Let $V_n = \{a, b, \ldots\}$ denote a vector space of dimension n over F with a symmetric bilinear form (x,y). If (a,a) = 0, the vector a is called isotropic.

If p = 2 and $n \geqslant 2$, V_n will contain two linearly independent vectors b and c. We may assume they are non-isotropic. The equation $\xi^2 = (b,b)/(c,c)$ has a solution $\xi \in F$. It follows that $(b + \xi c,\ b + \xi c) = (b,b) + 2\xi \cdot (b,c) + \xi^2 \cdot (c,c) =$ $(b,b) + \xi^2 \cdot (c,c) = 0.$

From now on let p > 2, $n \geqslant 3$. For every vector a let M_a denote the set of the norms $(\lambda a, \lambda a) = \lambda^2 (a,a)$ with $\lambda \neq 0$. Thus either a is isotropic or $M_a = G$ or $M_a = \overline{G}$.

We choose any three mutually orthogonal vectors $\neq 0$. if none of them is isotropic, two of them, say b and c satisfy $M_b = M_c$. We may assume (b,b) = (c,c). Thus

$$(b + \xi c,\ b + \xi c) = (b,b) + 2\xi \cdot (b,c) + \xi^2 \cdot (c,c)$$

$$= (b,b) + 2\xi \cdot 0 + \xi^2 \cdot (b,b) = (1 + \xi^2)(b,b).$$

Case (i): $-1 \in G$. Then let ξ be a solution of $1 + \xi^2 = 0$. The vector $b + \xi c$ will be isotropic.

Case (ii): $-1 \in \overline{G}$. By (1) there is a ξ such that $1 + \xi^2 \in \overline{G}$. Thus there is a vector d such that $M_b \neq M_d$.

Since $n \geqslant 3$, there is a vector $e \neq 0$ such that (e,b) = (e,d) = 0. Since M_e must be distinct from either M_b or M_d, we have found two vectors, say e and f such that (e,f) = 0, $M_e \neq M_f$. We may assume $1 \in M_e$, $-1 \in M_f$ and hence (e,e) = 1, (f,f) = -1. This yields (e + f, e + f) = (e,e) + (f,f) = 0.

NOTES

ON THE DISCRIMINANTS OF A BILINEAR FORM

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Let E denote a vector space of dimension n over a field of characteristic $\neq 2$. In E a symmetric bilinear form f(x,y) is given. Define E_f^* as the subspace of those vectors x for which f(x,y) = 0 for all $y \in E$. Thus rank f = n - dim E_f^*. Furthermore, define ind f = maximum dimension of a subspace in which

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f vanishes identically (cf. Jonathan Wild, Can. Math. Bull. 1(1958), 180). As every such subspace contains E_f^*, we have ind f $\geq$ dim E_f^*.

In the following let x_o be fixed; $f(x_o, x_o) \neq 0$. Let V denote the subspace of all x such that $f(x, x_o) = 0$. Thus $x_o \notin V$ and dim V = n - 1. Through

$$x \to z = f(x_o, x_o).x - f(x_o, x).x_o$$

E is mapped linearly onto V (The vector $z/f(x_o, x_o)$ is the projection of x into V parallel to x_o). The discriminant at x_o of f is the symmetric form

$$(1) \qquad g(x, y) = f(x_o, x_o).f(x, y) - f(x_o, x).f(x_o, y)$$

$$= f(f(x_o, x_o).x - f(x_o, x).x_o, y) = f(z, y).$$

It has recently been studied over the real field by Schwerdtfeger and Scherk (same J., 175-179 and 181-182). We wish to comment on its rank and index.

By (1), $g(x, y) = 0$ for given x and all y if and only if $z \in E_f^*$, i.e. if x lies in the space spanned by E_f^* and x_o. Thus

$$E_g^* = E_f^* + x_o .$$

In particular rank g = rank f - 1.

Obviously

$$(2) \qquad g(x_o, y) = 0 \qquad \text{for every } y$$

and

$$(3) \qquad g(x, y) = f(x_o, x_o).f(x, y) \qquad \text{if } x \in V.$$

Let W denote a subspace of maximal dimension in which f vanishes identically. By (3), g will vanish in $W \cap V$. Hence, by (2), g will vanish identically in the subspace spanned by x_o and $W \cap V$. This implies

$$(4) \qquad \text{ind } g \geq \text{ind } f \qquad \text{always},$$

$$(5) \qquad \text{ind } g \geq \text{ind } f + 1 \quad \text{if there is a } W \subset V.$$

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Conversely, let U be a subspace of maximal dimension in which g vanishes identically. By (2), g will also vanish in $U + x_0$. As U was to be maximal, we have $U = U + x_0$ or $x_0 \in U$. Hence $U \not\subset V$. By (3), f vanishes in $U \cap V$. Hence

$$\text{ind } f \geq \dim(U \cap V) = \dim U - 1 = \text{ind } g - 1$$

and (5) implies

(i) $\quad \text{ind } g = \text{ind } f + 1 \quad$ if there is a $W \subset V$.

If there is no subspace $W \subset V$, then $U \cap V$ cannot be a subspace W of maximal dimension in which f vanishes. This maximal dimension must therefore be greater than $\dim(U \cap V)$. Thus ind f > ind g - 1 or ind f $\geq$ ind g. Hence by (4)

(ii) $\quad \text{ind } g = \text{ind } f \quad$ if there is no $W \subset V$.

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