The geometry of $C^{1,\alpha}$ flat isometric immersions

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We show that any isometric immersion of a flat plane domain into $\mathbb{R}^3$ is developable provided it enjoys the little Hölder regularity $c^{1,2/3}$. In particular, isometric immersions of local $C^{1,\alpha}$ regularity with $\alpha > 2/3$ belong to this class. The proof is based on the existence of a weak notion of second fundamental form for such immersions, the analysis of the Gauss–Codazzi–Mainardi equations in this weak setting, and a parallel result on the very weak solutions to the degenerate Monge–Ampère equation analysed in [M. Lewicka and M. R. Pakzad. Anal. PDE 10 (2017), 695–727.].

1. Introduction

1.1. Background

It often happens, in differential geometry, as in other contexts where nonlinear PDEs appear, that the solutions to a given nonlinear system of equations satisfy also systems of higher order equations carrying significant geometric or analytic information. A very basic example is the Gauss–Codazzi–Mainardi equations satisfied by any smooth isometric immersion of a two-dimensional Riemannian manifold into $\mathbb{R}^3$. Note that, in their turn, these equations can be used to prove certain rigidity statements for the immersion, e.g. to show that any isometric immersion of a flat domain is developable, i.e. roughly speaking, at any given point p its image contains a segment of $\mathbb{R}^3$ passing through p.

The subtlety which is often glossed over in the context of classical differential geometry, where all the mappings are assumed smooth, or at least twice or thrice continuously differentiable, is that, due to nonlinearity, the passage from say a given

set of first order equations, e.g. the system of isometric immersion equations in our example, to the higher order equations requires a minimum of regularity. There is no guarantee that in the absence of this regularity the geometric information hidden in the higher order equations would be accessible. The above mentioned subtlety could be a source of confusion. Coming back to our example, it is often assumed that all isometric images of flat domains must be ruled, which is untrue. Already, at the dawn of modern differential geometry, no less figures than Lebesgue and Picard intuited the existence of a surface which could be flattened onto the plane with no distortion of relative distances, but nowhere contained a straight segment. To quote Picard[1]:

> According to general practice, we suppose in the preceding analysis, as in all infinitesimal geometry of curves and surfaces, the existence of derivatives which we need in the calculus. It may seem premature to entertain a theory of surfaces in which one does not make such hypotheses. However, a curious result has been pointed out by Mr Lebesgue (Comptes Rendus, 1899 and thesis)[2] ; according to which one may, by the aid of continuous functions, obtain surfaces corresponding to a plane, of such sort that every rectifiable line of the plane has a corresponding rectifiable line of the same length of the surface, nevertheless the surfaces obtained are no longer ruled. If one takes a sheet of paper, and crumples it by hand, one obtains a surface applicable to the plane and made up of a finite number of pieces of developable surfaces, joined two and two by lines, along which they form a certain angle. If one imagines that the pieces become infinitely small, the crumpling being pushed everywhere to the limit, one may arrive at the conception of surfaces applicable to the plane and yet not developable [in the sense that there is no envelope of a family of planes of one parameter and not ruled].

The suggested constructions by Lebesgue and then Picard fall short of being totally satisfactory as they point to continuous but yet not differentiable mappings. However, they were perhaps the heralds of a celebrated result due to Kuiper [**39**], who based on the work by John Nash [**47**, **48**] proved that any Riemannian manifold can be C^1 isometrically embedded into a Riemannian manifold of one dimension higher. (Actually, by Nash and Kuiper's construction, C^1 isometric embeddings can approximate any short immersion.) As a consequence, the flat disk can be C^1-isometrically embedded into any given small ball of $\mathbb{R}^3$ and hence the image cannot be ruled. To make the matters worse, these embeddings can be made through recurrent self-similar corrugations [**3**, **4**], whose images seem, without a proof at hand, not to include any straight segment of $\mathbb{R}^3$ in any scale. In contrast, and returning to our original developability statement in a higher regularity setting where the Gauss–Codazzi equations hold true, Hartman and Nirenberg [**28**] proved, incidentally almost at the same time Nash and Kuiper showcased their results, that a C^2-regular isometric immersion of a flat domain must have a locally ruled surface as its image. We have come full circle, and this discussed dichotomy leaves us with a fundamental question any analyst would like to ask: **At what regularity threshold or thresholds between C^1 and C^2 the unruled isometries transition into the ruled regime?**

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[1][**51**, Page 555], translated and quoted in [**7**].
[2][**40**]. See also [**22**, Pages 164-165] and the discussion therein of Lebesgue's heuristic construction.

1.2. Recent developments

To put the problem described above in a broader context, we note that the above dichotomy—known in the recent literature as *flexibility* vs. *rigidity*—is not specific to the case of isometric immersions. The Nash and Kuiper scheme for creating highly oscillatory anomalous solutions of typically lower regularity through iterations could be studied under the broader topic of convex integration [**25**, **54**] for the differential inclusions (or PDEs re-cast in this framework), and the involved density or flexibility-related results are usually referred to as h-principle. The existence of such h-principle is usually accompanied by a parallel rigidity property which indicates that the construction cannot be carried out in high regularity scales. A notable example is the recent discovery by the first author and Székelyhidi [**17**]: They showed that the system of Euler equations in fluid dynamics is given to the convex integration method and that Nash and Kuiper's iterations can be adapted in this case in order to create non-conservative compactly supported continuous flows approximating an open set of subsolutions (see [**18**, **55**] for a thorough discussion of the connection between the two problems of turbulence and isometric immersions). Their results, and the subsequent work in improving the regularity of the anomalous solutions stood in contrast with the rigidity result reflected in the conservation of energy for solutions passing a certain regularity threshold ($\alpha > 1/3$ for $C^{1,\alpha}$-Hölder continuous solutions) in [**12**, **20**], as conjectured by Onsager [**49**]. Their approach finally lead to the complete resolution of this conjecture by Isett in [**31**]; see [**19**] for a review of the history of the problem and the intermediate results.

Coming back to the question of isometric immersions, a parallel question is whether the images of isometric immersions into $\mathbb{R}^3$ of closed convex surfaces (i.e. 2-manifolds with no boundary and positive Gaussian curvature) are rigid motions. This result follows from the convexity of the image of such surfaces. This convexity fails for $C^{1,\alpha}$ isometric immersions of convex surfaces if $\alpha < 1/5$, in which regime the above mentioned h-principle for isometric immersions holds true, as shown in [**9**, **16**], improving on the results by [**6**, **13**]. On the other hand, following Borisov [**5**], Conti *et al.* proved in [**13**, **14**] that when $\alpha > 2/3$, the isometric image of a closed convex surface is convex (from which it follows that the immersion is a rigid motion), and that more generally the h-principle cannot hold true for isometric immersions of any elliptic 2-manifold with or without boundary. Mischa Gromov conjectures in [**26**, Section 3.5.5.C, Open Problem 34-36] that the transition threshold is $\alpha = 1/2$ in this case. The best evidence so far for this conjecture is provided in [**15**], where the authors show that $C^{1,1/2}$ regularity could be the borderline regularity for the identity of intrinsic covariant derivative of the 2-manifold and its geometric interpretation.

This article concerns the fundamental question of the sub-C^2 regularity scales for which, given an isometric immersion of a flat domain, one is able to make sense of fundamentally geometric objects such as the second fundamental form and the Gauss–Codazzi equations. The final goal is to show that the developability property of such isometric immersions (as defined below in Definition 1.1 and proven for C^2 isometries in [**28**]) survives for $C^{1,\alpha}$ isometric immersions if $\alpha > 2/3$. As we shall see, parts of the project, e.g. the definition of the second fundamental form, can be

carried over for $\alpha > 1/2$, but our analysis falls short of proving a weak variant of the Codazzi–Mainardi equations for $1/2 < \alpha < 2/3$. In this manner, the question of the optimal threshold remains open in the flat case as it is for the elliptic case.

Before proceeding, it should be mentioned that the developability of flat isometric immersions has been discussed in the literature in other regularity regimes. Pogorelev [52, Theorem 1, p. 603] proves this developability under the rather weak assumptions of C^1 regularity and the existence and vanishing of the total extrinsic curvature (defined as a measure). This result lies at the background of our conclusion through the analysis made in [41] and [37, 38]. On the other hand parallel results have been shown by the second author for $W^{2,2}$ isometric immersions [50], where a slight C^1 regularity gain in this case is also proven, and by Jerrard in [34] for the class of Monge–Ampère functions [24]. The former result was based on observations made by Kirchheim on solutions to the degenerate Monge–Ampère equation [36, Chapter 2], and lead to a subsequent statement regarding the smooth density of isometric immersions. These statements were generalized to higher dimensional domains for the co-dimension 1 case in [43]. The latter approach by Jerrard [33, 34], anchored in geometric measure theory, opened the path for proving C^1 regularity and a full range of developability results for isometric immersions of any dimension and co-dimension of sub-critical Sobolev regularity [35].

1.3. The main result

We first begin by a definition to clarify the notion of developabilty as we understand it.

DEFINITION 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain and $u : \Omega \to \mathbb{R}^k$ be C^1-smooth. We say u is developable if for all $x \in \Omega$, either u is affine in a neighbourhood of x, or there exists a line L_x containing x such that the Jacobian derivative ∇u is constant on l_x, the connected component of x in $L_x \cap \Omega$.*

REMARK 1.2. For bounded domains, l_x is always a segment whose both ends lie on $\partial\Omega$. We allow for unbounded domains, and hence l_x could be a complete line, a ray or an open segment. With some abuse of notation, we will refer to all of them as 'segments' throughout the paper.

There are other equivalent formulations for the developability whose equivalence we will show further on in propostion 2.1. In particular, local and global developability, as laid out in corollary 2.2, are equivalent. Also, from corollary 2.8 and lemma 2.9 it follows that when u is developable, the segments l_x are uniquely determined outside the maximal affine region; they do neither intersect each other nor pass through the local constancy regions of the Jacobian derivative.

In this article, we prefer to work with the little Hölder spaces. For our choice of notations and precise definitions of these spaces see § 3. Working with the little Hölder scale pinpoints the needed control—for the effectiveness of our arguments—on the oscillations of the mapping gradient. The corresponding results in the classical Hölder regime is then obtained as a corollary.

The main result of this article is the following theorem:

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain and $2/3 \leqslant \alpha < 1$. If $u \in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ is an isometric immersion, then u is developable. In particular, if $\alpha > 2/3$, we have $C^{1,\alpha}(\Omega) \subset c^{1,2/3}(\Omega)$ and therefore all isometric immersions $u : \Omega \to \mathbb{R}^3$ of local $C^{1,\alpha}$ regularity are developable.*

REMARK 1.4. Based on the observations made in this article, the proofs of [**13**, **14**] for the rigidity statements are also valid under little Hölder $c^{1,2/3}$ regularity assumption. A more difficult task would be to consider other types of regularity regimes, e.g. the fractional Sobolev one. For this direction, see [**42**] and the discussions therein.

Per [**52**, Theorem 1, p. 603], what we need to show is that the image of the Gauss map $\vec{n}$ is of measure zero. Following [**13**, Corollary 5] it is possible to prove that the Brouwer degree $\deg(y, V, \vec{n})$ of $\vec{n}$ of the immersion vanishes for all open sets $V \subset \Omega$ and $y \in \mathbb{R}^2$, when defined. However, this is not sufficient to conclude with the needed statement $|\vec{n}(\Omega)| = 0$ for developability: Through a similar approach as in [**44**, Section 5], for each $\alpha \in (0, 1)$, one can construct a map in $C^{0,\alpha}(\Omega, \mathbb{R}^2)$ whose local degree vanishes everywhere, but whose image is onto the unit square. Hence, the zero degree Gauss map could still have an image of positive measure and this obstacle necessitates another strategy. Indeed, our argument uses a slight improvement on the parallel result proved for the degenerate *very weak Monge–Ampère equation* in [**41**], which uses the degree formula for <u>both</u> the gradient of the solution, and its affine perturbations of positive degree, as follows:

The very weak Hessian determinant [**21**, **32**] of a given function $v \in W_{loc}^{1,2}(\Omega)$ is defined to be

$$\mathcal{D}et\, D^2 v := -\frac{1}{2}\left(\partial_{11}(\partial_2 v)^2 + \partial_{22}(\partial_1 v)^2 - 2\partial_{12}(\partial_1 v \partial_2 v)\right) = -\frac{1}{2}\mathrm{curl}^T\mathrm{curl}\,(\nabla v \otimes \nabla v)$$

in the sense of distributions. The operator $\mathcal{D}et\, D^2$ coinsides with the usual Monge–Ampère operator $\det \nabla^2$ when $v \in C^2$ and $\nabla^2 v$ stands for the Hessian matrix field of v. (A C^3 regularity is needed for a straightforward calculation.) The following statement regarding the degenerate Monge–Ampère equation is a crucial ingredient of our analysis (compare with [**41**, Theorem 1.3]).

THEOREM 1.5. *Let $\Omega \subset \mathbb{R}^2$ be an open domain and let $v \in c^{1,2/3}(\Omega)$ be such that*

$$\mathcal{D}et\, D^2 v = 0,$$

in Ω. Then v is developable.

REMARK 1.6. Note that per corollary 2.2 the assumption $v \in C^{1,\alpha}(\overline{\Omega})$ in [**41**, Theorem 1.3] can too be easily relaxed to the local regularity $v \in C^{1,\alpha}(\Omega)$ in any bounded or unbounded domain.

Theorem 1.5 can be proved following the exact footsteps of Lewicka and the second author in proving [**41**, Theorem 1.3], taking into account lemma 3.1 and

proposition 3.5 below in showing the following degree formulas for any open set $U \subset \Omega$:

$$\begin{array}{lll}
\forall y \in \mathbb{R}^2 \setminus \nabla v(\partial U) & & \deg(\nabla v, U, y) = 0, \\
\forall\, 0 < \delta \ll 1 \quad \forall y \in F_\delta(U) \setminus F_\delta(\partial U) & & \deg(F_\delta, U, y) \geqslant 1,
\end{array} \tag{1.1}$$

where $F_\delta(x_1, x_2) := \nabla v(x_1, x_2) + \delta(-x_2, x_1)$. See [**41**, Equations (7.6) and (7.9)]. The rest of the proof remains unchanged. We will leave the verification of details to the reader.

The main strategy of our proof of theorem 1.3 is hence to relate the given isometric immersion $u \in c^{1,2/3}$ to a scalar function v which satisfies the assumptions of theorem 1.5: In a first step, we show that a notion of the second fundamental form A in the weak sense exists for immersions of $C^{1,\alpha}$-regularity for $\alpha > 1/2$. Next, the Codazzi–Mainardi equation is used to prove that if u is isometric, A is curl free, implying that it must be the Hessian matrix of a scalar function, namely the sought after function v. We then show the required regularity for v, and proceed to prove using the Gauss equation that v satisfies the very weak degenerate Monge–Ampère equation as required by the assumptions of theorem 1.5. Finally, we need to prove that developability of v, as derived from theorem 1.5, implies the developability of the isometric immersion u. Apart from theorem 1.5, the $2/3$-Hölder exponent regularity is only required for proving that a weak version of Codazzi–Mainardi equations holds for isometric $u \in c^{1,2/3}$, i.e. when we need to show that A is curl free.

The article is organized as follows: In § 2, we will present and prove a few statements regarding the developability properties of C^1 mappings. In § 3, we will gather a few analytical tools which deal with properties of Hölder continuous functions and with quadratic differential expressions in terms of functions of low regularity. In particular, our basic proposition 3.8 will allow us to define a second fundamental form for the immersions of Hölder-type regularity. The subsequent section is dedicated to the definition and properties of this second fundamental form. In § 5, building on the previous sections, we complete the proof of theorem 1.3. Appendix A is dedicated to a side result (proposition A.1) on the developability of each component of the immersion, which can be shown independently with a shorter proof. Finally, in Appendix B the proof of some standard facts regarding little Hölder spaces are presented.

2. Preliminaries: developable mappings

We have gathered in this section a few statements we will need, and their proofs, regarding the developable mappings in two dimensions. Most of the material in this section are well known and can be found in one form or another in the literature on the topic [**11**, **28**–**30**, **34**–**38**, **41**, **43**, **46**, **50**, **52**, **56**], but the authors do not know of any instance where the following statements are explicitly formulated in concert. As already observed in [**29**], much of the developability properties of a mapping concern the level sets of its Jacobian derivative, so we formulate our statements having the mappings $f = \nabla u$ in mind, where u is a developable mapping.

2.1. Global vs local developability

Developability can be defined by local or global formulations, which turn out to be equivalent. There is a risk of confusion, which must be avoided, due to slight differences between the possible formulations (see e.g. the footnote on p. 875 in [**38**]). Also, it would be useful to have a set of equivalent statements at hand to streamline the arguments. The following proposition states three equivalent conditions which the Jacobian derivative of a C^1 mapping can satisfy to be developable:

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^2$ be any domain and let $f : \Omega \to \mathbb{R}^m$. If f is continuous, then the following three conditions are equivalent.*

(a) *For all $x \in \Omega$, either f is constant in a neighbourhood of x, or there exists a line L containing x such that f is constant on the connected component of x in $L \cap \Omega$.*

(b) *For all $x \in \Omega$, there exists an open disk $B_x \subset \Omega$ centred at x with the following property: To all $p \in B_x$, we can associate a line L_p containing p such that f is constant on the segment $\tilde{l}_p = L_p \cap B$. Moreover, for any y, $z \in B_x$, $\tilde{l}_y \cap \tilde{l}_z = \emptyset$, or $\tilde{l}_y = \tilde{l}_z$.*

(c) *For all $x \in \Omega$, there exists an open disk B_x centred at x in Ω such that for all $p \in B_x$, there exists a line $L \subset \mathbb{R}^2$ containing p so that f is constant on the segment $L \cap B_x$.*

COROLLARY 2.2. *$u \in C^1(\Omega, \mathbb{R}^k)$ is developable if and only if for any point in $x \in \Omega$, there exists a neighbourhood V of x on which the restriction $u|_V$ is developable.*

REMARK 2.3. Local regularity assumptions are sufficient in the assumptions of theorems 1.3 and 1.5 because of the equivalence of global and local developability as formulated above.

REMARK 2.4. The continuity of f is necessary for the implications (a) $\Rightarrow$ (b), (c) in proposition 2.1 as the following example demonstrates. The function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(s,t) := \begin{cases} 0 & st \neq 0 \\ 1 & st = 0, \end{cases}$$

satisfies condition (a) but not (b) or (c). Also, the geometric configuration of the level sets of f in [**50**, Figure 1] shows the necessity of the distinction between the points in the local constancy regions from other points in condition (a). In other words, a global formulation of condition (b) is not a statement equivalent to (b): A continuous function on a convex domain Ω can satisfy conditions (a) and (b) while there are points $x \in \Omega$ such that for no line L containing x the function is constant on the intersection $L \cap \Omega$.

REMARK 2.5. In the recent literature on similar problems, global (and hence inequivalent) versions of condition (b) are formulated as 'developability of f' in [**36**, Definition 2.29] and 'Ω-developability of f on Ω' in [**29**, Section 3.1]. On the

other hand condition (a) appears in the statement of [**50**, Theorem II], and then in [**46**] as the 'condition (L)'. It is also equivalent to f being 'countably developable' according to [**29**, Definition 1] and [**30**]. (Note that [**29**, Assumption (24)] is redundant for continuous f; see lemma 2.9.) Finally, it is condition (c) that is stated as the property of the gradients (with empty-interior images) of C^1 mappings in the main result of [**37**, **38**]. (This local result was an ingredient of the proof of theorem 1.5 in [**41**].) Generalizations of (a) to weaker regularity or higher dimensional settings can be found in [**34**, **35**, **56**].

REMARK 2.6. The developability of a C^1 mapping on Ω, as formulated in definition 1.1, translates therefore to its Jacobian derivative satisfying condition (a). To clarify some discrepancy in the literature, we emphasize that we prefer to reserve the term *developable* for those mappings whose Jacobian derivatives are constant along lower dimensional flat strata in one form or another [**35**, **43**], rather than directly using it for their derivative mappings as done in [**36**, Definition 2.29] or [**29**, **30**]. This is for historical reasons, since this term is conventionally used to refer to (ruled) smooth surfaces of zero Gaussian curvature [**10**].

Proof. (a) $\Rightarrow$ (b):

This is implicitly proved in [**43**, Lemma 3.7]. We pursue a slightly different strategy. For any $f : \Omega \to \mathbb{R}$, let C_f be the set on which f is locally constant:

$$C_f := \{x \in \Omega; f \text{ is constant in a neighbourhood of } x\}.$$

Note that f is constant on the connected components of the open set C_f. If f satisfies condition (a) of the proposition, for each $x \in \Omega \setminus C_f$ we choose a line L_x so that f is constant on the connected component x in $L_x \cap \Omega$, denoted by l_x.

We begin by the following simple but useful lemma. $\square$

LEMMA 2.7. *Let $f : \Omega \to \mathbb{R}^m$ be continuous and satisfy condition* (a) *in proposition* 2.1. *Let $\Delta \subset \Omega$ be a closed triangular domain. If f is constant on two edges of Δ, then f is constant on Δ.*

Proof. Let x, y, z be the vertices of Δ and assume that f is constant on $[x, y]$ and $[x, z]$. For any point $w \in \Delta \setminus C_f$, l_w must cross one of the two segments $[x, y]$ or $[x, z]$. This implies $f(w) = f(x)$. If $w \in \Delta \cap C_f$, and if the segment $[w, x] \subset C_f$, then once again $f(w) = f(x)$. Finally, if $[w, x] \not\subset C_f$, let $\tilde{w}$ be the closest element to w on the segment $[w, x]$ not in C_f. We already proved that $f(\tilde{w}) = f(x)$. But since $\tilde{w}$ is in the closure of the connected component of w in C_f, and f is continuous, we have also $f(w) = f(\tilde{w}) = f(x)$. We conclude that $f|_\Delta \equiv f(x)$. $\square$

We now make a crucial observation on the constancy directions L_x.

COROLLARY 2.8. *Let f be continuous and satisfy condition* (a) *in proposition* 2.1. *If $x \in \Omega \setminus C_f$, there exists only one line L for which f is constant on the connected component l of x in $L \cap \Omega$.*

Proof. Let $x \in \Omega \setminus C_f$ and assume that for two lines $L \neq \Lambda$, f is constant on the connected components l and λ of x in respectively $L \cap \Omega$ and $\Lambda \cap \Omega$. Choose a

disk $B(x, \rho) \subset B(x, 2\rho) \subset \Omega$ and note that $\partial B(x, \rho)$ intersects l and λ on 4 points x_1, $x_2 \in l$ and χ_1, $\chi_2 \in \lambda$. Now the assumptions of lemma 2.7 is satisfied for the four closed triangular domains with vertices x, x_i, χ_j, $i, j \in \{1, 2\}$. This implies that $x \in C_f$, which contradicts the first assumption on x. $\qquad\square$

As a consequence, when f is continuous, for each $x \in \Omega \setminus C_f$, there is only one choice for L_x and there is no ambiguity in the notation. We further observe:

LEMMA 2.9. *Let the continuous mapping $f : \Omega \to \mathbb{R}^m$ satisfy condition* (a) *in proposition* 2.1. *Then for all $x \in \Omega \setminus C_f$, $l_x \subset \Omega \setminus C_f$. Moreover, for all y, $z \in \Omega \setminus C_f$, $l_y \cap l_z = \emptyset$, or $l_y = l_z$.*

Proof. For showing the first conclusion, we argue by contradiction: Let $y \in C_f \cap l_x$ and let V be an open neighbourhood of y in Ω on which f is constant. We observe that for $\delta > 0$ small enough $B(x, \delta) \setminus l_x$ cannot entirely lie in C_f, since otherwise, $B(x, \delta) \setminus l_x$ having only two connected components on both sides of l_x, and f being continuous, the value of f would be constant on $B(x, \delta)$, contradicting $x \notin C_f$. Therefore, there is a sequence $x_k \in \Omega \setminus C_f$ converging to x such that $f(x_k) \neq f(x)$. Since the value of f on x_k and x differ, $l_{x_k} \cap l_x = \emptyset$ for all k. As $x_k \to x$, we deduce that l_{x_k} must locally uniformly converge to l_x. This implies that $l_{x_k} \cap V$ is non-empty for k large enough and hence $f(x_k) = f(y) = f(x)$, which is a contradiction.

To prove the second statement, also assume by contradiction that $l_y \neq l_z$ and that $x \in l_y \cap l_z \neq \emptyset$. Choose a disk $B(x, \rho) \subset B(x, 2\rho) \subset \Omega$ and note that $\partial B(x, \rho)$ intersects l_y and l_z on four points y_1, $y_2 \in l_y$ and z_1, $z_2 \in l_z$. Now the assumptions of lemma 2.7 are satisfied for the four closed triangular domains with vertices x, y_i, z_j, $i, j \in \{1, 2\}$. This implies that $x \in C_f$, which contradicts the first statement, as $y \notin C_f$ but $x \in l_y \cap C_f$. $\qquad\square$

We are now ready to prove (b) assuming (a). We first note that condition (b) is obvious if $x \in C_f$. If $x \notin C_f$, without loss of generality, and through rotations, dilations and translations, we can assume that $x = (0, 0)$ and that l_x is parallel to the horizontal axis. We show that we can find δ^-, $\delta^+ > 0$ such that condition (b) holds true for $B^+(x, \delta^+)$ and $B^-(x, \delta^-)$ in the upper and lower planes, where the open half disks are defined by

$$B^\pm(x, \delta) := B(x, \delta) \cap \{(s, t) \in \mathbb{R}^2; \ \pm t > 0\}.$$

Then we can choose $B = B(x, \min\{\delta^+, \delta^-\})$ and the proof is complete.

Without loss of generality we concentrate on the upper half plane. If there exists $\delta^+ > 0$ such that $B(x, \delta^+) \subset C_f$, the conclusion is obvious: We choose L_y always parallel to the horizontal axis. Otherwise there exists a sequence $x_k \notin C_f$ in the upper open half plane converging to x. Note that by lemma 2.9, any two (possibly unbounded) segments l_y, l_z do not intersect within Ω unless they are the same. This implies that $l_{x_k} \cap l_x = \emptyset$ and that the l_{x_k} must converge locally uniformly to

l_x. We choose $\rho > 0$ such that the closed rectangular box

$$S_\rho^+ := \{(s, t) \in \mathbb{R}^2; \ |s| \leqslant \rho, 0 \leqslant t \leqslant \rho\}$$

is a subset of Ω. We let $x_0^\pm := (\pm \rho, 0)$, $x^\pm = (\pm \rho, \rho)$ and choose k large enough such that for $y = x_k$, $l_y = l_{x_k}$ intersects the two segments $[x_0^\pm, x^\pm]$ in their respective interior points y^- and y^+. The closure of the convex open quadrilateral P created by the four vertices x_0^-, x_0^+, y^+, y^- lies in S_ρ^+ and we have that $[y^-, y^+] \subset l_y$ and $[x_0^-, x_0^+] \subset l_x$.

We claim that condition (b) is valid for P (standing for B_x), which completes the proof since there exists some $\delta^+ > 0$, such that $B^+(x, \delta^+) \subset P$. For any $z \in P$, if $z \notin C_f$, then we can choose $\tilde{l}_z = l_z \cap P$, which lies between $\tilde{l}_{x_0}$ and $\tilde{l}_y$. If, on the other hand, $z \in C_f$, let the points $z_1 \in [y^-, y^+]$ and $z_0 \in [x_0^-, x_0^+]$ be those points on ∂P which lie vertically above and below z. Since $z_0, z_1 \notin C_f$, we can choose the two closest elements of $P \setminus C_f$ to z on the segment $[z_0, z_1]$, which we name respectively by $\tilde{z}_0, \tilde{z}_1$. We observe that $l_{\tilde{z}_0}$ and $l_{\tilde{z}_1}$ cannot intersect l_x and l_y, which contain the upper and lower boundaries of P, and hence will intersect the two vertical boundaries of P between $x_0^\pm$ and $y^\pm$ in $\tilde{z}_i^\pm$, $i = 0, 1$. We claim that f is constant in the region X enclosed between $l_{\tilde{z}_0}$ (which is above $\tilde{l}_x$ and below $l_{\tilde{z}_1}$) and $l_{\tilde{z}_1}$ (which is above $\tilde{l}_{\tilde{z}_0}$ and below $\tilde{l}_y$) in P. Note that X is the convex quadrilateral created by $\tilde{z}_0^-, \tilde{z}_0^+, \tilde{z}_1^+, \tilde{z}_1^-$ and contains z in its interior.

The latter claim about the constancy of f on X, if proven, completes the proof of the former claim regarding P, since we are free to choose our non-intersecting constancy segments in this region X (in particular for z) in a manner that no such two segments intersect within P: If the directions $l_{\tilde{z}_0}$ and $l_{\tilde{z}_1}$ are parallel, choose the line L_p for all points $p \in X$ to be the line parallel to them passing from p. If, on the other hand, the extensions of $l_{\tilde{z}_0}$ and $l_{\tilde{z}_1}$ meet at a point q (outside of Ω), for all $p \in X$ we choose the line L_p to be the one passing through p and q. If $\tilde{l}_p$, for any $p \in X$, were to intersect any other $\tilde{l}_{\tilde{p}}$ for $\tilde{p} \in P \setminus X$, it would have to first intersect one of the two $\tilde{l}_{\tilde{z}_i}$, $i = 0, 1$, which does not happen by construction. Note that X is the connected component of $C_f \cap P$ containing z. We can therefore foliate P in constancy segments by exhausting all the connected components of $C_f \cap P$.

To prove our final claim, that is the constancy of f on the region X, we note that f is constant on the segment $\tilde{l}_{\tilde{z}_1} = [\tilde{z}_1^-, \tilde{z}_1^+]$, and on the segment $[\tilde{z}_0, \tilde{z}_1]$, whose interior lies in C_f. Since $\tilde{z}_1 \in [\tilde{z}_1^-, \tilde{z}_1^+]$, by lemma 2.7, f must be constant in the triangular domain created by $\tilde{z}_0, \tilde{z}_1^-, \tilde{z}_1^+$. But then this last assertion implies that f is constant on the segments $[\tilde{z}_0, \tilde{z}_1^-]$ and $[\tilde{z}_0, \tilde{z}_1^+]$, which combined with lemma 2.7 again, using the constancy of f on segments $[\tilde{z}_0, \tilde{z}_0^-]$ and $[\tilde{z}_0, \tilde{z}_0^+] \subset l_{\tilde{z}_0}$, proves the claim that f is constant on X. The proof of (a) $\Rightarrow$ (b) is complete.

(b) $\Rightarrow$ (c):

It is logically obvious.

(c) $\Rightarrow$ (a):

The proof can be found in the last page of [**41**]. We reproduce it here for completeness. Let $x \in \Omega$ and consider a Ball B_x over which condition (c) is satisfied. In particular, f is constant on $L_x \cap B_x$ for some line L_x passing through x. Now,

consider the (possibly infinite) segment

$$c_x := \text{The connected component of } x \text{ in } \{y \in L_x \cap \Omega; f(y) = f(x)\}.$$

We claim that c_x, which contains the segment $L_x \cap B_x$, is either equal to l_x, i.e. the connected component of x in $L_x \cap \Omega$, or $x \in C_f$. This proves that f is constant either on l_x, or in a neighbourhood of x, and so (a) holds true.

We assume that $c_x \neq l_x$, and prove $x \in C_f$. In this case c_x must admit at least one endpoint within Ω. Let $y \in \Omega$ be that endpoint. Consider the open disk B_y centred at y in which all points admit constancy segments within B_y. Let the line L_y passing through y be such that f is constant on the segment $L_y \cap B_y$ and let z, w be the endpoints of this segment. L_y and L_x cannot be parallel, since then c_x can be extended along $L_x = L_y$ to include either z or w, which contradicts the fact that it must be a maximal connected constancy subset of l_x. Therefore, we can choose an element $\tilde{y} \in B_y \cap c_x$ to form an open triangle Δ with vertices $\tilde{y}, z, w$. f is constant on $[y, z]$ and on $[\tilde{y}, y]$, where $y \in (z, w)$. Since no segment $L_p \cap B_y$ departing from a point $p \in \Delta$ can reach ∂B_y on its both ends without crossing $[w, z]$ or $[\tilde{y}, y]$, we deduce that $f|_\Delta \equiv f(y) = f(x)$.

We observe that C_f contains Δ, and hence also the open segment $(\tilde{y}, y)$. We have thus found a non-empty interval in c_x from which we can propagate our local constancy property and so reach to the desired conclusion of $x \in C_f$. We argue by contradiction: If $x \notin C_f$, let $\tilde{x}$ be the closest point to y on $[x, y]$ which is not in C_f. Certainly $C_f \supset (\tilde{x}, y) \supset (\tilde{y}, y)$ and $(x, y) \supset (\tilde{x}, y)$. Consider once more the open disk $B_{\tilde{x}}$ according to condition (c). Since C_f is open, for some $\hat{y} \in (\tilde{x}, y)$ close enough to $\tilde{x}$, f is constant on a segment $[\hat{z}, \hat{w}] \subset B_{\tilde{x}} \cap C_f$ orthogonal to c_x at $\hat{y}$, with $\hat{y} \in (\hat{z}, \hat{w})$. Also note that for $\hat{x} \in [x, \hat{y})$, f is constant on the portion $[\hat{x}, \hat{y}] = [x, \hat{y}] \cap B_{\tilde{x}} \supset [\tilde{x}, \hat{y}]$ of the segment $[x, \hat{y}] \subset c_x$. Since f takes the value $f(x)$ on both of $[\hat{x}, \hat{y}]$ and $[\hat{z}, \hat{w}]$, both within the set $B_{\tilde{x}}$, a similar argument as above shows that f is constant on the open triangle $\hat{\Delta}$ with vertices $\hat{x}, \hat{z}, \hat{w}$, and so $\hat{\Delta} \subset C_f$. But then $\tilde{x} \in (\hat{x}, \hat{y}) \subset \hat{\Delta} \subset C_f$, which contradicts its choice.

In view of proposition 2.1, and as already observed in [**36**, Proposition 2.30], we have:

COROLLARY 2.10. *Let $\Omega \subset \mathbb{R}^2$ be a domain and let the continuous mapping $f : \Omega \to \mathbb{R}^m$ satisfy any of the equivalent conditions in proposition 2.1. Then for all $x \in \Omega$, there exists $L > 0$, an open disk B centred at x, and a unit Lipschitz vector field $\vec{\eta} : B \to \mathbb{R}^2$ with $\mathrm{Lip}\,\vec{\eta} \leqslant L$, for which*

$$\forall y \in B \quad \vec{\eta}(y) = \vec{\eta}(y + s\vec{\eta}(y)) \text{ and } f(y) = f(y + s\vec{\eta}(y)) \text{ for all } s \text{ for which } y + s\vec{\eta}(y) \in B.$$

Proof. We choose a disk $B_x = B(x, 2\delta) \subset \Omega$ according to condition (b) and we let $B = B(x, \delta)$. For any $p \in B$, there exists a line L_p such that f is constant on $\tilde{l}_p = L_p \cap B_x$, and no two such lines meet within B_x. We define the mapping $\boldsymbol{\Lambda} : B \to \mathbb{RP}^1$ which associates to each point $p \in B$ the element of the real projective line $\mathbb{RP}^1$ determined by the direction of L_p, and we note that it is constant along the segments $L_p \cap B$ and continuous. Since B is simply connected, $\boldsymbol{\Lambda}$ can be lifted to a continuous mapping $\vec{\eta} : B \to \mathbb{S}^1$. By constancy of $\boldsymbol{\Lambda}$ along the segments $L_p \cap B$, $\vec{\eta}$

can only take two distinct values along them, and so its continuity implies that $\vec{\eta}$ is constant along these directions, which are now determined by $\vec{\eta}$ itself. Since two distinct lines L_p and L_q, for p, $q \in B$, do not meet except possibly at an at least a δ-distance from ∂B, we conclude that $\mathrm{Lip}\,\vec{\eta} \leqslant 1/\delta$.   □

2.2. Developability through test functions

The following lemma will allow us to translate the developability properties of $v \in C^1$ into a property for the distribution $\nabla^2 v$:

LEMMA 2.11. *Let $B \subset \mathbb{R}^2$ be an open disk and $f : B \to \mathbb{R}^m$ be continuous. Also, let $\vec{\eta} : B \to \mathbb{R}^2$ be a unit Lipschitz vector field such that the following property holds true:*

$$\forall x \in B \quad \vec{\eta}(x) = \vec{\eta}(x + s\vec{\eta}(x)) \text{ for all } s \text{ for which } x + s\vec{\eta}(x) \in B.$$

Then the following two properties are equivalent:

(a) $\forall x \in B \quad f(x) = f(x + s\vec{\eta}(x))$ *for all s for which $x + s\vec{\eta}(x) \in B$,*

(b) $\forall \psi \in C_c^\infty(B) \quad \displaystyle\int_B f \mathrm{div}(\psi\vec{\eta}) = 0.$

Proof. condition (b) states that the distributional derivative of f in direction of the vector field $\vec{\eta}$ vanishes. The proof shows that a regular enough change of variable reduces the problem to the case when $\vec{\eta}$ is constant.

Let $\vec{\xi} = \vec{\eta}^\perp$ and $x \in B$. Choose $T_0 > 0$ such that $\gamma : (-T_0, T_0) \to B$ is a solution to the ODE $\gamma'(t) = \vec{\xi}(\gamma(t))$ in B with initial value $\gamma(0) = x$. Let $L = \mathrm{Lip}\,\vec{\eta}$, $k(t) = -\gamma''(t) \cdot \vec{\eta}(\gamma(t))$ and note that $\|k\|_{L^\infty(-T_0, T_0)} \leqslant L$. It is straightforward to see that there exists $0 < t_0 \leqslant T_0/2$ such that for all t, $t' \in [-t_0, t_0]$, and all s, $s' \in \mathbb{R}$,

$$\gamma(t) + s\vec{\eta}(\gamma(t)), \gamma(t') + s'\vec{\eta}(\gamma(t')) \in B \implies \gamma(t) + s\vec{\eta}(\gamma(t)) \neq \gamma(t') + s'\vec{\eta}(\gamma(t')),$$

unless $t = t'$ and $s = s'$. Indeed, assume by contradiction that there exist sequences t_k, $t'_k \to 0$, s_k, s'_k, such that for all k, $(t_k, s_k) \neq (t'_k, s'_k)$, and

$$\gamma(t_k) + s_k\vec{\eta}(\gamma(t_k)) = \gamma(t'_k) + s'_k\vec{\eta}(\gamma(t'_k)) \in B.$$

Note that $t_k = t'_k$ implies $s_k = s'_k$, and hence we must have $t_k \neq t'_k$ for all k, implying on its turn that $\gamma(t_k) \neq \gamma(t'_k)$ for all k. On the other hand, by the main property of $\vec{\eta}$ we obtain

$$\vec{\eta}(\gamma(t_k)) = \vec{\eta}(\gamma(t_k) + s_k\vec{\eta}(\gamma(t_k))) = \vec{\eta}(\gamma(t'_k) + s'_k\vec{\eta}(\gamma(t'_k))) = \vec{\eta}(\gamma(t'_k)).$$

This yields

$$\gamma(t_k) - \gamma(t'_k) = (s'_k - s_k)\vec{\eta}(\gamma(t_k)),$$

implying that

$$\frac{\gamma(t_k) - \gamma(t'_k)}{t_k - t'_k} \cdot \vec{\xi}(\gamma(t_k)) = 0.$$

Passing to the limit we obtain $|\vec{\xi}(x)|^2 = \gamma'(0) \cdot \vec{\xi}(\gamma(0)) = 0$, which is a contradiction.

Since $\gamma([-t_0, t_0])$ is compact in B, we choose $0 < s_0 < 1/2L$ such that the image of $[-t_0, t_0] \times [-s_0, s_0]$ under

$$\Phi(t, s) := \gamma(t) + s\vec{\eta}(\gamma(t)), \ t \in [-t_0, t_0], s \in [-s_0, s_0],$$

lies compactly in B. We note that Φ is one-to-one and Lipschitz on $U := (-t_0, t_0) \times (-s_0, s_0)$ and that we have

$$\det \nabla\Phi(t, s) = 1 + sk(t) \geqslant \frac{1}{2} \text{ a.e. in } U.$$

We conclude that $\Phi : U \to \Phi(U)$ is a bilipschitz change of variable.

We let $V_x = \Phi(U)$, which is an open neighbourhood of x in B. We calculate for any $\psi \in C_c^\infty(V_x)$:

$$\mathrm{div}(\psi\vec{\eta}) = \partial_{\vec{\eta}}(\psi\vec{\eta}) \cdot \vec{\eta} + \partial_{\vec{\xi}}(\psi\vec{\eta}) \cdot \vec{\xi} = \nabla\psi \cdot \vec{\eta} + \psi\partial_{\vec{\xi}}\vec{\eta} \cdot \vec{\xi},$$

which gives for $\tilde{\psi} = \psi \circ \Phi$:

$$\mathrm{div}(\psi\vec{\eta})(\Phi(t, s)) = \partial_s\tilde{\psi} + \tilde{\psi}\frac{k(t)}{1 + sk(t)}.$$

Therefore, letting $\tilde{f}(t, s) = f \circ \Phi(s, t) = f(\gamma(t) + s\vec{\eta}(\gamma(t))$ we obtain

$$\begin{aligned}
\int_{V_x} f\mathrm{div}(\psi\vec{\eta}) &= \int_U f\mathrm{div}(\psi\vec{\eta}) \circ \Phi(s, t)(1 + sk(t)) \, \mathrm{d}s\mathrm{d}t \\
&= \int_U \tilde{f}(s, t)\left(\partial_s\tilde{\psi} + \tilde{\psi}\frac{k(t)}{1 + sk(t)}\right)(1 + sk(t)) \, \mathrm{d}s\mathrm{d}t \\
&= \int_U \tilde{f}(t, s)\left(k(t)\tilde{\psi} + \partial_s\tilde{\psi} + sk(t)\partial_s\tilde{\psi}\right) \, \mathrm{d}s\mathrm{d}t \\
&= \int_U \tilde{f}(t, s)\partial_s((1 + sk(t))\tilde{\psi}) \, \mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{2.1}$$

If (a) is satisfied, then $\tilde{f}(t, s) = \tilde{f}(t)$ for all values of s for which the function is defined, and hence, since $\tilde{\psi}$ is compactly supported in U, we obtain:

$$\forall \psi \in C_c^\infty(V_x) \quad \int_{V_x} f\mathrm{div}(\psi\vec{\eta}) = 0. \tag{2.2}$$

Now for $\psi \in C_c^\infty(B)$, we let $K := \mathrm{supp}(\psi) \subset B$. K is compact and so it admits a finite covering of open sets of the form V_{x_i}, for $i = 1, \cdots, n$. We consider a partition

of unity associated with this covering

$$\theta_i \in C_c^\infty(V_{x_i}), \quad \sum_{i=1}^n \theta_i = 1 \text{ on } K,$$

and we conclude with (b) by (2.2):

$$\int_B f \operatorname{div}(\psi\vec{\eta}) = \sum_{i=1}^n \int_B f \operatorname{div}(\theta_i\psi\vec{\eta}) = \sum_{i=1}^n \int_{V_{x_i}} f \operatorname{div}(\theta_i\psi\vec{\eta}) = 0.$$

To prove the converse, assume (b) and let $\varphi \in L^1(U)$, be such that

$$\int_{-s_0}^{s_0} \varphi(t,s)\, \mathrm{d}s = 0, \tag{2.3}$$

and define

$$\phi(t,s') := \frac{1}{1+s'k(t)} \int_{-s_0}^{s'} \varphi(t,s)\, \mathrm{d}s.$$

We can construct a sequence $\psi_k \in C_c^{0,1}(V_x)$ such that

$$\psi_k \circ \Phi \to \phi \quad \text{in } L^1(U) \quad \text{and} \quad \partial_s(\psi_k \circ \Phi) \to \partial_s\phi \quad \text{in } L^1(U).$$

Therefore by (b)

$$\int_{V_x} f \operatorname{div}(\psi\vec{\eta}) = 0 \,\, \forall \psi \in C_c^\infty(V_x) \implies \int_{V_x} f \operatorname{div}(\psi_k\vec{\eta}) = 0,$$

which in view of (2.1) implies

$$\int_U \tilde{f}\varphi\, \mathrm{d}s\mathrm{d}t = \int_U \tilde{f}(t,s)\partial_s((1+sk(t))\phi)\, \mathrm{d}s\mathrm{d}t$$

$$= \lim_{k\to\infty} \int_U \tilde{f}(t,s)\partial_s\left((1+sk(t))\psi_k \circ \Phi\right)\, \mathrm{d}s\mathrm{d}t = 0.$$

This fact being true for all $\varphi \in L^1(U)$ satisfying (2.3) implies that

$$\forall t \in (-t_0,t_0) \quad \forall s \in (-s_0,s_0) \quad \tilde{f}(t,s) = \fint_{-s_0}^{s_0} \tilde{f}(t,s)\, \mathrm{d}s.$$

This means that for $|s| < s_0$:

$$f(x+s\vec{\eta}(x)) = \tilde{f}(0,s) = \tilde{f}(0,0) = f(x).$$

Global property (a) is a direct consequence of this local property around each $x \in B$, and the fact that $\vec{\eta}$ itself is constant on $x + s\vec{\eta}(x) \in B$ for all s. $\qquad\square$

3. A Hölder continuity toolbox

3.1. Notations

For any open set $U \subset \mathbb{R}^n$, sufficiently differentiable function $f : U \to \mathbb{R}$, and nonnegative integer j we will denote by $[\cdot]_{j;U}$ the supremum norm of its jth (multi-index) derivatives over U and by

$$\|f\|_{k;U} := \sum_{j=0}^{k} [f]_{j;U}$$

its C^k norm. For $0 < \alpha \leqslant 1$, the corresponding Hölder seminorms and norms are identified by the standard conventions

$$[f]_{k,\alpha;U} := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|D^k f(x) - D^k f(y)|}{|x - y|^\alpha},$$

$$\|f\|_{k,\alpha;U} := \|f\|_{k;U} + [f]_{k,\alpha;U}.$$

The modulus of continuity of $f : U \to \mathbb{R}$ is defined by

$$\omega_{f;U}(r) := \sup_{\substack{x,y \in U \\ 0 < |x-y| \leqslant r}} |f(x) - f(y)|.$$

We also introduce the α-Hölder modulus of f for $0 < \alpha < 1$:

$$[f]_{0,\alpha;U|r} := \sup_{\substack{x,y \in U \\ 0 < |x-y| \leqslant r}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We will drop the subscript U when denoting these quantities for the specific case of an open set named U, hence $\| \cdot \|_{\cdot} = \| \cdot \|_{\cdot;U}$, etc.

For a fixed mollifying kernel $\phi \in C_c^\infty(B(0,1))$ with $\int \phi = 1$, and $x \in \mathbb{R}^n$, we denote the standard convolution of a given mapping f with ϕ over the length scale ε by

$$f_\varepsilon(x) := f * \phi_\varepsilon(x) = \int_{\mathbb{R}^n} f(x - y)\phi_\varepsilon(y)\mathrm{d}y,$$

where for all x

$$\phi_\varepsilon(x) := \varepsilon^{-n}\phi\left(\frac{x}{\varepsilon}\right).$$

Naturally $f_\varepsilon(x)$ is defined for ε small enough provided f is integrable in some neighbourhood of x.

Throughout this article, the universally bounded constant $c = O(1)$ might change but is independent of all the data, unless specified by an argument. We will also

use the usual little-o convention, i.e. for any two functions $b, \tilde{b} : [0, \varepsilon_0) \to \mathbb{R}^+$, we have:

$$\tilde{b}(\varepsilon) \text{ is } o(b(\varepsilon)) \Longleftrightarrow \forall \varepsilon \quad \tilde{b}(\varepsilon) \leqslant cb(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{\tilde{b}(\varepsilon)}{b(\varepsilon)} = 0.$$

We will repeatedly refer to the following convolution estimates in our arguments:

LEMMA 3.1. *Let $V \subset U \subset \mathbb{R}^n$ be open sets such that $\mathrm{dist}(\overline{V}, \partial U) > 0$. Assume that $f, h : U \to \mathbb{R}$ are locally integrable. Then for any $0 < \alpha < 1$ and $\varepsilon < \mathrm{dist}(\overline{V}, \partial U)$:*

(i) $\|f_\varepsilon - f\|_{0;V} \leqslant c[f]_{0,\alpha|\varepsilon}\varepsilon^\alpha,$

(ii) $\|\nabla f_\varepsilon\|_{0;V} \leqslant c[f]_{0,\alpha|\varepsilon}\varepsilon^{\alpha-1},$

(iii) $\|f_\varepsilon h_\varepsilon - (fh)_\varepsilon\|_{1;V} \leqslant c[f]_{0,\alpha|\varepsilon}[h]_{0,\alpha|\varepsilon}\varepsilon^{2\alpha-1}.$

Note the basic estimate

$$\forall x \in V \quad \forall y \in B(0,\varepsilon) \quad |f(x) - f(x-y)| \leqslant \omega_{f;U}(\varepsilon) \leqslant [f]_{0,\alpha|\varepsilon}\varepsilon^\alpha.$$

Based on this estimate, the proof follows the same lines as in [**13**, Lemma 1] and is left to the reader. The commutator estimate lemma 3.1-(iii) will be crucial for our analysis. It will be used to prove the quadratic estimate (4.5) ([**13**, Proposition 1]) for the pull-back metric of mollified isometric immersions. It is also a necessary ingredient of the proof of theorem 1.5 through establishing the degree formulas (1.1) for $v \in c^{1,2/3}$, following [**41**].

3.2. Little Hölder spaces

Our goal here is to define and characterize the elements of the little Hölder spaces $c^{1,\alpha}$. We denote by $C^k(U)$ the functions which are continuously differentiable up to the kth order in U, and by $C^k(\overline{U})$ the functions whose derivatives up to order k in U exist and admit a continuous extension to $\overline{U}$. For any open bounded weakly Lipschitz domain $U \subset \mathbb{R}^n$, we recall the definition of the Hölder space

$$C^{k,\alpha}(\overline{U}) := \{f \in C^k(U); \|f\|_{k,\alpha;U} < \infty\}.$$

By uniform continuity of derivatives, all the derivatives of f up to the order k have continuous extensions to $\overline{U}$ and so $C^{k,\alpha}(\overline{U}) \subset C^k(\overline{U})$. For an arbitrary open set $\Omega \subset \mathbb{R}^n$ we say $f \in C^{k,\alpha}(\Omega)$ if and only if for any point $x \in \Omega$, $f \in C^{k,\alpha}(\overline{U})$ for some open neighbourhood $U \subset \Omega$ of x.

REMARK 3.2. We have restricted our attention to weakly Lipschitz bounded domains U in order to insure the inclusion $C^1(\overline{U}) \subset C^{0,\alpha}(\overline{U})$ and to avoid unnecessary technical complications in what follows.

DEFINITION 3.3. *Let $0 < \alpha < 1$ and $U \subset \mathbb{R}^n$ be an open bounded weakly Lipschitz domain. The little Hölder space $c^{0,\alpha}(\overline{U})$ is the closure of $C^1(\overline{U})$ with respect to $\|\cdot\|_{0,\alpha;U}$ in $C^{0,\alpha}(\overline{U})$. For any open set $\Omega \subset \mathbb{R}^n$, we say $f \in c^{0,\alpha}(\Omega)$ if and only if for any point $x \in \Omega$, $f \in c^{0,\alpha}(\overline{U})$ for an open neighbourhood $U \subset \Omega$ of x. For $k \in \mathbb{N}$,*

the space $c^{k,\alpha}(\overline{U})$ (resp. $c^{k,\alpha}(\Omega)$) is defined to be the set of all functions $f \in C^k(U)$ (resp. $C^k(\Omega)$) for which the components of $D^k f$ belong to $c^{0,\alpha}(\overline{U})$ (resp. $c^{0,\alpha}(\Omega)$).

REMARK 3.4. Under the above regularity assumptions on U, it can be shown that $c^{k,\alpha}(\overline{U})$ is the closure of $C^\infty(\overline{U})$ in $C^{k,\alpha}(\overline{U})$. We will not use this fact in our arguments.

It is useful to identify the elements of the little Hölder spaces in terms of their modulus of continuity. In this regard we state the following well-known equivalence, whose proof is presented for the sake of completeness (see also Appendix B).

PROPOSITION 3.5. *Let U be a bounded weakly Lipschitz domain, $0 < \alpha < 1$. Then the following statements are equivalent.*

(i) $f \in c^{0,\alpha}(\overline{U})$.

(ii) $f : U \to \mathbb{R}$ *admits an extension* $\tilde{f} \in c^{0,\alpha}(\mathbb{R}^n)$ *and* $\tilde{f}_\varepsilon|_U \to f$ *in* $C^{0,\alpha}(\overline{U})$.

(iii) $\omega_{f;U}(r)$ *is* $o(r^\alpha)$.

(iv) $\lim_{r \to 0} [f]_{0,\alpha;U|r} = 0$.

Proof. (i) $\Rightarrow$ (iv): Assuming $f \in c^{0,\alpha}(\overline{U})$, let $f_k \in C^1(\overline{U})$ be a sequence of functions converging to f in $C^{0,\alpha}(\overline{U})$ as $k \to \infty$. We estimate:

$$[f]_{0,\alpha|r} \leqslant [f - f_k]_{0,\alpha|r} + [f_k]_{0,\alpha|r} \leqslant [f - f_k]_{0,\alpha} + c(U)\|\nabla f_k\|_0 r^{1-\alpha},$$

and a straightforward argument by the auxiliary argument k implies the required convergence.

(iv) $\Rightarrow$ (iii): This is straightforward since for all $r > 0$:

$$\frac{1}{r^\alpha} \omega_{f;U}(r) \leqslant [f]_{0,\alpha;U|r}.$$

(iii) $\Rightarrow$ (ii): By the assumption f is uniformly continuous in U. We extend f to $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by applying proposition B.2, obtaining that $\omega_{\tilde{f};\mathbb{R}^n}(r) \leqslant C\omega_{f;U}(r)$ is $o(r^\alpha)$. We conclude that

$$[\tilde{f}]_{0,\alpha;\mathbb{R}^n|r} = \sup_{0 \leqslant s \leqslant r} \frac{1}{s^\alpha} \omega_{\tilde{f};\mathbb{R}^n}(s)$$

is $o(1)$ as a function of r. Consider any bounded weakly Lipschitz domain $V \subset \mathbb{R}^n$ and note that $\tilde{f}|_V \in C^{0,\alpha}(\overline{V})$. Applying lemma B.1 to V implies that $\tilde{f}_\varepsilon|_V \to \tilde{f}|_V$ in $C^{0,\alpha}(\overline{V})$ as $\varepsilon \to 0$ and thus $\tilde{f}|_V \in c^{0,\alpha}(\overline{V})$. Both conclusions in (ii) follow.

(ii) $\Rightarrow$ (i): This statement trivially follows from definition 3.3. $\qquad\square$

COROLLARY 3.6. *Let $0 < \alpha < \beta < 1$ and let $U \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then $c^{1,\beta}(\overline{U}) \subsetneq C^{1,\beta}(\overline{U}) \subsetneq c^{1,\alpha}(\overline{U})$.*

Proof. The first inclusion is trivial. If $f \in C^{1,\beta}(\overline{U})$, then $\omega_{f;U}(r)$ is $O(r^\beta)$ and so the second inclusion follows by proposition 3.5. $\qquad\square$

3.3. A distributional product and a criteria for Hölder continuity

The following two propositions are practically known by the community at large. Similar or more general statements in the same spirit have appeared in various contexts, e.g. in the discussion of the Young integral [**58**] or of the paraproducts [**53**, Chapter 4]. For example, see [**2**, Theorem 2.52] and [**23**, Theorem 13.16], or compare with [**45**, Theorem 22]. Here, we have formulated and proved rather accessible straightforward versions for the little Hölder spaces which are more adapted to our needs.

DEFINITION 3.7. *Let $U \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain. We say that a distribution $T \in \mathcal{D}'(U, \mathbb{R}^n)$ satisfies the α-interpolation property on $W_0^{1,1}(U)$ whenever for $C > 0$*

$$\forall \psi \in C_c^\infty(U) \quad \forall j \in \{1, \cdots, n\} \quad |T_j[\psi]| \leqslant C\|\psi\|_{L^1}^\alpha \|\partial_j \psi\|_{L^1}^{1-\alpha}.$$

We say that T satisfies the fine α-interpolation property on $W_0^{1,1}(U)$ whenever for $C > 0$

$$\forall \sigma > 0 \quad \exists \delta = \delta(\sigma) > 0 \quad \forall \psi \in C_c^\infty(U) \quad \forall j \in \{1, \cdots, n\}$$
$$\|\psi\|_{L^1} \leqslant \delta\|\partial_j \psi\|_{L^1} \implies |T_j[\psi]| \leqslant C\|\psi\|_{L^1}^\alpha \|\partial_j \psi\|_{L^1}^{1-\alpha}\sigma, \tag{3.1}$$

with $\delta(1) = +\infty$

PROPOSITION 3.8. *Let $U \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain and $V \Subset U$ be an open set compactly contained in U. Let $0 < \alpha < 1$, $f \in C^{0,\alpha}(\overline{U})$ and $h \in c^{0,\alpha}(\overline{U})$. If $\alpha > 1/2$, then as $\varepsilon \to 0$, for all $j = 1, \cdots n$ the sequence $f_\varepsilon \partial_j h_\varepsilon$ converges in the sense of distributions to a distribution on V which we denote by $f\partial_j h$. More precisely, the convergence is in the dual of $W_0^{1,1}(V)$, i.e. if $\psi \in W_0^{1,1}(V)$*

$$f\partial_j h[\psi] := \lim_{\varepsilon \to 0} \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x) \mathrm{d}x$$

exists. Moreover for all $\psi \in C_c^\infty(V)$

$$\left| \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x)\mathrm{d}x - f\partial_j h[\psi] \right| \leqslant \|f\|_{0,\alpha}[h]_{0,\alpha}\left(o(\varepsilon^{2\alpha-1})\|\psi\|_{L^1} + o(\varepsilon^\alpha)\|\partial_j \psi\|_{L^1} \right), \tag{3.2}$$

and $T = (f\partial_1 h, \ldots, f\partial_n h)$ satisfies the fine α-interpolation property (3.1) on $W_0^{1,1}(U)$ with $C = c(U, V)\|f\|_{0,\alpha}[h]_{0,\alpha}$.

REMARK 3.9. *If h is merely in $C^{0,\alpha}(\overline{U})$, a similar result holds true with the following estimates for all $\psi \in W_0^{1,1}(U)$:*

$$\left| \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x)\mathrm{d}x - f\partial_j h[\psi] \right| \leqslant c\|f\|_{0,\alpha}[h]_{0,\alpha}\left(\varepsilon^{2\alpha-1}\|\psi\|_{L^1} + \varepsilon^\alpha\|\partial_j \psi\|_{L^1} \right)$$

and

$$|f\partial_j h[\psi]| \leqslant C\|\psi\|_{L^1}^\alpha \|\partial_j \psi\|_{L^1}^{1-\alpha},$$

i.e. $T = (f\partial_1 h, \ldots, f\partial_n h)$ satisfies the α-interpolation property on $W_0^{1,1}(U)$.

Proof. Let $\psi \in W_0^{1,1}(V)$, and for $\varepsilon < \text{dist}(\overline{V}, \partial U)$ we define:

$$a^{(\varepsilon)} := \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x) \mathrm{d}x.$$

We first prove that $a^{(\varepsilon)}$ is a Cauchy sequence. For any $0 < \varepsilon' < \varepsilon$ we have

$$|a^{(\varepsilon)} - a^{(\varepsilon')}| = \left| \int_V (f_\varepsilon \partial_j h_\varepsilon - f_{\varepsilon'} \partial_j h_{\varepsilon'}) \psi \right|$$

$$\leqslant \left| \int_V f_\varepsilon (\partial_j h_\varepsilon - \partial_j h_{\varepsilon'}) \psi \right| + \left| \int_V (f_\varepsilon - f_{\varepsilon'}) \partial_j h_{\varepsilon'} \psi \right|$$

$$= \left| -\int_V \partial_j f_\varepsilon (h_\varepsilon - h_{\varepsilon'}) \psi - \int_V f_\varepsilon (h_\varepsilon - h_{\varepsilon'}) \partial_j \psi \right| + \left| \int_V (f_\varepsilon - f_{\varepsilon'}) \partial_j h_{\varepsilon'} \psi \right|$$

$$\leqslant \left| \int_V \partial_j f_\varepsilon (h_\varepsilon - h_{\varepsilon'}) \psi \right| + \left| \int_V f_\varepsilon (h_\varepsilon - h_{\varepsilon'}) \partial_j \psi \right| + \left| \int_V (f_\varepsilon - f_{\varepsilon'}) \partial_j h_{\varepsilon'} \psi \right|$$

$$= I_1(\varepsilon, \varepsilon') + I_2(\varepsilon, \varepsilon') + I_3(\varepsilon, \varepsilon')$$

(3.3)

We estimate, using $\varepsilon' < \varepsilon$:

$$I_1(\varepsilon, \varepsilon') = \left| \int_V \psi(x) \left(\int_{\mathbb{R}^n} (f(x-y) \partial_j \phi_\varepsilon(y)) \mathrm{d}y \right) \right.$$

$$\left. \left(\int_{\mathbb{R}^n} (h(x-z)(\phi_\varepsilon(z) - \phi_{\varepsilon'}(z)) dz \right) \mathrm{d}x \right|$$

$$= \left| \int_V \psi(x) \int_{\mathbb{R}^n} (f(x-y) - f(x)) \partial_j \phi_\varepsilon(y) \mathrm{d}y \right.$$

$$\left. \int_{\mathbb{R}^n} (h(x-z) - h(x))(\phi_\varepsilon(z) - \phi_{\varepsilon'}(z)) dz \, \mathrm{d}x \right|$$

$$\leqslant \|\psi\|_{L^1} (c[f]_{0,\alpha} \varepsilon^{\alpha-1})(c[h]_{0,\alpha|\varepsilon} \varepsilon^\alpha) \leqslant \|\psi\|_{L^1} (c[f]_{0,\alpha} \varepsilon^{\alpha-1})([h]_{0,\alpha} o(\varepsilon^\alpha))$$

$$\leqslant \|\psi\|_{L^1} [h]_{0,\alpha} [f]_{0,\alpha} o(\varepsilon^{2\alpha-1}),$$

where we used lemma 3.1-(i) and proposition 3.5 in the third line. For $I_3(\varepsilon, \varepsilon')$, we note that $\varepsilon \nleqslant \varepsilon'$ and hence the same estimate as for I_1 is not obtained, but through the same calculations as for I_1, we obtain

$$I_3(\varepsilon, \varepsilon') \leqslant \|\psi\|_{L^1} [h]_{0,\alpha} [f]_{0,\alpha} \varepsilon'^{-1} o(\varepsilon'^\alpha) \varepsilon^\alpha.$$

In the same manner, we can also obtain an estimate for I_2:

$$
I_2(\varepsilon, \varepsilon') = \left| \int_V f_\varepsilon (h_\varepsilon - h_{\varepsilon'}) \partial_j \psi \right|
$$

$$
\leqslant \|\partial_j \psi\|_{L^1} \|f\|_0 \sup_{x \in V} \left| \int_{\mathbb{R}^n} h(x - y)(\phi_\varepsilon(y) - \phi_{\varepsilon'}(y)) \mathrm{d}y \right|
$$

$$
= \|\partial_j \psi\|_{L^1} \|f\|_0 \sup_{x \in V} \left| \int_{\mathbb{R}^n} (h(x - y) - h(x))(\phi_\varepsilon(y) - \phi_{\varepsilon'}(y)) \mathrm{d}y \right|
$$

$$
\leqslant \|\partial_j \psi\|_{L^1} \|f\|_0 [h]_{0,\alpha} o(\varepsilon^\alpha).
$$

Putting the three estimates together in view of (3.3), we obtain a first crude estimate:

$$
|a^{(\varepsilon)} - a^{(\varepsilon')}| \leqslant [h]_{0,\alpha} \left(\left[o(\varepsilon^{2\alpha-1}) + \varepsilon'^{-1} o(\varepsilon'^\alpha) \varepsilon^\alpha \right] [f]_{0,\alpha} \|\psi\|_{L^1} + o(\varepsilon^\alpha) \|f\|_0 \|\partial_j \psi\|_{L^1} \right).
\tag{3.4}
$$

This estimate is not enough to establish the Cauchy property of the sequence $a^{(\varepsilon)}$. Therefore, we proceed as follows. If $\varepsilon'/\varepsilon \geqslant 1/2$, we set $\rho = \varepsilon'/\varepsilon$, $m = 1$. Otherwise there exists $m \in \mathbb{N}$, depending on $\varepsilon'/\varepsilon < 1$, such that $1/4 \leqslant \rho = (\varepsilon'/\varepsilon)^{1/m} < 1/2$. We now write

$$
|a^{(\varepsilon)} - a^{(\varepsilon')}| \leqslant \sum_{k=1}^{m} |a^{(\rho^{k-1}\varepsilon)} - a^{(\rho^k \varepsilon)}|.
$$

We apply (3.4) successively to $0 < \rho^k \varepsilon < \rho^{k-1} \varepsilon$ (as the new ε' and ε) to obtain

$$
|a^{(\varepsilon)} - a^{(\varepsilon')}| \leqslant [h]_{0,\alpha} \left(o(\varepsilon^{2\alpha-1})[f]_{0,\alpha} \|\psi\|_{L^1}(1 + \rho^{\alpha-1}) \sum_{k=1}^{m} \rho^{(k-1)(2\alpha-1)} \right.
$$

$$
\left. + o(\varepsilon^\alpha) \|f\|_0 \|\partial_j \psi\|_{L^1} \sum_{k=1}^{m} \rho^{(k-1)\alpha} \right)
$$

$$
\leqslant (\rho^{\alpha-1} + 1) \frac{1 - \rho^{(2\alpha-1)m}}{1 - \rho^{2\alpha-1}} o(\varepsilon^{2\alpha-1})[h]_{0,\alpha}[f]_{0,\alpha} \|\psi\|_{L^1}
$$

$$
+ \frac{1 - \rho^{\alpha m}}{1 - \rho^\alpha} o(\varepsilon^\alpha)[h]_{0,\alpha} \|f\|_0 \|\partial_j \psi\|_{L^1}
$$

$$
\leqslant o(\varepsilon^{2\alpha-1})[h]_{0,\alpha}[f]_{0,\alpha} \|\psi\|_{L^1} + o(\varepsilon^\alpha)[h]_{0,\alpha} \|f\|_0 \|\partial_j \psi\|_{L^1},
$$

since by our choice either $1/2 \leqslant \rho < 1$ and $m = 1$, or $1/4 \leqslant \rho < 1/2$, independent of ε', ε. This implies that $a^{(\varepsilon)}$ is Cauchy sequence. Hence, the limit

$$
f \partial_j h[\psi] := a = \lim_{\varepsilon \to 0} a^{(\varepsilon)},
$$

exists.

Now, we estimate for a fixed $\varepsilon > 0$, as $\varepsilon' \to 0$:

$$\left| \int_V f_\varepsilon \partial_j h_\varepsilon \psi(x) \mathrm{d}x - f \partial_j h[\psi] \right| = |a^{(\varepsilon)} - a| \leqslant \|\psi\|_{L^1} [f]_{0,\alpha} [h]_{0,\alpha} o(\varepsilon^{2\alpha-1})$$
$$+ \|\partial_j \psi\|_{L^1} \|f\|_0 [h]_{0,\alpha} o(\varepsilon^\alpha),$$

which establishes (3.2). To obtain (3.1), applying lemma 3.1-(ii) and proposition 3.5 to $\partial_j h_\varepsilon$ we observe that:

$$|f \partial_j h[\psi]| \leqslant |a^{(\varepsilon)}| + |a^{(\varepsilon)} - a| \leqslant |a^{(\varepsilon)}| + [f]_{0,\alpha} [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{2\alpha-1})$$
$$+ \|f\|_0 [h]_{0,\alpha} \|\partial_j \psi\|_{L^1} o(\varepsilon^\alpha)$$
$$\leqslant \|f\|_0 [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{\alpha-1}) + [f]_{0,\alpha} [h]_{0,\alpha} \|\psi\|_{L^1} o(\varepsilon^{2\alpha-1})$$
$$+ \|f\|_0 [h]_{0,\alpha} \|\partial_j \psi\|_{L^1} o(\varepsilon^\alpha)$$
$$\leqslant \|f\|_{0,\alpha} [h]_{0,\alpha} \left(\|\psi\|_{L^1} o(\varepsilon^{\alpha-1}) + \|\partial_j \psi\|_{L^1} o(\varepsilon^\alpha) \right)$$

In other words,

$$\text{for any } \sigma > 0, \text{ there exists } \delta_0 = \delta_0(\sigma) > 0 \text{ such that}$$
$$\varepsilon \leqslant \min\{\delta_0, \mathrm{dist}(\overline{V}, \partial U)\} \implies |f \partial_j h[\psi]|$$
$$\leqslant c\|f\|_{0,\alpha} [h]_{0,\alpha} \left(\|\psi\|_{L^1} \varepsilon^{\alpha-1} + \|\partial_j \psi\|_{L^1} \varepsilon^\alpha \right) \sigma, \tag{3.5}$$

with $\delta_0(1) = +\infty$.

We conclude the proof of (3.1): Given, $\sigma > 0$, we let

$$\delta = \left(1 + \frac{\mathrm{diam}(V)}{\mathrm{dist}(\overline{V}, \partial U)} \right) \delta_0.$$

Assume $\|\psi\|_{L^1} \leqslant \delta \|\partial_j \psi\|_{L^1}$ for a given nonzero ψ, then letting

$$\varepsilon = \left(1 + \frac{\mathrm{diam}(V)}{\mathrm{dist}(\overline{V}, \partial U)} \right)^{-1} \frac{\|\psi\|_{L^1}}{\|\partial_j \psi\|_{L^1}},$$

we obtain $\varepsilon < \mathrm{dist}(\overline{V}, \partial U)$ by the Poincaré inequality $\|\psi\|_{L^1} \leqslant \mathrm{diam}(V) \|\partial_j \psi\|_{L^1}$ on V. On the other hand we also have $\varepsilon \leqslant \delta_0$ and therefore applying (3.5) we obtain as required:

$$|f \partial_j h[\psi]| \leqslant c \left(1 + \frac{\mathrm{diam}(V)}{\mathrm{dist}(\overline{V}, \partial U)} \right)^{1-\alpha} \|f\|_{0,\alpha} [h]_{0,\alpha} \|\psi\|_{L^1}^\alpha \|\partial_j \psi\|_{L^1}^{1-\alpha} \sigma.$$

$\square$

DEFINITION 3.10. *We say $U \subset \mathbb{R}^n$ is an open coordinate rectangular box when it is a rectangular cuboid with edges parallel to the coordinate axes, i.e.*

$$U = \prod_{k=1}^{n} I_k$$

for I_k an open interval in $\mathbb{R}$.

PROPOSITION 3.11. *Assume that $U \subset \mathbb{R}^n$ is an open coordinate rectangular box and that for $f \in L^1(U)$, $T = \nabla f$ satisfies the fine α-interpolation property on $W_0^{1,1}(U)$ according to definition 3.7 with $C = 1$. Then*

$$[f]_{0,\alpha;U|\delta(\sigma)} \leqslant 2^{n+1}\sigma.$$

In particular, by proposition 3.5, $f \in c^{0,\alpha}(\overline{U})$ and $[f]_{0,\alpha;U} \leqslant 2^{n+1}$.

Proof. The proof of the proposition follows classical ideas relating the decay of mean oscillations to pointwise behaviour of functions, as pioneered by Morrey and Campanato, see for instance [**8**]. We prove the statement by induction over the dimension.

First let $n = 1$ and $U \subset \mathbb{R}$ and be an open interval and let $I \subset U$ be any open subinterval. Note that for any $\phi \in C_c^\infty(I)$, we can construct a sequence $\tilde{\phi}_k \in C_c^\infty(I)$ such that $\|\tilde{\phi}_k\|_0 \leqslant 1 + 2\|\phi\|_0$, and $\tilde{\phi}_k$ converges strongly in L^1 to $\phi - \fint_I \phi$. Now fix $\varphi \in C_c^\infty(I)$ with $\fint_I \varphi = 1$ and let

$$\phi_k := \tilde{\phi}_k - \left(\fint_I \tilde{\phi}_k\right)\varphi \xrightarrow{L^1} \phi - \fint_I \phi.$$

By the dominated convergence theorem we have

$$\lim_{k \to \infty} \int_I f\phi_k = \int_I f\left(\phi - \fint_I \phi\right).$$

Let $\psi_k(t) = \int_{-\infty}^{t} \phi_k(s)\mathrm{d}s$ and note that $\psi_k \in C_c^\infty(I)$. We have therefore for all $\phi \in C_c^\infty(I)$ and $\sigma > 0$, provided $|I| \leqslant \delta(\sigma)$:

$$\left|\int_I \left(f - \fint_I f\right)\phi\right| = \left|\int_I f\left(\phi - \fint_I \phi\right)\right| = \lim_{k \to \infty}\left|\int_I f\phi_k\right| = \lim_{k \to \infty}\left|\int_I f\psi_k'\right| = \lim_{k \to \infty}\left|T[\psi_k]\right|$$

$$\leqslant \liminf_{k \to \infty} \|\psi_k\|_{L^1}^\alpha \|\psi_k'\|_{L^1}^{1-\alpha}\sigma \leqslant \liminf_{k \to \infty}\|\psi_k'\|_{L^1}|I|^\alpha\sigma$$

$$= \liminf_{k \to \infty}\|\phi_k - \fint_I \phi_k\|_{L^1}|I|^\alpha\sigma = \|\phi - \fint_I \phi\|_{L^1}|I|^\alpha\sigma \leqslant 2\|\phi\|_{L^1}|I|^\alpha\sigma,$$

where we used the Poincaré inequality on I. Since $(L^1)' = L^\infty$, and $C_c^\infty(I)$ is dense in $L^1(I)$, we conclude that

$$\forall \sigma > 0 \quad |I| \leqslant \delta(\sigma) \implies \|f - \fint_I f\|_{L^\infty(I)} \leqslant 2|I|^\alpha\sigma. \tag{3.6}$$

Now, let $y \in U$, and let $d = \operatorname{dist}(y, \partial U)$, and for $z \in (y - d/2, \, y + d/2)$ and $r < d/2$ define

$$h_r(z) := \fint_{z-r}^{z+r} f(x) \mathrm{d}x.$$

h_r is continuous in z and converges a.e. to f on $(y - d/2, \, y + d/2)$ as $r \to 0$. On the other hand, for all $r' < r < d/2$:

$$\left| h_r(z) - h_{r'}(z) \right| \leqslant \fint_{(z-r, z+r) \times (z-r', z+r')} |f(x) - f(x')| \mathrm{d}x \mathrm{d}x'$$
$$\leqslant 4|(z-r, z+r)|^\alpha = 2^{\alpha+2} r^\alpha,$$

where we applied bound (3.6) to $I = (z - r, \, z + r)$, $\sigma = 1$, $\delta(1) = +\infty$. As a consequence, h_r is a Cauchy sequence in the uniform norm and f is continuous as the uniform limit of the h_r for $r \to 0$. Now, applying once again (3.6) to $I = (x, y) \subset U$ we obtain that

$$\forall \sigma > 0 \quad \forall x, y \in U \quad |x - y| \leqslant \delta(\sigma) \implies |f(x) - f(y)| \leqslant 4|x - y|^\alpha \sigma,$$

which implies $[f]_{0,\alpha;U|\delta(\sigma)} \leqslant 4\sigma$, as required.

Now, assume that $n > 1$ and that the statement is true for $n - 1$. Let $Q \subset U$ be any coordinate rectangular box, i.e.

$$Q := \prod_{k=1}^{n} I_k \subset U,$$

with the open intervals $I_k \subset \mathbb{R}$. Let $\widehat{x} := (x_1, \cdots, x_{n-1})$ and note that for all $h \in L^1(Q)$,

$$\text{for} \quad \text{a.e. } \widehat{x} \in Q^{n-1} := \prod_{j=1}^{n-1} I_k, \quad \hat{h}(\widehat{x}) := \fint_{I_n} h(x) \mathrm{d}x_n$$

is well defined and belongs to $L^1(Q)$. We claim that:

$$\forall \sigma > 0 \quad |I_n| \leqslant \delta(\sigma) \implies \operatorname*{ess\,sup}_{x \in Q} \left| f(x) - \hat{f}(\widehat{x}) \right| \leqslant 2|I_n|^\alpha \sigma. \tag{3.7}$$

Let us first show that the conclusion holds assuming claim (3.7) is true for all coordinate boxes $Q \subset U$. For a, b the extremities of I_n, choose $0 \leqslant \theta_k \leqslant 1$ in $C_c^\infty(I_n)$ such that $\theta_k \equiv 1$ on $(a + 1/k, \, b - 1/k)$. Given $\sigma > 0$, for $1 \leqslant j \leqslant n - 1$, and $\psi \in C_c^\infty(Q^{n-1})$, assuming that $\|\psi\|_{L^1(Q^{n-1})} \leqslant \delta(\sigma) \|\partial_j \psi\|_{L^1(Q^{n-1})}$, we have by our main

assumption on f:

$$
\left| \int_{Q^{n-1}} \hat{f} \partial_j \psi \right| = \frac{1}{|I_n|} \left| \int_Q f(x) \partial_j \psi(\hat{x}) \mathrm{d}x \right| = \lim_{k \to \infty} \frac{1}{|I_n|} \left| \int_Q f(x) \theta_k(x_n) \partial_j \psi(\hat{x}) \mathrm{d}x \right|
$$

$$
= \lim_{k \to \infty} \frac{1}{|I_n|} \left| \int_Q f \partial_j(\theta_k \psi) \right| \leqslant \lim_{k \to \infty} \frac{1}{|I_n|} \|\theta_k \psi\|_{L^1(Q)}^\alpha \|\partial_j(\theta_k \psi)\|_{L^1(Q)}^{1-\alpha} \sigma
$$

$$
= \lim_{k \to \infty} \frac{1}{|I_n|} \|\theta_k \psi\|_{L^1(Q)}^\alpha \|\theta_k \partial_j \psi\|_{L^1(Q)}^{1-\alpha} \sigma = \frac{1}{|I_n|} \|\psi\|_{L^1(Q)}^\alpha \|\partial_j \psi\|_{L^1(Q)}^{1-\alpha} \sigma
$$

$$
= \|\psi\|_{L^1(Q^{n-1})}^\alpha \|\partial_j \psi\|_{L^1(Q^{n-1})}^{1-\alpha} \sigma,
$$

since $\theta_k \psi \in C_c^\infty(Q)$ and

$$
\|\theta_k \psi\|_{L^1(Q)} = \left(\int_a^b \theta_k \right) \|\psi\|_{L^1(Q^{n-1})} \leqslant \delta(\sigma) \left(\int_a^b \theta_k \right) \|\partial_j \psi\|_{L^1(Q^{n-1})}
$$

$$
= \delta(\sigma) \|\partial_j(\theta_k \psi)\|_{L^1(Q)}.
$$

Applying the induction assumption, we deduce that

$$
[\hat{f}]_{0,\alpha;Q^{n-1}|\delta(\sigma)} \leqslant 2^n. \tag{3.8}
$$

To prove that f is continuous, fix $y \in U$ and consider a box $Q_d = Q^{n-1} \times (y_n - d, y_n + d) \subset U$ containing y. For all $z \in Q^{n-1} \times (y_n - d/2, y_n + d/2)$, and $r < d/2$ we define

$$
h_r(z) := \fint_{z_n-r}^{z_n+r} f(\hat{z}, s) \mathrm{d}s
$$

Note that, if r is fixed, the vertical averages of f are continuous in $\hat{z}$ as established in (3.8), and so h_r is continuous in z. Applying (3.7) with $\sigma = 1$ to $Q = Q^{n-1} \times (z_n - r, z_n + r)$ we have for $r' < r < d/2$:

$$
\left| h_r(z) - h_{r'}(z) \right| \leqslant \fint_{(z-r,z+r)\times(z-r',z+r')} |f(\hat{z}, s) - f(\hat{z}, s')| \mathrm{d}s \mathrm{d}s'
$$

$$
\leqslant 4|(z - r, z + r)|^\alpha = 2^{\alpha+2} r^\alpha,
$$

which implies that h_r locally uniformly converge to their limit, which happens to be f. Hence f is continuous in U.

To obtain a Hölder estimate, let $x, y \in U$. First, assume $x_j = y_j$ for some $1 \leqslant j \leqslant n$. We re-arrange the coordinates so that $j = n$ and note that for any sequence of coordinate rectangular boxes Q_k of height $1/k$, containing x and y, such that $x_n = y_n$ is the midpoint of $I_{n,k}$, we have by (3.8)

$$
\forall \sigma > 0 \quad |x - y| \leqslant \delta(\sigma) \implies |f(x) - f(y)| = \lim_{k \to \infty} |\hat{f}_{Q_k}(\hat{x}) - \hat{f}_{Q_k}(\hat{y})|
$$

$$
\leqslant 2^n |\hat{x} - \hat{y}|^\alpha \sigma \leqslant 2^n |x - y|^\alpha \sigma.
$$

If, on the other hand, $x_j \neq y_j$ for all j, we can choose a coordinate rectangular box $Q \subset U$ which has x and y as its opposite vertices on the largest diameter. We obtain

by applying (3.7) and (3.8) to the now continuous f, provided $|x - y| \leqslant \delta(\sigma)$:

$$
\begin{aligned}
|f(x) - f(y)| &\leqslant |f(x) - \hat{f}(\widehat{x})| + |\hat{f}(\widehat{x}) - \hat{f}(\widehat{y})| + |f(y) - \hat{f}(\widehat{y})| \\
&\leqslant 4|x_n - y_n|^\alpha \sigma + 2^n |\widehat{x} - \widehat{y}|^\alpha \sigma \leqslant (4 + 2^n)|x - y|^\alpha \sigma \\
&\leqslant 2^{n+1}|x - y|^\alpha \sigma.
\end{aligned}
$$

In both cases we can conclude with the desired bound, i.e. $[f]_{0,\alpha;U|\delta(\sigma)} \leqslant 2^{n+1}\sigma$. Note that the sequence Q_k in the first case and the box Q in the second case exist since U is assumed to be a coordinate box itself.

It remains to prove claim (3.7). Given $\sigma > 0$ assume that $|I_n| \leqslant \delta(\sigma)$ as required in (3.7). For any $\phi \in C_c^\infty(Q)$, consider a sequence $\tilde{\phi}_k \in C_c^\infty(Q)$, $\|\tilde{\phi}_k\|_0 \leqslant 1 + 2\|\phi\|_0$, converging strongly in L^1 to $\phi - \hat{\phi}(\widehat{x})$. As a consequence,

$$
\lim_{k \to \infty} \int_{I_n} \tilde{\phi}_k(x) \mathrm{d}x_n = 0 \quad \text{in } L^1(Q^{n-1}).
$$

Choose a $\varphi \in C_c^\infty(I_n)$ such that $\fint_{I_n} \varphi = 1$ and define

$$
\phi_k(x) = \tilde{\phi}_k(x) - \left(\fint_{I_n} \tilde{\phi}_k(\widehat{x}, s) \mathrm{d}s \right) \varphi(x_n).
$$

Then $\phi_k \in C_c^\infty(Q)$ converges in L^1 to $\phi - \hat{\phi}(\widehat{x})$ and $\hat{\phi}_k(\widehat{x}) = \fint_{I_n} \phi_k(x) \mathrm{d}x_n = 0$. Letting $\psi_k(\widehat{x}, t) = \int_{-\infty}^t \phi_k(\widehat{x}, s) \mathrm{d}s$ we obtain for all $\phi \in C_c^\infty(Q)$, $\psi_k \in C_c^\infty(Q)$. We hence obtain

$$
\left| \int_Q (f - \hat{f}(\widehat{x}))\phi \right| = \left| \int_Q f(\phi - \hat{\phi}(\widehat{x})) \right| = \lim_{k \to \infty} \left| \int_Q f\phi_k \right| = \lim_{k \to \infty} \left| \int_Q f \partial_n \psi_k \right|
$$

$$
= \lim_{k \to \infty} \left| T_n[\psi_k] \right|
$$

$$
\leqslant \liminf_{k \to \infty} \|\psi_k\|_{L^1}^\alpha \|\partial_n \psi_k\|_{L^1}^{1-\alpha} \sigma \leqslant \liminf_{k \to \infty} |I_n|^\alpha \|\partial_n \psi_k\|_{L^1}^\alpha \|\partial_n \psi_k\|_{L^1}^{1-\alpha} \sigma
$$

$$
= \lim_{k \to \infty} |I_n|^\alpha \left\| \phi_k - \fint_{I_n} \phi_k(x) \mathrm{d}x_n \right\|_{L^1} \sigma = |I_n|^\alpha \Big\| \phi
$$

$$
- \fint_{I_n} \phi(x) \mathrm{d}x_n \Big\|_{L^1(Q)} \sigma
$$

$$
\leqslant 2|I_n|^\alpha \|\phi\|_{L^1(Q)} \sigma,
$$

where we used the Poincaré inequality

$$
\|\psi_k\|_{L^1(Q)} \leqslant |I_n| \|\partial_n \psi_k\|_{L^1(Q)} \leqslant \delta(\sigma) \|\partial_n \psi_k\|_{L^1(Q)}
$$

on I_n. Once again, using the fact that L^∞ is the dual of L^1, and the density of $C_c^\infty(Q)$ in $L^1(Q)$, we conclude with (3.7). $\qquad \square$

4. A second fundamental form for $c^{1,\alpha}$ isometries

Let u$\in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an immersion with $2/3 \leqslant \alpha < 1$ and let $V \Subset \Omega$ be any smooth simply connected domain compactly contained in Ω, e.g. an open disk. We intend to show the following three statements, which we will formulate more precisely later on. We shall prove:

1. A weak notion of the second fundamental form $A_{ij} = -\partial_i u \cdot \partial_j \vec{n}$ makes sense for the immersion u on V. $A = [A_{ij}]$ is symmetric. This part can be carried out for $\alpha > 1/2$.

2. If u is isometric, then A is curl free, and is equal to $\nabla^2 v$ for a scalar function $v \in c^{1,\alpha}(V)$.

3. $\mathcal{D}et\,(D^2 v) = -1/2\mathrm{curl}^T\mathrm{curl}\,(\nabla v \otimes \nabla v) = 0$ in $\mathcal{D}'(V)$.

We choose an open smooth domain U such that $V \Subset U \Subset \Omega$. Let $\phi \in C_c^\infty(B(0, 1))$ be a nonnegative function with $\int \phi = 1$. For $\varepsilon < \mathrm{dist}(\overline{V}, \partial U)$ we define

$$\vec{n}_\varepsilon := \vec{n} * \phi_\varepsilon$$

to be the mollification of the unit normal

$$\vec{n} = \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|}$$

of immersion u. Note that by assumption $C_0 := \|\partial_1 u \times \partial_2 u\|_{0;U} > 0$ and $\vec{n} \in c^{0,\alpha}(\Omega, \mathbb{R}^3)$. Let $u_\varepsilon := u * \phi_\varepsilon : V \to \mathbb{R}^3$ be the standard mollification of u on V. If ε is small enough u_ε is an immersion since ∇u_ε converges uniformly to ∇u on V, and

$$|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon| \geqslant \frac{1}{2} C_0.$$

We define

$$N^\varepsilon := \frac{\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon}{|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon|}$$

to be the unit normal corresponding to u_ε. A direct application of lemma 3.1 yields:

$$\|\partial_i u_\varepsilon - \partial_i u\|_{0;V} \leqslant \|u\|_{1,\alpha} o(\varepsilon^\alpha),\ \|u_\varepsilon\|_{2;V} \leqslant \|u\|_{1,\alpha} o(\varepsilon^{\alpha-1}); \quad \|\vec{n}_\varepsilon - \vec{n}\|_{0;V}$$
$$\leqslant \|u\|_{1,\alpha}^2 o(\varepsilon^\alpha). \tag{4.1}$$

We also note that

$$\|N^\varepsilon - \vec{n}\|_{0;V} \leqslant \frac{1}{C_0}\|u\|_{1,\alpha}^2 o(\varepsilon^\alpha), \quad \|\nabla N^\varepsilon\|_{0;V} \leqslant \frac{1}{C_0}\|u\|_{1,\alpha}^2 o(\varepsilon^{\alpha-1}). \tag{4.2}$$

To see the first estimate we observe that

$$\|N^\varepsilon - \vec{n}\|_{0;V} \leqslant \left\| \frac{\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon}{|\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon|} - \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|} \right\|_{0;V}$$

$$\leqslant \frac{2}{C_0^2} \Big\| |\partial_1 u \times \partial_2 u|(\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon) - |\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon|(\partial_1 u \times \partial_2 u) \Big\|_{0;V}$$

$$\leqslant \frac{2}{C_0} \left(\Big\| \partial_1 u_\varepsilon \times \partial_2 u_\varepsilon - \partial_1 u \times \partial_2 u \Big\|_{0;V} \right.$$

$$\left. + \Big\| |\partial_1 u_\varepsilon \times \partial_2 u_\varepsilon| - |\partial_1 u \times \partial_2 u| \Big\|_{0;V} \right)$$

$$\leqslant \frac{4}{C_0} \Big\| \partial_1 u_\varepsilon \times \partial_2 u_\varepsilon - \partial_1 u \times \partial_2 u \Big\|_{0;V} \leqslant \frac{1}{C_0} \|u\|_{1,\alpha}^2 o(\varepsilon^\alpha),$$

where we used (4.1). The second estimate also follows from (4.1) and a direct calculation.

We finally denote by the symmetric matrix field $A^\varepsilon := [A_{ij}^\varepsilon]$ the second fundamental form associated with the immersion u_ε in the local coordinates, i.e.

$$A_{ij}^\varepsilon := \partial_{ij} u_\varepsilon \cdot N^\varepsilon = -\partial_i u_\varepsilon \cdot \partial_j N^\varepsilon, \tag{4.3}$$

which by (4.1) satisfies the obvious bound

$$\|A^\varepsilon\|_{0;V} \leqslant [u_\varepsilon]_{2;V} \leqslant \|u\|_{1,\alpha} o(\varepsilon^{\alpha-1}). \tag{4.4}$$

4.1. Definition of the second fundamental form A

We are ready now to prove our main claims about the existence of a second fundamental form for the immersion u and its properties. In what follows the constant $C = c(U, V, \|u\|_{1,\alpha;U}, C_0)$ might change but it has a universal upper bound only depending on the stated quantities.

PROPOSITION 4.1. *If $\alpha > 1/2$, then A_{ij}^ε converges in the sense of distributions to a distribution $A_{ij} \in \mathcal{D}'(V)$. More precisely, the convergence is in the dual of $W_0^{1,1}(V)$, and we have:*

$$\forall \psi \in C_c^\infty(V) \quad |(A_{ij}^\varepsilon - A_{ij})[\psi]| \leqslant C \left(o(\varepsilon^{2\alpha-1}) \|\psi\|_{L^1} + o(\varepsilon^\alpha) \|\partial_j \psi\|_{L^1} \right).$$

Moreover, for each i, the distribution $T = (A_{ij})_{j=1}^n$ satisfies the fine α-interpolation property on $W_0^{1,1}(V)$ according to definition 3.7.

Proof. Applying proposition 3.8 to the components $f := \partial_i u^m$, $h := \vec{n}^m$, for $m \in \{1, 2, 3\}$, we deduce that for a distribution $A_{ij} \in \mathcal{D}'(V)$,

$$B_{ij}^\varepsilon := -\partial_i u_\varepsilon \cdot \partial_j \vec{n}_\varepsilon \to A_{ij} := -\partial_i u \cdot \partial_j \vec{n} \quad \text{in } \mathcal{D}'(V),$$

and

$$|B_{ij}^\varepsilon[\psi] - A_{ij}[\psi]| \leqslant C\left(o(\varepsilon^{2\alpha-1}) \|\psi\|_{L^1} + o(\varepsilon^\alpha) \|\partial_j \psi\|_{L^1} \right).$$

On the other hand, we have for all $\psi \in W_0^{1,1}(V)$:

$$|A_{ij}^\varepsilon[\psi] - A_{ij}[\psi]| \leqslant |B_{ij}^\varepsilon[\psi] - A_{ij}[\psi]| + \left| \int_B \partial_i u_\varepsilon(x) \cdot (\partial_j \vec{n}_\varepsilon(x) - \partial_j N^\varepsilon(x)) \psi(x) \mathrm{d}x \right|$$

$$\leqslant C\left(o(\varepsilon^{2\alpha-1}) \|\psi\|_{L^1} + o(\varepsilon^\alpha) \|\partial_j \psi\|_{L^1} \right)$$

$$+ \left| \int_V \partial_{ij} u_\varepsilon \cdot (\vec{n}_\varepsilon - N^\varepsilon) \psi \mathrm{d}x \right| + \left| \int_V \partial_i u_\varepsilon \cdot (\vec{n}_\varepsilon - N^\varepsilon) \partial_j \psi \right|$$

$$\leqslant C\left(o(\varepsilon^{2\alpha-1}) \|\psi\|_{L^1} + o(\varepsilon^\alpha) \|\partial_j \psi\|_{L^1} \right),$$

where we used proposition 3.8, (4.1) and (4.2). The stated fine α-interpolation property follows as in proposition 3.8. □

DEFINITION 4.2. *For $1/2 < \alpha < 1$, we define the weak second fundamental form of an immersion $u \in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ on V as the distribution*

$$A : C_c^\infty(V) \to \mathbb{R}^{2\times2}_{sym}, \quad A[\psi] := [A_{ij}[\psi]]_{2\times2}.$$

REMARK 4.3. The definition of A can be uniquely extended to a distribution globally defined on Ω, i.e. belonging to $\mathcal{D}'(\Omega, \mathbb{R}^{2\times2}_{sym})$, but we will not need this fact.

4.2. *A* satisfies Gauss-Codazzi-Mainardi equations (in a weak sense)

From now on we assume that the immersion u is also isometric, i.e. the pull back-metric $g := (\nabla u)^T (\nabla u)$ is the Euclidean metric $\mathscr{E}_2$ on Ω. We consider $\mathscr{G}^\varepsilon := (\nabla u_\varepsilon)^T \nabla u_\varepsilon = u_\varepsilon^* \mathscr{E}_3$, the pull-back metric induced by u_ε, where $\mathscr{E}_3$ is the Euclidean metric of $\mathbb{R}^3$. The commutator estimate lemma 3.1-(iii) implies

$$\|\mathscr{G}^\varepsilon - (g * \phi_\varepsilon)\|_{1;V} = \|(\nabla u_\varepsilon)^T \nabla u_\varepsilon - ((\nabla u)^T \nabla u)_\varepsilon\|_{1;V} \leqslant \|u\|_{1,\alpha}^2 o(\varepsilon^{2\alpha-1}).$$

So, in view of the fact that $\mathscr{E}_2 * \phi_\varepsilon = \mathscr{E}_2$ on V we obtain:

$$\|\mathscr{G}^\varepsilon - \mathscr{E}_2\|_{1;V} \leqslant \|u\|_{1,\alpha}^2 o(\varepsilon^{2\alpha-1}). \tag{4.5}$$

Compare with [**13**, Proposition 1], which is stated under more general settings and where the mollifier needs to be symmetric.

PROPOSITION 4.4. *We have the uniform estimate*

$$\|\operatorname{curl} A^\varepsilon\|_{0;V} \leqslant Co(\varepsilon^{3\alpha-2}). \tag{4.6}$$

In particular, if $\alpha \geqslant 2/3$, then $\operatorname{curl} A = 0$ in $\mathcal{D}'(V)$.

Proof. We recall that the Christoffel symbols associated with a metric g are given by

$$\Gamma^i_{jk}(g) = \frac{1}{2}g^{im}(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}),$$

where the Einstein summation convention is used. Hence, in view of (4.5), we have

$$\Gamma^{i,\varepsilon}_{kj} := \Gamma^i_{jk}(\mathscr{G}^\varepsilon) \to 0$$

uniformly, if $\alpha \geqslant 1/2$, with the estimate

$$\|\Gamma^{i,\varepsilon}_{kj}\|_{0;V} \leqslant Co(\varepsilon^{2\alpha-1}). \tag{4.7}$$

Writing the Codazzi–Mainardi equations [**27**, Equation (2.1.6)] for the immersion u_ε we have

$$\partial_2 A^\varepsilon_{11} - \partial_1 A^\varepsilon_{12} = A^\varepsilon_{11}\Gamma^{1,\varepsilon}_{12} + A^\varepsilon_{12}(\Gamma^{2,\varepsilon}_{12} - \Gamma^{1,\varepsilon}_{11}) - A^\varepsilon_{22}\Gamma^{2,\varepsilon}_{11},$$
$$\partial_2 A^\varepsilon_{12} - \partial_1 A^\varepsilon_{22} = A^\varepsilon_{11}\Gamma^{1,\varepsilon}_{22} + A^\varepsilon_{12}(\Gamma^{2,\varepsilon}_{22} - \Gamma^{1,\varepsilon}_{12}) - A^\varepsilon_{22}\Gamma^{2,\varepsilon}_{12}.$$

Hence, by (4.4) and (4.7),

$$\|\partial_2 A^\varepsilon_{i1} - \partial_1 A^\varepsilon_{i2}\|_{0;V} \leqslant Co(\varepsilon^{\alpha-1})o(\varepsilon^{2\alpha-1}),$$

which completes the proof. $\qquad\square$

Before proceeding to the next step, we need to establish a further property of the sequence A^ε. Observe that if $u \in C^2(\Omega, \mathbb{R}^3)$ were an isometric immersion, then the determinant of its second fundamental form A would vanish, since, in this particular case, it would be equal to the Gaussian curvature of the Euclidean metric. In the case considered here, u does not enjoy the sufficient regularity for $\det A$ to be defined. However, we can show that for $\alpha \geqslant 1/2$, $\det A^\varepsilon$ converges in the sense of distributions to 0 as $\varepsilon \to 0$, that with a rate which will be crucial in our analysis. The following statement is a slight variant of [**13**, Proposition 7] recast for our purposes:

PROPOSITION 4.5. *Let A^ε be as defined in (4.3). Then for all $\psi \in C^\infty_c(V)$,*

$$\left|\int_V (\det A^\varepsilon)\psi\right| \leqslant C\|\psi\|_{W^{1,1}}o(\varepsilon^{2\alpha-1}).$$

Proof. In view of the formula for the $(0, 4)$-Riemann curvature tensor [**27**, Equation (2.1.2)]

$$R_{iljk} = g_{lm}(\partial_k \Gamma_{ij}^m - \partial_j \Gamma_{ik}^m + \Gamma_{ks}^m \Gamma_{ij}^s - \Gamma_{js}^m \Gamma_{ik}^s),$$

and by the Gauss equation [**27**, Equation (2.1.7)], (4.5) and (4.7) we have on V:

$$\begin{aligned}
\det A^\varepsilon = R_{1212}(\mathscr{G}^\varepsilon) &= \mathscr{G}_{1m}^\varepsilon(\partial_1 \Gamma_{22}^{m,\varepsilon} - \partial_2 \Gamma_{21}^{m,\varepsilon} + \Gamma_{1s}^{m,\varepsilon}\Gamma_{22}^{s,\varepsilon} - \Gamma_{2s}^{m,\varepsilon}\Gamma_{21}^{s,\varepsilon}) \\
&= \partial_1(\mathscr{G}_{1m}^\varepsilon \Gamma_{22}^{m,\varepsilon}) - \partial_2(\mathscr{G}_{1m}^\varepsilon \Gamma_{21}^{m,\varepsilon}) + o(\varepsilon^{2\alpha-1}) \\
&= 2\partial_{12}\mathscr{G}_{12}^\varepsilon - \partial_{11}\mathscr{G}_{22}^\varepsilon - \partial_{22}\mathscr{G}_{11}^\varepsilon + o(\varepsilon^{2\alpha-1}) \\
&= -\mathrm{curl}^T \mathrm{curl}\, \mathscr{G}^\varepsilon + o(\varepsilon^{2\alpha-1}).
\end{aligned}$$

The conclusion follows by an integration by parts involving the first term and applying (4.5). □

4.3. *A* as the Hessian of a scalar function

PROPOSITION 4.6. *Let $\alpha \geqslant 2/3$, $u \in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an isometric immersion and V be a smooth simply connected domain compactly contained in Ω. There exists $v \in c^{1,\alpha}(V) \cap W^{1,2}(V)$ such that A, the second fundamental form of u on V, equals $\nabla^2 v$ in the sense of distributions. Moreover*

$$\mathcal{D}et\, D^2 v = -\frac{1}{2}\mathrm{curl}^T\, \mathrm{curl}(\nabla v \otimes \nabla v) = 0 \ in\ \mathcal{D}'(V).$$

REMARK 4.7. In our setting, it can also be shown that $v \in c^{1,\alpha}(\overline{V})$. In view of remark 4.3, one can find such $v \in c^{1,\alpha}(\Omega)$ provided Ω is simply connected. We will not need these facts.

Proof. Let A^ε be as defined in (4.3) and let F^ε be the solution to the Neumann problem

$$\begin{cases} \Delta F^\varepsilon = \mathrm{div}\, A^\varepsilon & \text{in}\quad V \\ \partial_\nu F^\varepsilon = A^\varepsilon \cdot \nu & \text{on}\quad \partial V \end{cases}$$

Since $\mathrm{div}(\nabla F^\varepsilon - A^\varepsilon) = 0$, there exists a vector field E^ε such that $A^\varepsilon = \nabla F^\varepsilon + \nabla^\perp E^\varepsilon$. E^ε solves

$$\begin{cases} \Delta E^\varepsilon = \mathrm{curl}(A^\varepsilon - \nabla F^\varepsilon) = \mathrm{curl}\, A^\varepsilon & \text{in}\quad V \\ E^\varepsilon = const. & \text{on}\quad \partial V \end{cases}$$

Since A^ε is symmetric, we have $\partial_2(F_1^\varepsilon + E_2^\varepsilon) = \partial_1(F_2^\varepsilon - E_1^\varepsilon)$, and thus we derive the existence of a scalar function $v^{(\varepsilon)} \in C^\infty(V)$ satisfying

$$\partial_1 v^{(\varepsilon)} = F_1^\varepsilon + E_2^\varepsilon, \quad \partial_2 v^{(\varepsilon)} = F_2^\varepsilon - E_1^\varepsilon,$$

i.e. $\nabla v^{(\varepsilon)} = F^\varepsilon + (E_1^\varepsilon, -E_2^\varepsilon)$. As a consequence

$$A^\varepsilon = \nabla^2 v^{(\varepsilon)} - \nabla(E^\varepsilon)^\perp + \nabla^\perp E^\varepsilon.$$

Standard elliptic estimates imply that for any $p < \infty$

$$\|\nabla E^\varepsilon\|_{W^{1,p}(V)} \leqslant c(p,V)\|\operatorname{curl} A^\varepsilon\|_{0;V}. \tag{4.8}$$

As a consequence, fixing $p > 2$, and in view of (4.6), $\|A^\varepsilon - \nabla^2 v^{(\varepsilon)}\|_{0;V} \leqslant Co(\varepsilon^{3\alpha-2}) \to 0$ and therefore $\nabla^2 v^{(\varepsilon)}$ converges to A in the sense of distributions.

Now, for all $\varphi \in C_c^\infty(V)$, there exists a vector field Ψ in V, vanishing on ∂V, such that $\operatorname{div}\Psi = \varphi - \fint_V \varphi$, for which $\|\Psi\|_{W^{1,2}(V)} \leqslant c(V)\|\varphi\|_{L^2(V)}$ (see e.g. [1]). By adjusting the $\nabla v^{(\varepsilon)}$ so that it is of average 0 over V, and in view of proposition 4.1, we obtain that for all $\varepsilon, \varepsilon' \leqslant \delta$:

$$\left| \int_V (\nabla v^{(\varepsilon)} - \nabla v^{(\varepsilon')})\varphi \right| = \left| \int_V (\nabla v^{(\varepsilon)} - \nabla v^{(\varepsilon')})(\varphi - \fint_V \varphi) \right|$$

$$= \left| \int_V (\nabla v^{(\varepsilon)} - \nabla v^{(\varepsilon')})\operatorname{div}\Psi \right|$$

$$= \left| \int_V (\nabla^2 v^{(\varepsilon)} - \nabla^2 v^{(\varepsilon')})\Psi \right|$$

$$\leqslant \left| \int_V (A^\varepsilon - A^{\varepsilon'})\Psi \right| + \left| \int_V (\nabla^2 v^{(\varepsilon)} - A^\varepsilon)\Psi \right|$$

$$+ \left| \int_V (\nabla^2 v^{(\varepsilon')} - A^{\varepsilon'})\Psi \right|$$

$$\leqslant C\|\Psi\|_{W^{1,1}}o(\delta^{3\alpha-2}) \leqslant C\|\varphi\|_{L^2(V)}o(\delta^{3\alpha-2}).$$

By duality, we conclude that $\nabla v^{(\varepsilon)}$ converges strongly in $L^2(V)$ to a vector field F and that $\nabla F = A$. It is now straightforward to see that, if necessary by adjusting the $v^{(\varepsilon)}$ by constants, $v^{(\varepsilon)}$ converges strongly in $W^{1,2}$ to some $v \in W^{1,2}(V)$ and that $A = \nabla^2 v$.

To complete the proof, it is only necessary to show that (a) $\mathcal{D}et\, D^2 v = 0$ and (b) $v \in c^{1,\alpha}(V)$.

In order to show (a), we first observe that since $\nabla v^{(\varepsilon)} \to \nabla v$ in $L^2(V)$, for all $\psi \in C_c^\infty(V)$ we obtain:

$$-\frac{1}{2}\operatorname{curl}^T \operatorname{curl}(\nabla v \otimes \nabla v)[\psi] = \lim_{\varepsilon \to 0} -\frac{1}{2}\operatorname{curl}^T \operatorname{curl}(\nabla v^{(\varepsilon)} \otimes \nabla v^{(\varepsilon)})[\psi]$$

$$= \lim_{\varepsilon \to 0} \det\nabla^2 v^{(\varepsilon)}[\psi]$$

$$= \lim_{\varepsilon \to 0} \int_V \det(A^\varepsilon + \nabla(E^\varepsilon)^\perp - \nabla^\perp E^\varepsilon)\psi$$

$$= \lim_{\varepsilon \to 0} \int_V (\det A^\varepsilon) \psi$$

$$+ \lim_{\varepsilon \to 0} \sum_{\text{some indices } i,j,k,l} \int_V A^\varepsilon_{ij} (\partial_k E^\varepsilon_l) \psi$$

$$+ \lim_{\varepsilon \to 0} \int_V \det(\nabla(E^\varepsilon)^\perp - \nabla^\perp E^\varepsilon) \psi.$$

The third term obviously converges to 0, and so does the first term by proposition 4.5. For the second term, we have, using propositions 4.1, (4.6) and (4.8)

$$\left| \int_V A^\varepsilon_{ij}(\partial_k E^\varepsilon_l) \psi \right| \leqslant \left| (A^\varepsilon_{ij} - A_{ij})[(\partial_k E^\varepsilon_l)\psi] \right| + \left| A[(\partial_k E^\varepsilon_l)\psi] \right|$$

$$\leqslant C\|(\partial_k E^\varepsilon_l)\psi\|_{W^{1,1}} o(\varepsilon^{2\alpha-1}) + C\|(\partial_k E^\varepsilon_l)\psi\|_{W^{1,1}}$$

$$\leqslant C\|\psi\|_1 o(\varepsilon^{2\alpha-1}) o(\varepsilon^{3\alpha-2}) + C\|\psi\|_1 o(\varepsilon^{3\alpha-2}) \xrightarrow{\varepsilon \to 0} 0.$$

This completes the proof of (a).

It remains to prove (b) $v \in c^{1,\alpha}(V)$. Since $\nabla^2 v = A$ is symmetric in i, j, we have $\partial_{ji} v = \partial_{ij} v = A_{ij}$, as distributions. Proposition 4.1 implies that, for each i, $\nabla(\partial_i v)$ satisfies the fine α-interpolation inequality on $W^{1,1}_0(V)$. For any $x \in V$, we apply proposition 3.11 to a coordinate rectangular box containing x and compactly included in V to conclude that $\partial_i v \in c^{0,\alpha}(V)$ for $i = 1$, 2, which yields $v \in c^{1,\alpha}(V)$. $\qquad\square$

5. Developability: a proof of theorem 1.3

Let $u \in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an isometric immersion for $2/3 \leqslant \alpha < 1$. In order to prove our main theorem, and in view of proposition 2.1, it is sufficient to show that ∇u satisfies condition (c). We first fix an open disk V containing x and compactly contained in Ω and note the existence of the function $v \in c^{1,\alpha}(V) \subset c^{1,2/3}(V)$ as defined in proposition 4.6. We apply the key developability result theorem 1.5 to v to obtain that ∇v satisfies any of the equivalent conditions of proposition 2.1 in V. The developability of u is a consequence of corollary 2.2 and proposition 5.1 below. The second conclusion follows from corollary 3.6.

PROPOSITION 5.1. *Let $1/2 < \alpha < 1$ and $u \in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an immersion, with the second fundamental form A defined as in proposition 4.1 on $V \Subset \Omega$. Assume moreover that $A = \nabla^2 v$, where $v \in C^1(V)$. Then $u|_V$ is developable if and only if v is developable.*

Proof. We consider the following identity [**27**, Equation (2.1.3)] (also known as the Gauss equation in the literature), which is valid for the mollified sequence of smooth immersions u_ε:

$$\partial_{ij} u_\varepsilon = \Gamma^{k,\varepsilon}_{ij} \partial_k u_\varepsilon + A^\varepsilon_{ij} N^\varepsilon, \tag{5.1}$$

where the standard Einstein summation convention is used.

Let $B \subset V$ be a given disk, $\psi \in C_c^\infty(B)$ and Lipschitz unit vector field $\vec{\eta} : B \to \mathbb{R}^2$. We denote the coefficients of u and N^ε respectively by u^m and $N^{\varepsilon,m}$ and calculate for $m = 1, 2, 3$,

$$
\begin{aligned}
\int_B \operatorname{div}(\psi\vec{\eta})(\nabla u^m) &= \lim_{\varepsilon \to 0} \int_B \operatorname{div}(\psi\vec{\eta})(\nabla u_\varepsilon^m) = \lim_{\varepsilon \to 0} - \int_B \psi\vec{\eta} \cdot \nabla^2 u_\varepsilon^m \\
&= \lim_{\varepsilon \to 0} - \int_B \psi\vec{\eta} \cdot \left(\Gamma^{k,\varepsilon} \partial_k u_\varepsilon^m + A^\varepsilon N^{\varepsilon,m} \right) \\
&= \lim_{\varepsilon \to 0} - \int_B \psi\vec{\eta} \cdot (A^\varepsilon N^{\varepsilon,m}),
\end{aligned}
\tag{5.2}
$$

where we used (4.7). We have

$$
\int_B \psi\vec{\eta} \cdot \left(A^\varepsilon N^{\varepsilon,m} \right) = A^\varepsilon[N^{\varepsilon,m}\psi\vec{\eta}] = (A^\varepsilon - A)[N^{\varepsilon,m}\psi\vec{\eta}] + A[N^{\varepsilon,m}\psi\vec{\eta}].
$$

By proposition 4.1 we have:

$$
\begin{aligned}
\left| (A^\varepsilon - A)[N^{\varepsilon,m}\psi\vec{\eta}] \right| &\leqslant C \left(\|N^{\varepsilon,m}\psi\vec{\eta}\|_{L^1} o(\varepsilon^{2\alpha-1}) + \|\nabla(N^{\varepsilon,m}\psi\vec{\eta})\|_{L^1} o(\varepsilon^\alpha) \right) \\
&\leqslant C \|\psi\vec{\eta}\|_{L^1} o(\varepsilon^{2\alpha-1}) + \|\nabla(\psi\vec{\eta})\|_{L^1} o(\varepsilon^\alpha) \to 0 \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}
$$

where we used (4.2). Note also that in view of $A = \nabla^2 v$, we have for all ε, in view of the fact that $N^{\varepsilon,m}\psi \in C_c^\infty(B)$:

$$
A[N^{\varepsilon,m}\psi\vec{\eta}] = - \int_B \operatorname{div}(N^{\varepsilon,m}\psi\vec{\eta})\nabla v.
$$

Therefore, (5.2) implies

$$
\int_B \operatorname{div}(\psi\vec{\eta})\nabla u^m = \lim_{\varepsilon \to 0} \int_B \operatorname{div}(N^{\varepsilon,m}\psi\vec{\eta})\nabla v.
\tag{5.3}
$$

If v is developable, we apply corollary 2.10 to ∇v, and choose accordingly the open disk B around $x \in V$ and the Lipschitz unit vector field $\vec{\eta} : B \to \mathbb{R}^2$. Therefore,

$$
\forall y \in B \quad \vec{\eta}(y) = \vec{\eta}(y + s\vec{\eta}(y)) \text{ and } \nabla v(y) = \nabla v(y + s\vec{\eta}(y)) \text{ for all } s \text{ for which } y
$$
$$
+ s\vec{\eta}(y) \in B.
\tag{5.4}
$$

We apply lemma 2.11 to $f = \nabla v$ and $B = B_x$ to obtain for any $\psi \in C_c^\infty(B)$:

$$
\int_B \operatorname{div}(N^{\varepsilon,m}\psi\vec{\eta})\nabla v = 0, \quad m = 1, 2, 3,
$$

where we used the fact that $N^{\varepsilon,m}\psi \in C_c^\infty(B)$. By (5.3) we conclude that

$$
\int_B \operatorname{div}(\psi\vec{\eta})\nabla u = 0.
$$

Once again applying lemma 2.11 and (5.4) implies that the Jacobian derivative ∇u of the isometric immersion u is constant along the segments generated by the vector

field $\vec{\eta}$ in B. Therefore, ∇u satisfies condition (c) of proposition 2.1, which implies the developability of $u|_V$. As already mentioned, this part of the proof concludes the proof of theorem 1.3 for $u \in c^{1,\alpha}$, $2/3 \leqslant \alpha < 1$.

To finish the proof of proposition 5.1 we need to prove the converse statement. If, on the other hand, $u|_V$ is assumed to be developable, we proceed in a similar manner. According to corollary 2.10 and lemma 2.11, for all $x \in V$ there exist a disk $B \subset V$ centred at x and a unit Lipschitz vector field $\vec{\eta} : B \to \mathbb{R}^2$ such that for all $\psi \in C_c^\infty(B)$:

$$\forall y \in B \quad \vec{\eta}(y) = \vec{\eta}(y + s\vec{\eta}(y)) \text{ and } \int_B \operatorname{div}(N^{\varepsilon,m}\psi\vec{\eta})\nabla u^m = 0, m = 1, 2, 3.$$

Hence, using the fact that $A_{ij}^\varepsilon = \partial_{ij} u_\varepsilon \cdot N^\varepsilon$ we deduce for all $\psi \in C_c^\infty(B)$:

$$\int_B \operatorname{div}(\psi\vec{\eta})(\nabla v) = -A[\psi\vec{\eta}] = \lim_{\varepsilon \to 0} A^\varepsilon[\psi\vec{\eta}] = \lim_{\varepsilon \to 0} \int_B \sum_{m=1}^3 (N^{\varepsilon,m}\nabla^2 u_\varepsilon^m)\psi\vec{\eta}$$

$$= \lim_{\varepsilon \to 0} -\int_B \sum_{m=1}^3 \operatorname{div}(N^{\varepsilon,m}\psi\vec{\eta})\nabla u_\varepsilon^m$$

$$= \lim_{\varepsilon \to 0} -\int_B \sum_{m=1}^3 \operatorname{div}(N^{\varepsilon,m}\psi\vec{\eta})(\nabla u_\varepsilon^m - \nabla u^m) = 0,$$

where we used (4.1) and (4.2) to obtain a convergence of order $o(\varepsilon^{2\alpha-1})$ in the last line. Following the same line of argument as before v is hence developable in view of lemma 2.11. □

Acknowledgements

Appendix A. Developabiltiy of each component can be shown independently

PROPOSITION A.1. *Assume $\Omega \subset \mathbb{R}^2$ is an arbitrary domain, $2/3 \leqslant \alpha < 1$ and let $u \in c^{1,\alpha}(\Omega, \mathbb{R}^3)$ be an isometric immersion. Then for each $m = 1, 2, 3$, the component u^m is developable in Ω.*

REMARK A.2. Obviously this statement does not guarantee that the constancy lines or regions of ∇u^m are the same for the three components. If this fact is independently shown (e.g. through geometric arguments) theorem 1.3 will also be proved through this approach.

Proof. In view of thereom 1.5, it suffices to show that $\mathcal{D}et\, D^2 u^m = 0$ for each m. Once again, we consider the mollifications $u_\varepsilon = u * \phi_\varepsilon$ defined on a suitably chosen domain $V \Subset \Omega$ and use (5.1) to calculate for m=1,2,3 and $\psi \in C_c^\infty(V)$:

$$
\begin{aligned}
-\frac{1}{2}\mathrm{curl}^T \mathrm{curl}(\nabla u^m \otimes \nabla u^m)[\psi] &= \lim_{\varepsilon \to 0} -\frac{1}{2}\mathrm{curl}^T \mathrm{curl}(\nabla u_\varepsilon^m \otimes \nabla u_\varepsilon^m)[\psi] \\
&= \lim_{\varepsilon \to 0} \det \nabla^2 u_\varepsilon^m[\psi] \\
&= \lim_{\varepsilon \to 0} \int_\Omega \det(\Gamma^{k,\varepsilon}\partial_k u_\varepsilon^m + A^\varepsilon N^{\varepsilon,m})\psi \\
&= \lim_{\varepsilon \to 0} \int_\Omega \det(A^\varepsilon)(N^{\varepsilon,m})^2 \psi \\
&\quad + \lim_{\varepsilon \to 0} \int_\Omega \det(\Gamma^{k,\varepsilon}\partial_k u_\varepsilon^m)\psi \\
&\quad + \lim_{\varepsilon \to 0} \sum_{\substack{\text{some indices } i,j,r,s}} \int_\Omega A_{ij}^\varepsilon N^{\varepsilon,m}(\Gamma_{rs}^{k,\varepsilon}\partial_k u_\varepsilon^m)\psi.
\end{aligned}
$$

The second term obviously converges to 0 in view of (4.7). For the third term we use (4.1), (4.4) and (4.7) to prove:

$$
\left| \int_\Omega A_{ij}^\varepsilon N^{\varepsilon,m}(\Gamma_{rs}^{k,\varepsilon}\partial_k u_\varepsilon^l)\psi \right| \leqslant C\|A^\varepsilon\|_{0;V}\|\Gamma^{k,\varepsilon}\|_{0;V}\|\psi\|_{L^1} \leqslant C\|\psi\|_{L^1} o(\varepsilon^{3\alpha-2}) \to 0.
$$

Finally, the vanishing of the first term follows from Propostion 4.5 and (4.2) since

$$
\begin{aligned}
\left| \int_\Omega \det(A^\varepsilon)(N^{\varepsilon,m})^2 \psi \right| &\leqslant C\|(N^{\varepsilon,m})^2 \psi\|_{W^{1,1}} o(\varepsilon^{2\alpha-1}) \\
&\leqslant C\left(\|\psi\|_{W^{1,1}} o(\varepsilon^{2\alpha-1}) + \|\nabla N^\varepsilon\|_{0;V}\|\psi\|_{L^1} o(\varepsilon^{2\alpha-1}) \right) \\
&\leqslant C\|\psi\|_{W^{1,1}} o(\varepsilon^{3\alpha-2}) \to 0. \qquad \square
\end{aligned}
$$

Appendix B. Little Hölder spaces and modulus of continuity

Here we will prove two auxiliary statements regarding little Hölder spaces.

LEMMA B.1. *Let $V \subset U \subset \mathbb{R}^n$ be open sets such that $\mathrm{dist}(\overline{V}, \partial U) > 0$ and $f : U \to \mathbb{R}$. If for $0 < \alpha < 1$, $[f]_{0,\alpha;U|r}$ is $o(1)$ as a function of r, then $\|f_\varepsilon - f\|_{0,\alpha;V} \to 0$ as $\varepsilon \to 0$.*

Proof. Let $\varepsilon < \mathrm{dist}(\overline{V}, \partial U)$. We first note that by lemma 3.1-(i) we have

$$\|f_\varepsilon - f\|_{0;V} \leqslant c[f]_{0,\alpha;U|\varepsilon}\varepsilon^\alpha. \tag{B.1}$$

We therefore estimate

$$
\begin{aligned}
[f_\varepsilon - f]_{0,\alpha;V} \quad &\leqslant [f_\varepsilon - f]_{0,\alpha;V|\varepsilon} + \sup_{\substack{x,y \in V \\ \varepsilon < |x-y|}} \frac{|f_\varepsilon(x) - f(x) + f_\varepsilon(y) - f(y)|}{|x-y|^\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant [f_\varepsilon - f]_{0,\alpha;V|\varepsilon} + c\|f_\varepsilon - f\|_{0;V}\varepsilon^{-\alpha} \\
&\leqslant [f_\varepsilon]_{0,\alpha;V|\varepsilon} + [f]_{0,\alpha;V|\varepsilon} + c[f]_{0,\alpha;U|\varepsilon} \\
&\leqslant [f_\varepsilon]_{0,\alpha;V|\varepsilon} + c[f]_{0,\alpha;U|\varepsilon}.
\end{aligned}
$$

We now observe that

$$
\begin{aligned}
[f_\varepsilon]_{0,\alpha;V|\varepsilon} = \sup_{\substack{x,y \in V \\ 0 < |x-y| \leqslant \varepsilon}} \frac{\left|\int_{\mathbb{R}^n} (f(x-z) - f(y-z))\phi_\varepsilon(z)\,dz\right|}{|x-y|^\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\leqslant \sup_{\substack{x,y \in V \\ 0 < |x-y| \leqslant \varepsilon}} \int_{\mathbb{R}^n} \frac{|f(x-z) - f(y-z)|}{|x-y|^\alpha}|\phi_\varepsilon(z)|\,dz
\end{aligned}
$$

$$
\leqslant c[f]_{0,\alpha;U|\varepsilon}.
$$

Combining the latter two estimates we hence obtain

$$[f_\varepsilon - f]_{0,\alpha;V} \leqslant c[f]_{0,\alpha;U|\varepsilon}.$$

In view of (B.1) and the main assumption, we conclude with the desired approximation of f by $f_\varepsilon \in C^\infty(\overline{V})$ in $C^{0,\alpha}$. □

PROPOSITION B.2. *Let $U \subset \mathbb{R}^n$ be a bounded weakly Lipschitz domain and let $f : U \to \mathbb{R}$ be uniformly continuous. Then f admits an extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ with $\omega_{\tilde{f};\mathbb{R}^n} \leqslant c(U)\omega_{f;U}$.*

Proof. U is a bounded weakly Lipschitz domain, i.e. by the standard definition, there is a covering of $\overline{U}$ of open sets U_i and charts $\psi_i : U_i \to \mathbb{R}^n$, $0 \leqslant i \leqslant N$ such that $U_0 \subset U$ and for $0 \leqslant i \leqslant N$, ψ_i is a bilipschitz diffeomorphism between U_i and an open cylinder $V_i := B_i \times (-1, 1) \subset \mathbb{R}^n$ such that $\psi_i(U \cap U_i) = V_i^+ := B_i \times (0, 1)$ and $\psi_i(\partial U \cap U_i) = B_i \times \{0\}$, where $B_i \subset \mathbb{R}^{n-1}$ is the unit ball.

For each $1 \leqslant i \leqslant N$, consider the function $h_i := f \circ \psi^{-1} : V_i^+ \to \mathbb{R}$. We can estimate the modulus of continuity of each h_i from above by the modulus of continuity

of f using the maximum bilipschitz constant of the ψ_i, which depends on U:

$$\omega_i := \omega_{h_i;V_i^+} \leqslant c(U)\omega_{f;U}.$$

Note that ω_i is necessarily nondecreasing by definition. Since each V_i^+ is convex, ω_i is also subadditive, i.e. for all $r_1, r_2 \geqslant 0$,

$$\omega_i(r_1 + r_2) \leqslant \omega_i(r_1) + \omega_i(r_2).$$

Moreover the uniform continuity of f implies that

$$\lim_{r \to 0} \omega_i(r) = 0.$$

A direct application of [**57**, Theorem 13.16] implies that h_i admits an extension $\tilde{h}_i$, given by

$$\tilde{h}_i(x) := \sup\{h_i(y) - \omega_i(|x - y|); \ y \in V_i^+\},$$

to the whole $\mathbb{R}^n$ also satisfying

$$\omega_{\tilde{h}_i;\mathbb{R}^n} \leqslant \omega_i \leqslant c(U)\omega_{f;U}.$$

Let $\{\theta_i\}_{0 \leqslant i \leqslant N}$ be a partition of unity corresponding to the covering U_i, i.e.

$$\forall \, 0 \leqslant i \leqslant N \quad \theta_i \in C_c^\infty(U_i, [0,1]) \quad \text{and} \quad \sum_{i=0}^{N} \theta_i = 1.$$

We extend $\theta_0 f : U_0 \to \mathbb{R}$ to $\tilde{f}_0 : \mathbb{R}^n \to \mathbb{R}$, and $\theta_i(\tilde{h}_i \circ \psi_i) : U_i \to \mathbb{R}$ to $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ by the trivial zero extension and write

$$\tilde{f} := \sum_{i=0}^{N} \tilde{f}_i.$$

The conclusion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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