_n is not derivable from $\{x(yz) = (xy)z\} \cup \{\epsilon_k : 2 < k < n\}$. The following statement will be established by induction on derivations:

If $\{x(yz) = (xy)z\} \cup \{\epsilon_k : 2 < k < n\} \vdash x_0x_1 \ldots x_nx_n \ldots x_1x_0 = \varphi$ then $\varphi = x_0x_{i_1} \ldots x_{i_n}x_{i_n} \ldots x_{i_1}x_0$ where $(i_1, \ldots, i_n)$ is a permutation of $\{1, \ldots, n\}$.

By permuting the names of the variables $x_1, \ldots, x_n$ it is clear that we need only consider derivations of length 2. But the only subterm of $x_0 \ldots x_nx_n \ldots x_0$ of the form $\theta_0 \ldots \theta_k\theta_k \ldots \theta_0$ for $k < n$ is $x_{n-k} \ldots x_nx_n \ldots x_{n-k}$ and so φ is $x_0 \ldots x_{n-k-1}x_nx_{n-1} \ldots x_{n-k}x_{n-k} \ldots x_{n-1}x_nx_{n-k-1} \ldots x_0$ for some $k < n$. Thus x_0 is both the leftmost and rightmost variable in φ. This is clearly the case if the derivation sequence is only one symbol long (since then φ is $x_0x_1 \ldots x_nx_n \ldots x_1x_0$). Suppose the statement is true for derivations of length m and that $x_0x_1 \ldots x_nx_n \ldots x_1x_0 = \varphi$ can be derived in $m + 1$ steps. Then there is a term ψ (the one occurring at step m) such that x_0 is both the leftmost and the rightmost variable in ψ, the variables $x_0, x_1, \ldots, x_n$ are exactly the variables appearing in ψ (by Claim 4) and each of these appears twice, ψ is non-trivial, and there

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is an identity ϵ_k, $2 < k < n$, so that φ is obtained from ψ by replacing one side of a substitution instance of ϵ_k by the other side. Suppose $\theta_0\theta_1 \ldots \theta_k\theta_k \ldots \theta_1\theta_0$ is a subterm of ψ. Then $\theta_0\theta_1 \ldots \theta_k\theta_k \ldots \theta_1\theta_0$ is non-trivial. Hence, each θ_i is a variable and the whole string then is of length $2k$. Since $2n > 2k$, x_0 cannot occur in $\theta_0\theta_1 \ldots \theta_k\theta_k \ldots \theta_1\theta_0$ and therefore x_0 is both the leftmost and the rightmost variable in φ as it was in ψ. The induction is complete and the claim follows.

Let S be any set of natural numbers greater than two. Define $T = \Delta \cup \{\epsilon_n : n \notin S \text{ and } n > 2\}$. For $n > 2$, let $T^n = \Delta \cup \{\epsilon_m : m \notin S \text{ and } 2 < m < n\}$.

Claim 6. *For* $n > 2$, $[T^n] = [T]_n$

Proof. Clearly $[T^n] \subseteq [T]_n$. On the other hand, if $m \geqq n$ then any substitution instance of ϵ_m with no more than n variables must be trivial and therefore derivable from Δ. Consequently, $[T]_n \subseteq [T^n]$, establishing Claim 6.

Claim 7. $[T]_n = [T]_{n+1}$ *if and only if* $n \in S$

Proof. If $n \in S$ then $[T^n] = [T^{n+1}]$ and, by Claim 6, $[T]_n = [T]_{n+1}$. If $n \notin S$ and $n > 2$, then $\epsilon_n \in [T^{n+1}] = [T]_{n+1}$ but $\epsilon_n \notin [T^n] = [T]_n$ by Claim 5. Therefore $[T]_n \neq [T]_{n+1}$. If $n = 0$, $x(xx) = (xx)x$ is in $[T]_1$ but not in $[T]_0$. Hence $[T]_0 \neq [T]_1$. If $n = 1$, $x(yx) = (xy)x$ is in $[T]_2$ but not in $[T]_1$. Hence $[T]_1 \neq [T]_2$. If $n = 2$, $x(yz) = (xy)z$ is in $[T]_3$ but not in $[T]_2$. Hence $[T]_2 \neq [T]_3$.

Claim 8. $[T]$ *is finitely based if and only if* S *is cofinite.*

Proof. This is immediate from Claim 5.

Claim 9. $[T]$ *is recursive if and only if* S *is recursive.*

Proof. Suppose $[T]$ is recursive. By Claim 5, $n \in S$ if and only if $\epsilon_n \notin [T]$. So S is recursive.

Now suppose S is recursive. Then T is recursive and hence the set of finite sequences of terms which are derivations from T is also recursive. Suppose φ and ψ are both non-trivial and every variable occurs exactly the same number of times in φ as in ψ. Now there are only finitely many terms, θ, so that every variable occurs exactly the same number of times in φ as in θ. Consequently, there are only finitely many sequences of such terms which begin with φ and end with ψ and in which no term appears twice — in fact the number of such sequence is computable from φ. $\varphi = \psi \in [T]$ if and only if one of these sequences is a derivation from T. The only other identities in $[T]$ are the trivial identities and all of them are in $[T]$. So $[T]$ is recursive.

Now let V be the variety of algebras satisfying T. V has all the properties demanded by Theorem 2, as Claims 7, 8, and 9 assert.

Problem 2. What patterns of proper inclusion in descending chains are possible without the restriction that $V^0 \neq V^1 \neq V^2 \neq V^3$?

Problem 3. What patterns of proper inclusion are possible in ascending (descending) varietal chains of variety of groups? lattices? rings?

Problem 4. What patterns of proper inclusion along both ascending and descending chains of a single variety are possible?

Problem 5. What patterns of proper inclusion are possible for descending varietal chains of minimal (equationally complete) varieties?

Problem 6. For any fixed pattern of proper inclusions along the descending (ascending or both simultaneously) chain how many varieties can realize it?

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