Canad. J. Math. Vol. **57** (5), 2005 pp. 983–1011

A Symmetric Imprimitivity Theorem for Commuting Proper Actions

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Abstract. We prove a symmetric imprimitivity theorem for commuting proper actions of locally compact groups H and K on a C^*-algebra.

1 Introduction

Until recently, the various symmetric imprimitivity theorems in the literature have all been associated with commuting free and proper actions of two locally compact groups H and K on the left and right of a locally compact space P. The original theorem, due to Green and Rieffel [23], says that the crossed products $C_0(P/H) \rtimes K$ and $C_0(K\backslash P) \rtimes H$ are Morita equivalent; the most powerful generalizations give Morita equivalences between crossed products of induced C^*-algebras [14, 20] or crossed products of $C_0(P)$-algebras [12].

There are two ways to prove a symmetric imprimitivity theorem. The first, used in [23, 20], is to build module actions and inner products on spaces of compactly supported functions and then complete to get a bimodule over the (complete) crossed products; for the Green–Rieffel theorem, the bimodule $Z = Z_K^H$ is a completion of $C_c(P)$. The second, used in [2, 14], starts from the one-sided equivalence involving just one group, and bootstraps up by taking crossed products and tensor products; for the Green–Rieffel theorem, we start with $X := Z_{\{e\}}^H$ and $Y := Z_K^{\{e\}}$, and the tensor product

$$(X \rtimes K) \otimes_{C_0(P)\rtimes(K\times H)} (Y \rtimes H)$$

implements the equivalence. In applications, the first bimodule Z is more convenient for direct calculations, and the tensor-product bimodule is useful when we want to bootstrap results from the one-sided case. It is known that the two bimodules are in fact isomorphic as imprimitivity bimodules [12], and this isomorphism is useful, for example, in settling questions of amenability [11, 12].

Both constructions, though, ultimately use imprimitivity bimodules constructed from algebras of functions, and the algebraic structure in these bimodules was found by pretty much *ad hoc* methods. In [24], Rieffel proposed an alternative, more systematic approach for building imprimitivity bimodules, based on abstracting the concepts of proper and free actions to the noncommutative setting. He described a class of proper saturated actions of a locally compact group G on a non-commutative

Received by the editors February 1, 2002; revised November 3, 2003.

This research was supported by grants from the Australian Research Council, the National Science Foundation, Dartmouth College, the University of Denver, and the University of Newcastle.

AMS subject classification: Primary: 46L05; secondary: 46L08, 46L55.

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https://doi.org/10.4153/CJM-2005-038-2 Published online by Cambridge University Press

C^*-algebra C for which there is a Morita equivalence between the (reduced) crossed product $C \rtimes_r G$ and a generalized fixed-point algebra C^G of C; this one-sided equivalence is implemented by a bimodule which is constructed by completing a dense subalgebra of C in a very particular way.

Pask and Raeburn have recently proved a symmetric imprimitivity theorem for commuting actions on the Cuntz–Krieger algebras of directed graphs [19, Theorem 2.1], which appears to be quite independent of the machinery developed in [12, 20, 23]. However, the actions considered in [19] are proper in Rieffel's sense, and [19, Theorem 2.1] can be formulated as a Morita equivalence of crossed products of generalized fixed-point algebras. It is therefore tempting to look for a symmetric imprimitivity theorem for commuting proper actions on a C^*-algebra, and the purpose of the present paper is to formulate and prove such a theorem. Thus we consider commuting actions $\tau\colon H \to \operatorname{Aut} C$ and $\sigma\colon K \to \operatorname{Aut} C$ of two groups on the same C^*-algebra C, and aim to prove that if both actions are proper and saturated in Rieffel's sense, then we have a Morita equivalence between the crossed products $C^\tau \rtimes_{\sigma,r} K$ and $C^\sigma \rtimes_{\tau,r} H$. We want a complete theory: we want a tensor-product bimodule which is good for bootstrapping arguments, a bimodule which is obtained by completing a dense subalgebra of C, and an isomorphism between these bimodules. If this new symmetric imprimitivity theorem requires extra hypotheses, we want to know that the hypotheses are satisfied in the key examples.

The first step is relatively straightforward. Under some mild continuity hypotheses which ensure that the various crossed products make sense, we can start with two applications of Rieffel's theorem from [24] and use the usual bootstrap arguments to obtain a tensor-product bimodule (Proposition 2.2). The second and third steps are achieved in §3 using the results of our earlier paper [13]. We show that the natural action of K on the bimodule implementing the equivalence between C^τ and $C \rtimes_{\tau,r} H$ is proper and saturated in the sense of [13], and identify the generalized fixed-point algebra $(C \rtimes_{\tau,r} H)^K$ with $C^\sigma \rtimes_{\tau,r} H$, so that the main theorems of [13] give the desired Morita equivalence between $C^\tau \rtimes_{\sigma,r} K$ and $C^\sigma \rtimes_{\tau,r} H$ and the isomorphism with the tensor-product bimodule (Theorem 3.7 and Corollary 3.8). The proof of Theorem 3.7 raises substantial technical problems involving vector-valued integrals whose treatment we defer to two appendices.

Theorem 3.7 requires substantial hypotheses of the sort needed by Rieffel in [24]. In the final section, we show that the hypotheses in §3 are often automatically satisfied. More specifically, we show that if there is an underlying free and proper space ${}_K P_H$ such that $C_0(P)$ maps bi-equivariantly into $M(C)$, then Theorem 3.7 applies. Although it is a little against the spirit of Rieffel's theory to assume the existence of an underlying proper action on a space, it is a fact that in all main examples of proper actions of G there is such an underlying space ${}_G P$. We discuss this in our concluding Remark 4.5, and also speculate on possible implications for nonabelian duality.

2 The Tensor-Product Imprimitivity Bimodule

Let $\tau\colon H \to \operatorname{Aut} C$ and $\sigma\colon K \to \operatorname{Aut} C$ be commuting actions of locally compact groups on a C^*-algebra C. We assume that both τ and σ are proper and saturated with respect to the same dense invariant $*$-subalgebra C_0 of C in the sense of [24].

https://doi.org/10.4153/CJM-2005-038-2 Published online by Cambridge University Press

Applying [24, Corollary 1.7] to τ gives a $C \rtimes_{\tau,r} H - C^\tau$ imprimitivity bimodule $\overline{C_0}$, where C^τ denotes the generalized fixed-point algebra. Recall that, by definition, C^τ is the closure of

$$D_0 := \operatorname{span}\{\, \langle b\,, c\rangle_{C^\tau} : b, c \in C_0 \,\} \subset M(C)^\tau,$$

where each $\langle b\,, c\rangle_{C^\tau}$ is a uniquely determined element of $M(C)^\tau$ such that for every $a \in C_0$

$$\int_H a\tau_s(b^*c)\,ds = a\langle b\,, c\rangle_{C^\tau}.$$

Since τ is saturated,

$$E_0 := \operatorname{span}\{\, s \mapsto \Delta_H(s)^{-1/2} b\tau_s(c^*) : b, c \in C_0 \,\}$$

is dense in $C \rtimes_{\tau,r} H$.

Throughout we will denote by X a module isomorphic to the dual of ${}_{C\rtimes_{\tau,r}H}(\overline{C_0})_{C^\tau}$, so that X is a $C^\tau - (C \rtimes_{\tau,r} H)$-imprimitivity bimodule. Formally, X is obtained by completing $X_0 := C_0$, where X_0 is the left D_0-module with $d \cdot x := dx$ and ${}_{C^\tau}\langle x\,, y\rangle = \langle x^*\,, y^*\rangle_{C^\tau}$; one can easily check that the map $\varphi\colon \flat(c) \mapsto c^*$ is an isomorphism of the dual of ${}_{C^\tau}(\overline{C_0})$ onto ${}_{C^\tau}\overline{X_0}$. We use the same isomorphism φ to work out what the formula for the $C \rtimes_{\tau,r} H$-valued inner product on X_0 should be:

$$\begin{aligned}
\langle x\,, y\rangle_{C\rtimes_{\tau,r}H}(s) &= \big\langle (x^*)^*\,, (y^*)^*\big\rangle_{C\rtimes_{\tau,r}H}(s)\\
&= \big\langle \varphi(\flat(x^*))\,, \varphi(\flat(y^*))\big\rangle_{C\rtimes_{\tau,r}H}(s)\\
&= \big\langle \flat(x^*)\,, \flat(y^*)\big\rangle_{C\rtimes_{\tau,r}H}(s)\\
&= {}_{C\rtimes_{\tau,r}H}\big\langle x^*\,, y^*\big\rangle(s) = \Delta_H(s)^{-1/2} x^*\tau_s(y).
\end{aligned}$$

Now $X_0 = C_0$ completes to give a Morita equivalence between C^τ and $C \rtimes_{\tau,r} H$, and for $x, y, z \in X_0$, $d \in D_0$ and $e \in E_0 \subset L^1(H, C)$ the actions and inner products are given by

$$d \cdot x = dx \text{ is multiplication in } M(C); \tag{2.1}$$

$$x \cdot e = \int_H \tau_s^{-1}(xe(s))\Delta_H(s)^{-1/2}\,ds; \tag{2.2}$$

$${}_{C^\tau}\langle x\,, v\rangle \text{ is characterized by } {}_{C^\tau}\langle x\,, v\rangle \cdot z = \int_H \tau_s(xv^*)z\,ds; \tag{2.3}$$

$$\langle x\,, v\rangle_{C\rtimes_{\tau,r}H}(s) = \Delta_H(s)^{-1/2} x^*\tau_s(v). \tag{2.4}$$

The first step to obtaining the tensor-product version of the symmetric imprimitivity theorem is to show that the natural extension $\bar\sigma$ of σ to $M(C)$ leaves C^τ invari-

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ant: if $x, v, w \in X_0$ then, using (2.3), we have

$$\begin{aligned}{}_{C^\tau}\big\langle\, \sigma_t(x)\,,\, \sigma_t(v)\big\rangle \cdot w &= \int_H \tau_s(\sigma_t(x)\sigma_t(v)^*)w\,ds = \int_H \sigma_t(\tau_s(xv^*))w\,ds \\ &= \sigma_t\Big(\int_H \tau_s(xv^*)\sigma_t^{-1}(w)\,ds\Big) = \sigma_t\big({}_{C^\tau}\langle x\,,v\rangle \cdot \sigma_t^{-1}(w)\big) \\ &= \bar\sigma_t\big({}_{C^\tau}\langle x\,,v\rangle\big) \cdot w.\end{aligned}$$

Similarly, we use (2.4) to show that

$$\begin{aligned}\big\langle\, \sigma_t(x)\,,\, \sigma_t(v)\big\rangle_{C\rtimes_{\tau,r}H}(s) &= \Delta_H(s)^{-1/2}\sigma_t(x)^*\tau_s(\sigma_t(v)) = \Delta_H(s)^{-1/2}\sigma_t(x^*\tau_s(v)) \\ &= \sigma_t(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}(s)) = (\sigma \rtimes \mathrm{id})_t(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H})(s).\end{aligned}$$

These two calculations show that, algebraically at least, $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ is a potential candidate for an action of K on the $C^\tau - C \rtimes_{\tau,r} H$-imprimitivity bimodule X.

To form the crossed-product bimodule, we need conditions implying that the action $\sigma\colon K \to \operatorname{Aut} C$ induces appropriately continuous actions $\bar\sigma, \sigma$ and $\sigma \rtimes \mathrm{id}$ on C^τ, X and $C \rtimes_{r,\tau} H$ respectively. We start by observing that $\sigma \rtimes \mathrm{id}\colon K \to \operatorname{Aut}(C \rtimes_{\tau,r} H)$ is always continuous. To see this, fix $\epsilon > 0$ and $f \in C_c(H, C)$. Note that $L := \{\, f(s) : s \in \operatorname{supp} f\,\}$ is a compact subset of C. Since $\sigma\colon K \to \operatorname{Aut} C$ is continuous, there exists a neighborhood U of e_K such that $t \in U$ implies that $\|\sigma_t(c) - c\| < \epsilon/\mu_H(\operatorname{supp} f)$ for all $c \in L$. Thus $t \in U$ implies

$$\|(\sigma \rtimes \mathrm{id})_t(f) - f\| \le \int_H \|\sigma_t\big(f(s)\big) - f(s)\|\,ds < \epsilon.$$

Lemma 2.1 *Suppose that the map* $t \mapsto \big\langle\, \sigma_t(x)\,,x\big\rangle_{C\rtimes_{\tau,r}H}$ *is continuous for each fixed* $x \in X_0 = C_0$. *Then* $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ *gives a continuous action of* K *on* ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$.

Proof If $x \in X_0$ then

$$\begin{aligned}\|\sigma_t(x) - x\|^2 &= \|\big\langle\, \sigma_t(x) - x\,,\, \sigma_t(x) - x\big\rangle_{C\rtimes_{\tau,r}H}\| \\ &= \Big\|\,(\sigma \rtimes \mathrm{id})_t(\big\langle\, x\,,x\big\rangle_{C\rtimes_{\tau,r}H}) - \big\langle\, \sigma_t(x)\,,x\big\rangle_{C\rtimes_{\tau,r}H} \\ &\qquad - \big\langle\, x\,,\sigma_t(x)\big\rangle_{C\rtimes_{\tau,r}H} + \big\langle\, x\,,x\big\rangle_{C\rtimes_{\tau,r}H}\Big\| \\ &\le \|(\sigma \rtimes \mathrm{id})_t\big(\langle x\,,x\rangle_{C\rtimes_{\tau,r}H}\big) - \langle x\,,x\rangle_{C\rtimes_{\tau,r}H}\| \\ &\qquad + \|\langle x\,,x\rangle_{C\rtimes_{\tau,r}H} - \big\langle\, \sigma_t(x)\,,x\big\rangle_{C\rtimes_{\tau,r}H}\| \\ &\qquad + \|\langle x\,,x\rangle_{C\rtimes_{\tau,r}H} - \big\langle\, x\,,\sigma_t(x)\big\rangle_{C\rtimes_{\tau,r}H}\|,\end{aligned}$$

so that $\|\sigma_t(x)-x\| \to 0$ as $t \to e_K$, using the assumption and the continuity of $\sigma \rtimes \mathrm{id}$. Thus $\sigma\colon K \to \operatorname{Aut} X$ is continuous. Since

$$\|\bar{\sigma}_t\big({}_{C^\tau}\langle x\,,v\rangle\big) - {}_{C^\tau}\langle x\,,v\rangle\| \le \|x\|\|\sigma_t(v)-v\| + \|v\|\|\sigma_t(x)-x\|$$

the continuity of $\bar{\sigma}\colon K \to \operatorname{Aut} C^\tau$ follows from the continuity of σ. ■

Of course, by symmetry, the action $(\tau \rtimes \mathrm{id}, \tau, \bar{\tau})$ of H on ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$ is continuous provided $s \mapsto {}_{C\rtimes_{\sigma,r}K}\big\langle \tau_s(y)\,,y\big\rangle$ is continuous for each fixed $y \in Y_0 = C_0$.

Proposition 2.2 *Suppose that the action $(\bar{\sigma}, \sigma, \sigma \rtimes \mathrm{id})$ of K on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ is continuous and that the action $(\tau \rtimes \mathrm{id}, \tau, \bar{\tau})$ of H on ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$ is continuous. Let $B := C \rtimes_{\sigma\rtimes\tau,r} (H \times K)$. Then $\big(X \rtimes_{\sigma,r} K\big) \otimes_B \big(Y \rtimes_{\tau,r} H\big)$ is a $C^\tau \rtimes_{\bar{\sigma},r} K$–$C^\sigma \rtimes_{\bar{\tau},r} H$-imprimitivity bimodule.*

Proof Note that whenever the actions $(\bar{\sigma}, \sigma, \sigma \rtimes \mathrm{id})$ of K on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ and $(\tau \rtimes \mathrm{id}, \tau, \bar{\tau})$ of H on ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$ are continuous it makes sense to form the Combes $C^\tau \rtimes_{\bar{\sigma},r} K$–$(C \rtimes_{\tau,r} H) \rtimes_{\sigma\rtimes\mathrm{id},r} K$ and $(C \rtimes_{\sigma,r} K) \rtimes_{\tau\rtimes\mathrm{id},r} H$–$C^\sigma \rtimes_{\bar{\tau},r} H$-imprimitivity bimodules $X \rtimes_{\sigma,r} K$ and $Y \rtimes_{\tau,r} H$ [1, Remark, p. 300]. We identify $(C \rtimes_{\tau,r} H) \rtimes_{\sigma\rtimes\mathrm{id},r} K$ and $(C \rtimes_{\sigma,r} K) \rtimes_{\tau\rtimes\mathrm{id},r} H$ with $B := C \rtimes_{\sigma\rtimes\tau,r} (H \times K)$ since they are all naturally isomorphic. Now the internal tensor product over B, (see, for example, [22, Proposition 3.16]), is a $C^\tau \rtimes_{\bar{\sigma},r} K$–$C^\sigma \rtimes_{\bar{\tau},r} H$-imprimitivity bimodule. ■

3 The Concrete Imprimitivity Bimodule

To get a concrete version of the imprimitivity bimodule obtained in Proposition 2.2 we will use the tools developed in [13], where we looked at a notion of proper action on an imprimitivity bimodule which generalizes Rieffel's in [24]. (Although there are other notions of proper action, for example [7, 17, 18, 25], we are closest to [24] in spirit.)

Definition 3.1 ([13, Definitions 2.1 and 2.15]) If (X, G, γ) is a Morita equivalence between two dynamical systems (A, G, α) and (B, G, β), then the action γ of G on ${}_AX_B$ is *proper* if there are an invariant subspace X_0 of X and invariant $*$-subalgebras A_0 of A and B_0 of B, such that ${}_{A_0}(X_0)_{B_0}$ is a pre-imprimitivity bimodule with completion ${}_AX_B$, and such that

(1) for every $x, y \in X_0$, both $s \mapsto \Delta(s)^{-1/2}{}_A\big\langle x\,,\gamma_s(y)\big\rangle$ and $s \mapsto {}_A\big\langle x\,,\gamma_s(y)\big\rangle$ are in $L^1(G, A)$,

(2) for every $b \in B_0$ and $x \in X_0$, both $s \mapsto \gamma_s(x)\cdot b$ and $s \mapsto \Delta(s)^{-1/2}\gamma_s(x)\cdot b$ are in $L^1(G, X)$,

(3) for every $x, y \in X_0$, there is a multiplier $\langle x\,,y\rangle_{B^\beta}$ in $M(B_0)^\beta$ such that $z\cdot\langle x\,,y\rangle_{B^\beta} \in X_0$ for all $z \in X_0$, and

$$\int_G b\beta_s\big(\langle x\,,y\rangle_B\big)\,ds = b\langle x\,,y\rangle_{B^\beta} \quad \text{for all } b \in B_0. \tag{3.1}$$

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If, in addition, the span in $L^1(G, A)$ of the functions $s \mapsto \Delta(s)^{-1/2}{}_A\langle x, \gamma_s(y)\rangle$ with $x, y \in X_0$ is dense in $A \rtimes_{\alpha,r} G$, then γ is called saturated.

If γ is proper and saturated, then $(X_0)_{B^\beta}$ completes to give an imprimitivity bimodule implementing a Morita equivalence between $A \rtimes_{\alpha,r} G$ and a generalized fixed-point algebra B^β of B [13, Theorem 2.16]. Furthermore, in [13, Theorem 3.1] we showed that the action β on B is proper with respect to $B_1 := \langle X_0, X_0\rangle_B$, so B_1 completes to a $B \rtimes_{\beta,r} G - B^\beta$-imprimitivity bimodule. Finally, a linking algebra argument from [13] implies that there is a natural imprimitivity bimodule-isomorphism of

$$(X \rtimes_{\alpha,r} G) \otimes_{B \rtimes_{\beta,r} G} \overline{B_1} \quad \text{onto} \quad \overline{X_0}.$$

Note that if ${}_{A_0}(X_0)_{B_0} = {}_{B_0}(B_0)_{B_0}$ then Definition 3.1 reduces to that of Rieffel and [13, Theorem 2.16] reduces to [24, Corollary 1.7].

In our situation, we want the action σ of K on ${}_{C^\tau}X_{C \rtimes_{\tau,r} H}$ to be proper with respect to ${}_{(D_0,\bar{\sigma})}(X_0, \sigma)_{(E_0, \sigma \rtimes \mathrm{id})}$. We will then obtain the concrete version of the symmetric imprimitivity theorem, as well as the desired isomorphism onto the tensor-product version, along the following lines: if the action on X is proper and saturated, then first, [13, Theorem 2.16] implies that $\overline{X_0}$ is a $C^\tau \rtimes_{\bar{\sigma},r} K - (C \rtimes_{\tau,r} H)^{\sigma \rtimes \mathrm{id}}$-imprimitivity bimodule. On the other hand, [13, Theorem 3.1] implies $\sigma \rtimes \mathrm{id}$ is saturated and proper with respect to $E_0 \subset C \rtimes_{\tau,r} H$ so that $\overline{E_0}$ is a $(C \rtimes_{\tau,r} H) \rtimes_{\sigma \rtimes \mathrm{id},r} K - (C \rtimes_{\tau,r} H)^{\sigma \rtimes \mathrm{id}}$-imprimitivity bimodule. Let Y be the $C \rtimes_{\sigma,r} K - C^\sigma$-imprimitivity bimodule based on $Y_0 := C_0$ coming from the proper and saturated action σ of K on C. Then the Combes bimodule $Y \rtimes_{\tau,r} H$ is a $(C \rtimes_{\sigma,r} K) \rtimes_{\sigma \times \mathrm{id}} H - C^\sigma \rtimes_{\bar{\tau},r} H$-imprimitivity bimodule. We can identify $(C \rtimes_{\sigma,r} K) \rtimes_{\sigma \times \mathrm{id}} H$ with $C \rtimes_{\tau \times \sigma} (K \times H)$, and, with a bit more work, we will show that $(C \rtimes_{\tau,r} H)^{\sigma \rtimes \mathrm{id}}$ and $C^\sigma \rtimes_{\bar{\tau},r} H$ are isomorphic. With these identifications, we will show that $\overline{E_0}$ is isomorphic to $Y \rtimes_{\tau,r} H$, and then [13, Theorem 3.1] gives an isomorphism

$$(X \rtimes_{\sigma,r} K) \otimes_{C \rtimes_{\sigma \times \tau,r}(K \times H)} (Y \rtimes_{\tau,r} H) \cong \overline{X_0}$$

as $C^\tau \rtimes_{\bar{\sigma},r} K - C^\sigma \rtimes_{\bar{\tau}} H$-imprimitivity bimodules. This gives us both the desired concrete Morita equivalence and the isomorphism onto the tensor-product version.

Thus our first step is to find conditions which ensure that the three items of Definition 3.1 hold in our situation so that the action σ of K on ${}_{C^\tau}X_{C \rtimes_{\tau,r} H}$ is proper with respect to ${}_{(D_0,\bar{\sigma})}(X_0, \sigma)_{(E_0, \sigma \rtimes \mathrm{id})}$.

For Definition 3.1(1) we need to assume that for every $x, v \in X_0$ the functions

$$t \mapsto {}_{C^\tau}\big\langle x, \sigma_t(v)\big\rangle \quad \text{and} \quad t \mapsto \Delta_K(t)^{-1/2}{}_{C^\tau}\big\langle x, \sigma_t(v)\big\rangle \tag{3.2}$$

are in $L^1(K, C^\tau)$. Note that if $w \in X_0$ then $\sigma_t(x) \cdot \langle v, w\rangle_{C \rtimes_{\tau,r} H} = {}_{C^\tau}\langle \sigma_t(x), v\rangle \cdot w$, so that the integrability of the functions in (3.2) implies the integrability of $t \mapsto \sigma_t(x) \cdot e$ and its product with $\Delta_K(t)^{-1/2}$ for all $x \in X_0$ and $e \in E_0$. So, Definition 3.1(1) and (2) hold, provided the functions in (3.2) are integrable. For Definition 3.1(3), we need to assume that the function $s \mapsto \Delta_H(s)^{-1/2}\langle x, \tau_s(v)\rangle_{C^\sigma}$ is in $L^1(H, C^\sigma)$ whenever $x, v \in X_0$ and, using the C^σ-valued inner product for Y, define

$$\big\langle x, v\big\rangle_{(C \rtimes_{\tau,r} H)^{\sigma \rtimes \mathrm{id}}}(s) := \Delta_H(s)^{-1/2}\big\langle x, \tau_s(v)\big\rangle_{C^\sigma}; \tag{3.3}$$

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now we still need to find conditions which ensure that

(1) $\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}$ is a multiplier of $C\rtimes_{\tau,r}H$;
(2) $\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}$ multiplies E_0 and is invariant under $\sigma\rtimes\mathrm{id}$;
(3) equation (3.1) is satisfied; and
(4) $w\cdot\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}$ is back in X_0.

Item (1) is an immediate consequence of Proposition 3.2 below, which allows us to view elements of $L^1(H,C^\sigma)$ as multipliers of $C\rtimes_\sigma H$ via convolution. Although essentially straightforward and presumably known, its proof requires some intricacies of vector-valued integration which are certainly far from the subject at hand. We provide a detailed proof in Appendix A.

Proposition 3.2 *Let (A,G,α) be a dynamical system. Suppose that B is a C^*-subalgebra of $M(A)$ such that $(B,G,\bar{\alpha})$ is a dynamical system. If $g\in L^1(G,B)$, then there is a unique multiplier T_g in $M(A\rtimes_\alpha G)$ such that for all $f\in L^1(G,A)$ both T_gf and fT_g are in $L^1(G,A)$ (viewed as a subalgebra of $A\rtimes_\alpha G$), and for almost all s,*

$$T_gf(s)=\int_G g(r)\alpha_r\big(f(r^{-1}s)\big)\,dr \tag{3.4}$$

and

$$fT_g(s)=\int_G f(r)\bar{\alpha}_r\big(g(r^{-1}s)\big)\,dr. \tag{3.5}$$

Note that if $B=A$, then Proposition 3.2 reduces to two things. First, the familiar formula for convolutions of functions in $C_c(G,A)$ extends to functions in $L^1(G,A)$, and second, that convolution has the usual $*$-algebraic properties.

For (2), we seek conditions which ensure that $\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}$ multiplies E_0. Using (3.5) we compute:

$$\begin{aligned}
\langle u\,,w\rangle_{C\rtimes_{\tau,r}H}\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}(s)&=\int_H\langle u\,,w\rangle_{C\rtimes_{\tau,r}H}(r)\bar{\tau}_r\big(\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}(r^{-1}s)\big)\,dr\\
&=\Delta_H(s)^{-1/2}u^*\int_H\tau_r(w)\bar{\tau}_r\big(\langle x\,,\tau_{r^{-1}s}(v)\rangle_{C^\sigma}\big)\,dr\\
&=\Delta_H(s)^{-1/2}u^*\tau_s\Big(\int_H\tau_{s^{-1}r}(w)\langle\tau_{s^{-1}r}(x)\,,v\rangle_{C^\sigma}\,dr\Big)\\
&=\Delta_H(s)^{-1/2}u^*\tau_s\Big(\int_H\tau_r(w)\langle\tau_r(x)\,,v\rangle_{C^\sigma}\,dr\Big)
\end{aligned}$$

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by the change of variable $s^{-1}r \mapsto r$. Since

$$\begin{aligned}\int_H \tau_r(w)\big\langle \tau_r(x)\,,v\big\rangle_{C^\sigma}\,dr &= \int_H {}_{C\rtimes_{\sigma,r}K}\big\langle \tau_r(w)\,,\tau_r(x)\big\rangle\cdot v\,dr\\ &= \int_H\int_K {}_{C\rtimes_{\sigma,r}K}\big\langle \tau_r(w)\,,\tau_r(x)\big\rangle(t)\sigma_t(v)\Delta_K(t)^{1/2}\,dt\,dr\\ &= \int_H\int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\,dt\,dr.\end{aligned}$$

Thus $\langle u\,,w\rangle_{C\rtimes_{\tau,r}H}\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}$ is back in E_0 provided

$$\int_H\int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\,dt\,dr \in X_0 \tag{3.6}$$

whenever $u, v, w, x \in X_0$.

For (3), we need to show that

$$\int_K e(\sigma\rtimes\mathrm{id})_t\big(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}\big)\,dt = e\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}.$$

To do this we need the following lemma. For the sake of clarity, we have decorated our integrals with the space in which the integral takes values; thus we write $\int_H^C f(s)\,ds$ for the C-valued integral of $f \in L^1(H, C)$. Again, the proof of the lemma requires some gymnastics with vector-valued integration, so we relegate it to Appendix B.

Lemma 3.3 *Assume that for all $u, v, w, x \in X_0$, the function*

$$(r, s, t) \mapsto u\tau_r(v^*)\sigma_t\big(\tau_r(w^*)\tau_s(x)\big)\,\Delta_H(s)^{-\frac{1}{2}}$$

is integrable as a function from $H\times H\times K$ to C. Then for all $e_i \in E_0$ the function $t \mapsto e_1(\sigma\rtimes\mathrm{id})_t(e_2)$ is integrable as a function from K to $C\rtimes_{\tau,r}H$. Further, the integral

$$\int_K^{C\rtimes_{\tau,r}H} e_1(\sigma\rtimes\mathrm{id})_t(e_2)\,dt \tag{3.7}$$

takes values in $L^1(H, C)$ viewed as a subalgebra of $C\rtimes_{r,\tau}H$, and a representative for (3.7) is given by

$$s \mapsto \int_K^C e_1(\sigma\rtimes\mathrm{id})_t(e_2)(s)\,dt. \tag{3.8}$$

Using Lemma 3.3 we obtain

$$\begin{aligned}\int_K e(\sigma\rtimes\mathrm{id})_t\big(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}\big)\,dt(s) &= \int_K e(\sigma\rtimes\mathrm{id})_t\big(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}\big)(s)\,dt\\ &= \int_K\int_H e(r)\tau_r\big((\sigma\rtimes\mathrm{id})_t(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}(r^{-1}s)\big)\,dr\,dt\\ &= \int_K\int_H e(r)\tau_r\sigma_t(x^*\tau_{r^{-1}s}(v))\Delta_H(r^{-1}s)^{-1/2}\,dr\,dt.\end{aligned}$$

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Since our assumptions guarantee that the previous integrand is an integrable function of (r,t) provided $e \in E_0$ (see Remark A.2), is, by Fubini's Theorem, equal to

$$\begin{aligned}
&\int_H \int_K e(r)\sigma_t\big(\tau_r(x)^*\tau_s(v)\big)\,\Delta_H(r^{-1}s)^{-1/2}\,dt\,dr\\
&= \int_H e(r)\big\langle \tau_r(x)\,,\tau_s(v)\big\rangle_{C^\sigma}\Delta_H(r^{-1}s)^{-1/2}\,dr\\
&= \int_H e(r)\bar\tau_r\big(\langle x\,,\tau_{r^{-1}s}(v)\rangle_{C^\sigma}\big)\,\Delta_H(r^{-1}s)^{-1/2}\,dr\\
&= \int_H e(r)\bar\tau_r\big(\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}(r^{-1}s)\big)\;dr\\
&= e\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}(s)
\end{aligned}$$

as required.

To establish (4), we claim that

$$w\cdot\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}} = \int_K w\cdot(\sigma\rtimes\mathrm{id})_t(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H})\,dt. \tag{3.9}$$

To see this, first note that

$$\begin{aligned}
\|w\cdot(\sigma\rtimes\mathrm{id})_t\big(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}\big)\| &= \|w\cdot\big\langle\sigma_t(x)\,,\sigma_t(v)\big\rangle_{C\rtimes_{\tau,r}H}\|\\
&\le \|{}_{C^\tau}\big\langle w\,,\sigma_t(x)\big\rangle\|\|\sigma_t(v)\|\\
&\le \|{}_{C^\tau}\big\langle w\,,\sigma_t(x)\big\rangle\|\|v\|,
\end{aligned}$$

and the latter is integrable by Definition 3.1(1). It is not hard to see that

$$\Big\langle z\,,\int_K w\cdot(\sigma\rtimes\mathrm{id})_t\big(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}\big)\;dt\Big\rangle_{C\rtimes_{\tau,r}H} = \Big\langle z\,,w\cdot\langle x\,,v\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\Big\rangle_{C\rtimes_{\tau,r}H},$$

and (3.9) follows. We use (2.2) to write the left-hand side of (3.9) as

$$\begin{aligned}
&\int_K\int_H \tau_s^{-1}\Big(w\big((\sigma\rtimes\mathrm{id})_t\big(\langle x\,,v\rangle_{C\rtimes_{\tau,r}H}\big)(s)\big)\Big)\,\Delta_H(s)^{-1/2}\,ds\,dt\\
&\qquad= \int_K\int_H \tau_s^{-1}\Big(w\sigma_t\big(x^*\tau_s(v)\big)\Big)\,\Delta_H(s)^{-1}\,ds\,dt\\
&\qquad= \int_K\int_H \tau_s^{-1}(w)\sigma_t\big(\tau_{s^{-1}}(x^*)v\big)\,\Delta_H(s)^{-1}\,ds\,dt\\
&\qquad= \int_K\int_H \tau_s(w)\sigma_t(\tau_s(x^*)v)\,ds\,dt,
\end{aligned}$$

and we already assumed in Equation (3.6) above that this double integral is back in X_0.

We can restate our conclusions above as

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Proposition 3.4 *Suppose that the action $(\bar{\sigma}, \sigma, \sigma \rtimes \mathrm{id})$ of K on ${}_{C^\tau}X_{C \rtimes_{\tau,r} H}$ is continuous. If for all $u, v, w, x \in X_0$*

(1) *the function $t \mapsto {}_{C^\tau}\langle x\,, \sigma_t(v)\rangle$ and its product with $\Delta_K(t)^{-1/2}$ are in $L^1(K, C^\tau)$,*

(2) *the function $s \mapsto \Delta_H(s)^{-1/2}\langle x\,, \tau_s(v)\rangle_{C^\sigma}$ is in $L^1(H, C^\sigma)$,*

(3) *the integral $\int_H \int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\,dt\,dr$ is in X_0, and*

(4) *the function $(r, s, t) \mapsto u\tau_r(v^*)\sigma_t(\tau_r(w^*)\tau_s(x))\Delta_H(s)^{-1/2}$ is integrable,*

then σ is a proper action of K on ${}_{C^\tau}X_{C \rtimes_{\tau,r} H}$ with respect to ${}_{(D_0,\bar{\sigma})}(X_0, \sigma)_{(E_0,\sigma\rtimes\mathrm{id})}$.

In the situation of Proposition 3.4 above, we want the action of K to be saturated with respect to X_0, so that X_0 completes to a $C^\tau \rtimes_{\bar{\sigma},r} K - (C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}}$-imprimitivity bimodule by [13, Theorem 2.16]. Since σ is a proper action of K on ${}_{C^\tau}X_{C \rtimes_{\tau,r} H}$ with respect to ${}_{(D_0,\bar{\sigma})}(X_0, \sigma)_{(E_0,\sigma\rtimes\mathrm{id})}$, [13, Theorem 3.1] says that $\sigma \rtimes \mathrm{id}$ is a proper action on $C \rtimes_{\tau,r} H$ with respect to $\langle X_0\,, X_0\rangle_{C\rtimes_{\tau,r}H}$. We have set things up so that $\langle X_0\,, X_0\rangle_{C\rtimes_{\tau,r}H} = E_0$, and thus E_0 completes to an I–J-imprimitivity bimodule, where I is an ideal in $(C \rtimes_{\tau,r} H) \rtimes_{\sigma\rtimes\mathrm{id}} K$ and J is a generalized fixed point algebra of $C \rtimes_{\tau,r} H$, by [24, Theorem 1.5]. But [13, Theorem 3.1] implies that J is an ideal in $(C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}}$, and if $\sigma \rtimes \mathrm{id}$ is saturated with respect to E_0 then σ is saturated with respect to X_0, and $J = (C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}}$. In applications we expect that it will be easy to check that $\sigma \rtimes \mathrm{id}$ is saturated with respect to E_0, and then we have

Proposition 3.5 *Suppose that the action $(\bar{\sigma}, \sigma, \sigma\rtimes\mathrm{id})$ of K on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ is continuous and proper with respect to ${}_{(D_0,\bar{\sigma})}(X_0, \sigma)_{(E_0,\sigma\rtimes\mathrm{id})}$, and that the proper action $\sigma \rtimes \mathrm{id}$ of K on $C \rtimes_{\tau,r} H$ is saturated with respect to E_0. Then E_0 completes to a $(C \rtimes_{\tau,r} H) \rtimes_{\sigma\rtimes\mathrm{id},r} K - (C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}}$-imprimitivity bimodule.*

In the next proposition we identify the module $\overline{E_0}$ and the fixed point algebra $(C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}}$; again we add substantial hypotheses. The set E_0 is always a subset of $L^1(H, C_0)$. Even though the completion Y of C_0 is a $C \rtimes_{\sigma,r} K - C^\sigma$-imprimitivity bimodule, this does not in general imply that E_0 is contained in $L^1(H, Y)$ (the norm of Y is not related to the norm of C).

Proposition 3.6 *Suppose that the action $(\tau\rtimes\mathrm{id}, \tau, \bar{\tau})$ of H on ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$ is continuous so that $Y \rtimes_{\tau,r} H$ is a $(C \rtimes_{\sigma,r} K) \rtimes_{\tau\rtimes\mathrm{id},r} H - C^\sigma \rtimes_{\bar{\tau},r} H$-imprimitivity bimodule. Also suppose that the action $\sigma\rtimes\mathrm{id}$ of K on $C\rtimes_{\tau,r}H$ is proper and saturated with respect to E_0. If $E_0 \subset L^1(H, Y)$ and if for all $u, v, w, x \in X_0$ the function given in Proposition* 3.4(4) *is integrable, then $(C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}} \cong C^\sigma \rtimes_{\bar{\tau},r} H$ and*

$$ {}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id}}K}\overline{E_0}_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}} \cong {}_{(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H}(Y \rtimes_{\tau,r} H)_{C^\sigma\rtimes_{\bar{\tau},r}H} $$

as imprimitivity bimodules.

Proof As observed in [5, p. 428], if $h \in C_c(H, Y)$ then the Cauchy–Schwartz inequality gives $\|h\| \leq \int_H \|h(s)\|\,ds = \|h\|_1$. It follows that $L^1(H, Y)$ is dense in $Y \rtimes_{\tau,r} H$ and the actions and inner products on $C_c(H, Y) \subset Y \rtimes_{\tau,r} H$ given, for example, in

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[13, (3.5)–(3.8)], extend to integrable functions. That the formulas themselves extend to integrable functions can be seen by including $Y \rtimes_{\tau,r} H$ in the linking algebra $L(Y \rtimes_{\tau,r} H)$ and doing the computations in the relevant bit of the C^*-algebra.

By assumption E_0 is contained in $L^1(H,Y)$; we let $\iota\colon E_0 \to Y \rtimes_{\tau,r} H$ be the inclusion map. Let $\varphi\colon (C \rtimes_{\tau,r} H) \rtimes_{\sigma\rtimes\mathrm{id},r} K \to (C \rtimes_{\sigma,r} K) \rtimes_{\tau\rtimes\mathrm{id},r} H$ be the isomorphism such that $\varphi(f)(s)(t) = f(t)(s)$ for $f \in L^1(K, L^1(H,C))$. We will show that $\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}} \mapsto \langle e\,,f\rangle_{C^\sigma\rtimes_{\tilde\tau,r}H}$ extends to an isomorphism ψ of $(C \rtimes_{r,\tau} H)^{\sigma\rtimes\mathrm{id}}$ onto $C^\sigma \rtimes_{\tilde\tau} H$ and that (φ,ι,ψ) extends to an imprimitivity bimodule isomorphism of $\overline{E_0}$ onto $Y \rtimes_{\tau,r} H$.

If $e, f \in E_0$, then

$$\begin{aligned}
{}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle e\,,f\rangle(t)(s) &= \Delta_K(t)^{-1/2}\,{}_{C\rtimes_{\tau,r}H}\big\langle e\,,(\sigma\rtimes\mathrm{id})_t(f)\big\rangle(s)\\
&= \Delta_K(t)^{-1/2} e * (\sigma\rtimes\mathrm{id})_t(f)^*(s)\\
&= \Delta_K(t)^{-1/2}\int_H e(r)\tau_r((\sigma\rtimes\mathrm{id})_t(f^*)(r^{-1}s))\,dr\\
&= \Delta_K(t)^{-1/2}\int_H e(r)\tau_s\sigma_t(f(s^{-1}r)^*)\Delta_H(s^{-1}r)\,dr\\
&= \int_H \Delta_K(t)^{-1/2}e(r)\sigma_t(\tau_s(f(s^{-1}r))^*)\Delta_H(s^{-1}r)\,dr\\
&= \int_H {}_{C\rtimes_{\sigma,r}K}\big\langle e(r)\,,\tau_s(f(s^{-1}r))\big\rangle(t)\Delta_H(s^{-1}r)\,dr\\
&= {}_{(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H}\langle e\,,f\rangle(s)(t)
\end{aligned}$$

so that $\varphi\big({}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle e\,,f\rangle\big) = {}_{(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H}\big\langle \iota(e)\,,\iota(f)\big\rangle$. Next, we show that

$$\iota\big({}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle e\,,f\rangle\cdot g\big) = \varphi\big({}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle e\,,f\rangle\big)\cdot\iota(g)$$

whenever $e, f, g \in E_0$. First,

$$\begin{aligned}
{}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle e\,,f\rangle\cdot g(s) &= e\cdot\langle f\,,g\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}(s)\\
&= \int_K e * ((\sigma\rtimes\mathrm{id})_t(f^* * g))\,dt(s)
\end{aligned}$$

which, by Lemma 3.3, is

$$\begin{aligned}
&= \int_K e * ((\sigma\rtimes\mathrm{id})_t(f^* * g))(s)\,dt\\
&= \int_K\int_H e(r)\tau_r\big((\sigma\rtimes\mathrm{id})_t(f^* * g)(r^{-1}s)\big)\,drdt\\
&= \int_K\int_H\int_H e(r)\sigma_t\tau_{ru^{-1}}\big(f(u)^*g(ur^{-1}s)\big)\,dudrdt. \qquad (3.10)
\end{aligned}$$

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On the other hand,

$$\begin{aligned}\varphi\big({}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle e\,,f\rangle\big)\cdot\iota(g) &= {}_{(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H}\langle e\,,f\rangle\cdot g(s)\\ &= e\cdot\langle f\,,g\rangle_{C^\sigma\rtimes_{\bar\tau,r}H}(s)\end{aligned}$$

which, using the Combes action, is

$$\begin{aligned}&= \int_H e(r)\cdot\bar\tau_r\big(\langle f\,,g\rangle_{C^\sigma\rtimes_{\bar\tau,r}H}(r^{-1}s)\big)\,dr\\ &= \int_H e(r)\bar\tau_r\Big(\int_H \bar\tau_u^{-1}\big(\big\langle f(u)\,,g(ur^{-1}s)\big\rangle_{C^\sigma}\big)\Big)\,dudr\\ &= \int_H\int_H\int_K e(r)\sigma_t\tau_{ru^{-1}}\big(f(u)^*g(ur^{-1}s)\big)\,dtdudr\end{aligned}$$

which is the same as (3.10) by two applications of Fubini's Theorem.

So far we have shown that

$$(\varphi,\iota)\colon {}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}E_0 \to {}_{(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H}(Y\rtimes_{\tau,r}H)$$

extends to a monomorphism of left-Hilbert modules. To produce

$$\psi\colon (C\rtimes_{r,\tau}H)^{\sigma\rtimes\mathrm{id}}\to C^\sigma\rtimes_{\bar\tau}H$$

taking

$$\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\quad\text{to}\quad\langle\iota(e)\,,\iota(f)\rangle_{C^\sigma\rtimes_{\bar\tau,r}H},$$

let π be a faithful representation of C on $\mathcal{H}_\pi$ and $\bar\pi$ its extension to $M(C)$. By our assumptions, $\bar\tau\colon H\to\operatorname{Aut}C^\sigma$ is continuous, so that the regular representation $(\widetilde\pi,\lambda)$ of (C,H,τ) extends to a covariant representation $(\widetilde{\bar\pi},\lambda)$ of $(C^\sigma,H,\bar\tau)$. The representations

$$\overline{\widetilde\pi\rtimes\lambda}\colon (C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}\to B(L^2(H,\mathcal{H}_\pi))\quad\text{and}\quad\widetilde{\bar\pi}\rtimes\lambda\colon C^\sigma\rtimes_{\bar\tau,r}H\to B(L^2(H,\mathcal{H}_\pi))$$

are faithful. We will show that for $e,f\in E_0$

$$\overline{\widetilde\pi\rtimes\lambda}\big(\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\big)=\widetilde{\bar\pi}\rtimes\lambda\big(\langle\iota(e)\,,\iota(f)\rangle_{C^\sigma\rtimes_{\bar\tau,r}H}\big).\tag{3.11}$$

It then follows that $\psi:=(\widetilde{\bar\pi}\rtimes\lambda)^{-1}\circ\overline{\widetilde\pi\rtimes\lambda}$ is an injective homomorphism satisfying

$$\psi\big(\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\big)=\langle\iota(e)\,,\iota(f)\rangle_{C^\sigma\rtimes_{\bar\tau,r}H}.\tag{3.12}$$

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As the first step we show that $\iota(g\cdot\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}})=\iota(g)\cdot\langle\iota(e)\,,\iota(f)\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}$ whenever $e,f,g\in E_0$; this will then also give us that (ι,ψ) extends to a right-Hilbert module homomorphism. Well,

$$\begin{aligned}\iota\big(g\cdot\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\big)&=\iota\big({}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle g\,,e\rangle\cdot f\big)\\&=\varphi\big({}_{(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K}\langle g\,,e\rangle\big)\cdot\iota(f)\\&={}_{(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H}\big\langle\,\iota(g)\,,\iota(e)\big\rangle\cdot\iota(f)\\&=\iota(g)\cdot\langle\iota(e)\,,\iota(f)\rangle_{C^\sigma\rtimes_{\tilde\tau,r}H}.\end{aligned}$$

Note that for all $g\in E_0$ and $\xi,\eta\in L^2(H,\mathcal{H}_\pi)$

$$\begin{aligned}&\Big(\overline{\tilde\pi\rtimes\lambda}\big(\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\big)\,\xi\ \big|\ \tilde\pi\rtimes\lambda(g^*)\eta\Big)\\&\quad=\Big(\tilde\pi\rtimes\lambda\big(g\cdot\langle e\,,f\rangle_{(C\rtimes_{\tau,r}H)^{\sigma\rtimes\mathrm{id}}}\big)\,\xi\ \big|\ \eta\Big)\\&\quad=\Big(\tilde\pi\rtimes\lambda\big(\iota(g)\cdot\big\langle\iota(e)\,,\iota(f)\big\rangle_{C^\sigma\rtimes_{\tilde\tau,r}H}\big)\,\xi\ \big|\ \eta\Big)\\&\quad=\Big(\tilde{\tilde\pi}\rtimes\lambda\big(\big\langle\iota(e)\,,\iota(f)\big\rangle_{C^\sigma\rtimes_{\tilde\tau,r}H}\big)\,\xi\ \big|\ \tilde\pi\rtimes\lambda(g^*\Big)\,\eta),\end{aligned}$$

so that (3.12) follows from the nondegeneracy of $\tilde\pi\rtimes\lambda$.

We now know that the triple (φ,ι,ψ) is an isomorphism of $\overline{E_0}$ into $Y\rtimes_{\tau,r}H$; to see that this map is onto $Y\rtimes_{\tau,r}H$, we will show that $\iota(E_0)$ is invariant under the right action of C^σ and H, and then must be invariant under the action of $C^\sigma\rtimes_{\tilde\tau,r}H$ as well. This suffices because φ maps $(C\rtimes_{\tau,r}H)\rtimes_{\sigma\rtimes\mathrm{id},r}K$ onto $(C\rtimes_{\sigma,r}K)\rtimes_{\tau\rtimes\mathrm{id},r}H$ and the Rieffel correspondence then implies that $\overline{\iota(E_0)}=Y\rtimes_{\tau,r}H$. Therefore $\psi\colon(C\rtimes_{r,\tau}H)^{\sigma\rtimes\mathrm{id}}\to C^\sigma\rtimes_{\tilde\tau,r}H$ is an isomorphism.

Consider $e\in E_0$, say $e(s)=\Delta_H(s)^{-1/2}x\tau_s(y^*)$, and $\langle v\,,w\rangle_{C^\sigma}$ where $v,w,x,y\in X_0$. Then

$$\begin{aligned}\big(e\cdot\langle v\,,w\rangle_{C^\sigma}\big)(s)=e(s)\tau_s\big(\langle v\,,w\rangle_{C^\sigma}\big)&=\Delta_H(s)^{-1/2}x\tau_s(y^*)\tau_s\big(\langle v\,,w\rangle_{C^\sigma}\big)\\&=\Delta_H(s)^{-1/2}x\tau_s\big(y^*\langle v\,,w\rangle_{C^\sigma}\big)\end{aligned}$$

so that $e\cdot\langle v\,,w\rangle_{C^\sigma}$ is back in E_0 because $\langle v\,,w\rangle_{C^\sigma}$ multiplies X_0. Also,

$$(e\cdot i_H(r))(s)=e(sr^{-1})\Delta_H(r^{-1})=\Delta_H(sr^{-1})^{-1/2}\big(\Delta_H(r^{-1})x\tau_{sr^{-1}}(y^*)\big)$$

so $e\cdot i_H(r)$ is back in E_0. ■

We can summarize our above discussions as follows.

Theorem 3.7 *Suppose that $\tau\colon H\to\operatorname{Aut}C$ and $\sigma\colon K\to\operatorname{Aut}C$ are commuting actions which are proper and saturated with respect to the same dense $*$-subalgebra C_0.*

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(1) *(Continuity) If the maps* $t \mapsto \langle \sigma_t(x), x\rangle_{C\rtimes_{\tau,r}H}$ *and* $s \mapsto {}_{C\rtimes_{\sigma,r}K}\langle \tau_s(x), x\rangle$ *are continuous for all* $x \in X_0 = Y_0 = C_0$ *then the actions* $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ *of* K *on* ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ *and* $(\tau \rtimes \mathrm{id}, \tau, \bar\tau)$ *of* H *on* ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$ *are continuous. In particular,* $C^\tau \rtimes_{\bar\sigma,r} K$ *and* $C^\sigma \rtimes_{\bar\tau,r} H$ *are Morita equivalent.*

(2) *(Properness) The action* $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ *of* K *on* ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ *is proper with respect to* ${}_{(D_0,\bar\sigma)}(X_0, \sigma)_{(E_0,\sigma\rtimes\mathrm{id})}$ *if for all* $u, v, w, x \in X_0$

(a) *the function* $t \mapsto {}_{C^\tau}\langle x, \sigma_t(v)\rangle$ *and its product with* $\Delta_K(t)^{-1/2}$ *are in* $L^1(K, C^\tau)$*;*

(b) *the function* $s \mapsto \Delta_H(s)^{-1/2}\langle x, \tau_s(v)\rangle_{C^\sigma}$ *is in* $L^1(H, C^\sigma)$*;*

(c) *the integral* $\int_H \int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\, dt\, dr$ *is in* X_0*;*

(d) *the function* $(r, s, t) \mapsto u\tau_r(v^*)\sigma_t\big(\tau_r(w^*)\tau_s(x)\big)\, \Delta_H(s)^{-1/2}$ *is integrable.*

(3) *(Tensor decomposition isomorphism) If in addition to* (1) *and* (2) *above,*

(a) *the action* $\sigma \rtimes \mathrm{id}$ *of* K *on* $C \rtimes_{\tau,r} H$ *is saturated with respect to* E_0*, and*

(b) $E_0 \subset L^1(H, Y)$

then X_0 *completes to a* $C^\tau \rtimes_{\bar\sigma,r} K – C^\sigma \rtimes_{\bar\tau,r} H$*-imprimitivity bimodule and*

$$(X \rtimes_{\sigma,r} K) \otimes (Y \rtimes_{\tau,r} H) \cong \overline{X_0}$$

as $C^\tau \rtimes_{\bar\sigma,r} K – C^\sigma \rtimes_{\bar\tau,r} H$*-imprimitivity bimodules.*

Proof Item (1) follows from Proposition 2.2, item (2) from Proposition 3.4, and item (3) from Proposition 3.6 and [13, Theorem 3.1]. ∎

Corollary 3.8 *Suppose that* $\tau \colon H \to \operatorname{Aut} C$ *and* $\sigma \colon K \to \operatorname{Aut} C$ *are commuting actions which are proper and saturated with respect to the same dense ∗-subalgebra* C_0*, and suppose that the hypotheses of Theorem* 3.7(1)–(3) *are satisfied. Then* C_0 *completes to a* $C^\tau \rtimes_{\bar\sigma,r} K – C^\sigma \rtimes_{\bar\tau,r} H$*-imprimitivity bimodule where the actions and inner products are given on dense objects by equations* 3.13–3.16 *below.*

Proof The action $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ of K on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ is continuous by item (1) and is proper with respect to ${}_{(D_0,\bar\sigma)}(C_0, \sigma)_{(E_0,\sigma\rtimes\mathrm{id})}$ by item (2). The action $\sigma \rtimes \mathrm{id}$ of K on $C \rtimes_{\tau,r} H$ is saturated with respect to E_0 by item (3a), and since $E_0 = \langle C_0, C_0\rangle_{C\rtimes_{\tau,r}H}$, it follows from [13, Theorem 3.1] that the action σ on X is saturated with respect to C_0. By [13, Theorem 2.16] the completion of C_0 is a $C^\tau \rtimes_{\bar\sigma,r} K – (C \rtimes_{\bar\tau,r} H)^{\sigma\rtimes\mathrm{id}}$-imprimitivity bimodule. Finally, items (2d), (3b), and Proposition 3.6 allow us to identify $(C \rtimes_{\tau,r} H)^{\sigma\rtimes\mathrm{id}}$ and $C^\sigma \rtimes_{\bar\tau,r} H$.

Chasing through the construction and identification above we can write down the actions and inner products for the $C^\tau \rtimes_{\bar\sigma,r} K – C^\sigma \rtimes_{\bar\tau,r} H$-imprimitivity bimodule obtained by competing C_0. Let $x, y, c \in C_0$. By [13, Lemma 2.17],

$$\begin{aligned} {}_{C^\tau\rtimes_{\bar\sigma,r}K}\langle x, y\rangle \cdot c &= \int_K {}_{C^\tau\rtimes_{\bar\sigma,r}K}\langle x, y\rangle(t)\sigma_t(c)\Delta_K(t)^{-1/2}\, dt \\ &= \int_K {}_{C^\tau}\langle x, \sigma_t(y)\rangle\sigma_t(c)\, dt \\ &= \int_K \int_H \tau_s(x\sigma_t(y^*))\sigma_t(c)\, ds dt, \end{aligned} \tag{3.13}$$

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and now

$$c \cdot \langle x\,,\, y\rangle_{C^\sigma \rtimes_{\bar\tau,r} H} = {}_{C^\tau \rtimes_{\bar\sigma,r} K}\langle c\,,\, x\rangle \cdot y = \int_K \int_H \tau_s(c\sigma_t(x^*))\sigma_t(y)\,dsdt. \tag{3.14}$$

By [13, Theorem 2.16] the left inner product is

$${}_{C^\tau \rtimes_{\bar\sigma,r} K}\langle x\,,\, y\rangle(t) = \Delta_K(t)^{-1/2}{}_{C^\tau}\langle x\,,\, \sigma_t(y)\rangle, \tag{3.15}$$

and the right inner product is defined using the isomorphism of $(C \rtimes_{\tau,r} H)^{\sigma \rtimes \mathrm{id}}$ onto $C^\sigma \rtimes_{\bar\tau,r} H$ and equation 3.3:

$$\langle x\,,\, y\rangle_{C^\sigma \rtimes_{\bar\tau,r} H}(s) = \Delta_H(s)^{-1/2}\langle x\,,\, \tau_s(y)\rangle_{C^\sigma}. \tag{3.16}$$

■

4 Examples

Our examples are based on the proper actions constructed in [25, Section 5]. There Rieffel starts with a proper action of G on the left of a locally compact Hausdorff space P, a nondegenerate homomorphism $\theta\colon C_0(P) \to M(A)$, and an action α of G on A such that

$$\alpha_s(\theta(f)a) = \theta(\mathrm{lt}_s(f))\alpha_s(a).$$

Rieffel proves in [25, Theorem 5.7] that α is proper in the sense of [24] with respect to the subalgebra

$$A_0 := \theta(C_c(P))A\theta(C_c(P)) = \mathrm{span}\{\,\theta(f)a\theta(g) : a \in A,\ f, g \in C_c(P)\,\}.$$

The homomorphism θ does *not* necessarily have range in $ZM(A)$, and consequently this setup includes striking examples (see Remark 4.5). If θ is central then A is a $C_0(P)$-algebra, and Proposition 4.2 and Theorem 4.4 below reduce to results in [12].

Rieffel also says in [25] that the action is saturated if the action of G on P is free; this is the content of the following lemma which we prove in Appendix C.

Lemma 4.1 (Rieffel) *Suppose that G acts freely and properly on a locally compact Hausdorff space P, and that there is a nondegenerate homomorphism $\theta\colon C_0(P) \to M(A)$ and an action α of G on A such that $\alpha_s(\theta(f)a) = \theta(\mathrm{lt}_s(f))\alpha_s(a)$. Then the proper action α of G on A is saturated with respect to $A_0 := \theta(C_c(P))A\theta(C_c(P))$.*

For our example, consider commuting proper actions of K and H on the left and right of P. Suppose that we have a nondegenerate homomorphism $\theta\colon C_0(P) \to M(C)$, and that we have commuting actions $\sigma\colon K \to \mathrm{Aut}\, C$ and $\tau\colon H \to \mathrm{Aut}\, C$ satisfying

$$\bar\sigma_t(\theta(f)) = \theta(\mathrm{lt}_t(f)) \text{ for } t \in K, \quad \text{and} \quad \bar\tau_s(\theta(f)) = \theta(\mathrm{rt}_s(f)) \text{ for } s \in H. \tag{4.1}$$

Then [25, Theorem 5.7] implies that both σ and τ are proper with respect to the same subalgebra $C_0 := \theta(C_c(P))C\theta(C_c(P))$.

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Proposition 4.2 *Suppose that $\sigma\colon K \to \operatorname{Aut} C$ and $\tau\colon H \to \operatorname{Aut} C$ are commuting actions and that the actions of K and H on P are free and proper. If $C_0(P)$ maps bi-equivariantly into $M(C)$ then $C^\tau \rtimes_{\bar\sigma,r} K$ and $C^\sigma \rtimes_{\bar\tau,r} H$ are Morita equivalent.*

By Theorem 3.7(1), it suffices to verify that the actions $(\bar\sigma, \sigma, \sigma \times \mathrm{id})$ and $(\tau \rtimes \mathrm{id}, \tau, \bar\tau)$ of K and H on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ and ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$ are continuous. Furthermore, if $B := C \rtimes_{\sigma\times\tau,r} (H \times K)$, then $(X \rtimes_{\sigma,r} K) \otimes_B (Y \rtimes_{\tau,r} H)$ is a $C^\tau \rtimes_{\bar\sigma,r} K$ – $C^\sigma \rtimes_{\bar\tau,r} H$-imprimitivity bimodule by Proposition 2.2. We retain the notation from Sections 2 and 3, and we will drop all mention of θ — we must remember that $fc \neq cf$. The key to our calculations is the following lemma.

Lemma 4.3 *For $x = fbg$ and $y = hck$ in C_0, the functions $s \mapsto x\tau_s(y)$ and $t \mapsto x\sigma_t(y)$ have compact support depending only on $\operatorname{supp} g$ and $\operatorname{supp} h$.*

Proof The consistency conditions (4.1) imply that $x\tau_s(y) = fbg\big(\mathrm{rt}_s(h)\big)\,\tau_s(ck)$. But $g\,\mathrm{rt}_s(h)$ is nonzero only if

$$\operatorname{supp} g \cap \operatorname{supp}(\mathrm{rt}_s(h)) = \operatorname{supp} g \cap (\operatorname{supp} h)s^{-1}$$

is nonempty. Therefore the support of $s \mapsto x\tau_s(y)$ is contained in

$$\{s \in H : \operatorname{supp} g \cap (\operatorname{supp} h)s^{-1} \neq \varnothing\}$$

which is compact because H acts properly on P. The other part is similar. ∎

Proof of Proposition 4.2 We want to show that $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ is a continuous action on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$, and Theorem 3.7(1) implies it suffices to see that for each fixed $x \in X_0 = C_0$ the map $t \mapsto \langle \sigma_t(x)\,,x\rangle_{C\rtimes_{\tau,r}H}$ is continuous. Note that

$$\begin{aligned}(4.2)\qquad \big\|\langle \sigma_t(x)\,,x\rangle_{C\rtimes_{\tau,r}H} - \langle x\,,x\rangle_{C\rtimes_{\tau,r}H}\big\| &\le \int_H \big\|\langle \sigma_t(x) - x\,,x\rangle_{C\rtimes_{\tau,r}H}(s)\big\|\,ds \\ &= \int_H \|(\sigma_t(x) - x)^*\tau_s(x)\|\Delta_H(s)^{-1/2}\,ds.\end{aligned}$$

We claim that the integrand in (4.2) has support in a compact set L whenever t is in a sufficiently small neighborhood M of e, so that

$$(4.2) \le \|\sigma_t(x) - x\|\|\tau_s(x)\Delta_H(s)^{-1/2}\|_\infty \mu_H(L) \to 0 \text{ as } t \to 0$$

because $\sigma\colon K \to \operatorname{Aut} C$ and $s \mapsto \tau_s(x)\Delta_H(s)^{-1/2}$ are continuous.

To prove our claim, set $x = fcg$, and choose $h, k \in C_c(P)$ such that $h = 1$ on a neighborhood of $\operatorname{supp} f$ and $k = 1$ on a neighborhood of $\operatorname{supp} g$. Then there exists a neighborhood M of e in K such that if $t \in M$ then $h = 1$ on $\operatorname{supp} \mathrm{lt}_t(f)$ and $k = 1$ on $\operatorname{supp} \mathrm{lt}_t(g)$, so that

$$\sigma_t(x) - x = h(\sigma_t(x) - x)k.$$

Now, by Lemma 4.3, when $t \in M$ the support of $(\sigma_t(x) - x)^*\tau_s(x)$ is compact and depends only on k and f. This proves the claim. Note that by the symmetry of our situation the action $(\tau \rtimes \mathrm{id}, \tau, \bar\tau)$ is continuous on ${}_{C\rtimes_{\sigma,r}K}Y_{C^\sigma}$. ∎

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In fact, our Theorem 3.7 gives us two $C^\tau \rtimes_{\bar\sigma,r} K$–$C^\sigma \rtimes_{\bar\tau,r} H$-imprimitivity bimodules and an isomorphism between the two:

Theorem 4.4 *Suppose that $\sigma\colon K \to \operatorname{Aut} C$ and $\tau\colon H \to \operatorname{Aut} C$ are commuting actions and that the actions of K and H on P are free and proper. If $C_0(P)$ maps bi-equivariantly into $M(C)$, then all the hypotheses of Theorem 3.7 are satisfied. That is, the two $C^\tau \rtimes_{\bar\sigma,r} K - C^\sigma \rtimes_{\bar\tau,r} H$-imprimitivity bimodules $(X \rtimes_{\sigma,r} K) \otimes_B (Y \rtimes_{\tau,r} H)$ and $\overline{X_0}$ are isomorphic.*

Proof In view of the proof of Proposition 4.2 we only need to verify items (2)and (3) of Theorem 3.7.

(2) (Properness) To see that $(\bar\sigma, \sigma, \sigma \rtimes \mathrm{id})$ is a proper action on ${}_{C^\tau}X_{C\rtimes_{\tau,r}H}$ with respect to ${}_{D_0}(X_0)_{E_0}$ we need to check items (a)–(d) of Theorem 3.7(2). For parts (a) and (b), note that if $x, v \in X_0$, then $t \mapsto x\sigma_t(v)$ has compact support by Lemma 4.3. Since

$$
{}_{C^\tau}\langle x\,, \sigma_t(v)\rangle c = \int_H \tau_s(x\sigma_t(v^*))c\,ds
$$

for every $c \in C_0$, it follows that the continuous function $t \mapsto {}_{C^\tau}\langle x\,, \sigma_t(v)\rangle$ has compact support with norm at most $\|x\|\|v\|$. Thus it and its product with $\Delta_K(t)^{-1/2}$ are integrable. By the symmetry of our situation the function $s \mapsto \Delta_H(s)^{-1/2}\langle x\,, \tau_s(v)\rangle_{C^\sigma}$ is in $L^1(H, C^\sigma)$.

For (c), we need

$$
\int_H \int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\,dt\,dr \in X_0 \quad \text{whenever } v, w, x \in X_0. \tag{4.3}
$$

Note that $r \mapsto \tau_r(x^*)v$ has compact support L, say, by Lemma 4.3. Let $w = fcg$ and choose $h \in C_c(P)$ such that $h = 1$ on $(\operatorname{supp} f) \cdot L^{-1}$. Then $\tau_r(w) = h\tau_r(w)$ for $r \in L$ so that the left-hand side of (4.3) equals

$$
h \int_H \int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\,dt\,dr,
$$

and is therefore an element of $C_c(P)C$. Since $t \mapsto w\sigma_t(x^*)$ is also compactly supported, we just repeat the argument for the right side of the integral to get that $\int_H \int_K \tau_r(w)\sigma_t(\tau_r(x^*)v)\,dt\,dr \in X_0 = C_c(P)CC_c(P)$.

For part (d), we check that

$$
(r, s, t) \mapsto u\tau_r(v^*)\sigma_t\big(\tau_r(w^*)\tau_s(x)\big)\,\Delta_H(s)^{-1/2}
$$

is integrable whenever $u, v, w, x \in X_0$. Again, by Lemma 4.3, the maps $r \mapsto u\tau_r(v^*)$ and $t \mapsto v^*\sigma_t(w^*)$ have compact supports. Thus it suffices to see that for r in a compact set L the function $s \mapsto \tau_r(w^*)\tau_s(x)$ has compact support. If $w^* = fcg$ and $x = kdh$ where $f, g, k, h \in C_c(P)$, then $\tau_r(w^*)\tau_s(x) = \tau_r(fc)\tau_r(g)\tau_s(k)\tau_s(dh)$ and $\tau_r(g)\tau_s(k)$ is nonzero if and only if $(\operatorname{supp} g) \cdot L^{-1} \cap (\operatorname{supp} k) \cdot s^{-1} \neq \varnothing$. Since the

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action of H on P is proper, the set $\{s \in H : (\operatorname{supp} g) \cdot L^{-1} \cap (\operatorname{supp} k) \cdot s^{-1} \neq \varnothing\}$ is compact.

(3) (Tensor decomposition isomorphism) That the action of $\sigma \rtimes \mathrm{id}$ of K on $C \rtimes_{\tau,r} H$ is saturated with respect to E_0 follows by noting that $E_0 = C_c(P)E_0C_c(P)$ and applying Lemma 4.1 with $A_0 = E_0$. To see that $E_0 = C_c(P)E_0C_c(P)$, let $x = fcg$ and $y = kdh \in C_0$ and let L be the compact support of $x\tau_s(y)$. Choose $l \in C_c(P)$ such that l is identically one on $(\operatorname{supp} h) \cdot L^{-1}$ and on $\operatorname{supp} f$. Then $x\tau_s(y) = fcg\tau_s(kdh) = l\big(fcg\tau_s(kdh)\big)\, l$.

Finally, if $e \in E_0$ is given by $e(s) = \Delta_H(s)^{-1/2} x\tau_s(y^*)$, then

$$
\begin{aligned}
(4.4)\quad \|e\|_{L^1(H,Y)} &= \int_H \|e(s)\|_Y \, ds = \int_H \|{}_{C \rtimes_{\sigma,r} K}\langle e(s)\, , e(s)\rangle\|^{1/2} \, ds \\
&\leq \int_H \int_K \|{}_{C \rtimes_{\sigma,r} K}\langle e(s)\, , e(s)\rangle(t)\|_C^{1/2} \, dt ds \\
&= \int_H \int_K \|e(s)\sigma_t(e(s)^*)\|_C^{1/2} \Delta_K(t)^{-1/4} \, dt ds \\
&= \int_H \int_K \|x\tau_s(y^*)\sigma_t(\tau_s(y)x^*)\|_C^{1/2} \Delta_H(s)^{-1/2} \Delta_K(t)^{-1/4} \, dt ds < \infty
\end{aligned}
$$

because the integrand is continuous with compact support (because $s \mapsto x\tau_s(y^*)$ and $t \mapsto y^*\sigma_t(y)$ have compact supports). Hence $E_0 \subset L^1(H, Y)$ as required. ∎

Remark 4.5 It has apparently not been noticed that Rieffel's construction in [25, Theorem 5.7] implies that the dual action on any crossed product by a coaction is proper. To see this, suppose $\delta\colon A \to M(A \otimes C^*(G))$ is a coaction of a locally compact group G on a C^*-algebra A. Then the crossed product $A \rtimes_\delta G$ is generated by a universal covariant representation $(j_A, j_{C(G)})$ of $(A, C_0(G))$ in $M(A \rtimes_\delta G)$. Since the dual action $\hat{\delta} : G \to \operatorname{Aut}(A \rtimes_\delta G)$ is characterized by

$$\hat{\delta}_s\big(j_A(a) j_{C(G)}(f)\big) = j_A(a) j_{C(G)}(\mathrm{rt}_s(f)),$$

the homomorphism $j_{C(G)}$ is equivariant for the actions $\hat{\delta}$ and $\mathrm{rt}\colon G \to \operatorname{Aut}(C_0(G))$. Applying [25, Theorem 5.7] to $j_{C(G)}$ shows that $\hat{\delta}$ is a proper action. More generally, it shows that $\hat{\delta}|_H$ is proper for any closed subgroup H of G; this improves a result of Mansfield [16, Theorem 30] for normal amenable H.

We can therefore apply Rieffel's original theorem from [24] to obtain a Morita equivalence between $(A \rtimes_\delta G) \rtimes_{\hat{\delta},r} H$ and a generalized fixed-point algebra $(A \rtimes_\delta G)^H$. Since $A \rtimes_\delta G$ is generated by the universal covariant representation $(j_A, j_{C(G)})$, and is even spanned by elements of the form $j_A(a) j_{C(G)}(f)$ (see [21, §2]), it is tempting to guess that $(A \rtimes_\delta G)^H$ is at least generated by elements of the form $j_A(a) j_{C(G)}(f)$ for $f \in C_0(G/H)$, and hence coincides with the candidate for the crossed product $A \rtimes_\delta G/H$ by the homogeneous space discussed in [6]. This is indeed the case if G is discrete [3]. Thus, Rieffel's theorem could give an extension of Mansfield's imprimitivity theorem to coactions of arbitrary homogeneous spaces (as opposed to quotients by normal amenable subgroups as in [16]; see [15] for a discussion of this

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problem for non-amenable normal subgroups). Unfortunately it does not seem to be easy to write a typical element of $C_c(G)(A \rtimes_\delta G)C_c(G)$ in the form $j_A(a)\, j_{C(G)}(f)$, or to do so approximately in such a way that one can verify the existence of the multiplier $\langle x\,,y\rangle_D$. Indeed, it is the content of one of Mansfield's main theorems [16, Theorem 19], that there are such multipliers when H is normal and amenable and x, y lie in a dense subalgebra $\mathcal{D}$ of $A \rtimes_\delta G$, and the proof of this theorem relies on some very subtle estimates. This analysis is a crucial ingredient in the proof of [16, Theorem 30].[1]

Pask and Raeburn showed in [19] that a free action on a directed graph E induces a proper action on the associated graph algebra $C^*(E)$. In their result, too, there is an underlying proper G-space P together with a non-central equivariant map of $C_0(P)$ into $M(C^*(E))$: just take P to be the set of vertices of the graph with the discrete topology.

Thus all the main examples of proper actions come with the existence of an underlying proper action on a space.

A Proof of Proposition 3.2

The object of these appendices is to make sense of certain manipulations with vector-valued integrals needed to give careful proofs of Proposition 3.2 and Lemma 3.3. If A is a C^*-algebra, then the collection of Bochner-integrable functions from G to A will be denoted by $\mathcal{L}^1(G, A)$, and the Banach space of equivalence classes of integrable functions agreeing almost everywhere will be denoted by $L^1(G, A)$.

For motivation, recall that we can realize $A \rtimes_\alpha G$ as the enveloping C^*-algebra of the Banach $*$-algebra $L^1(G, A)$. The product and involution are given by the usual formulas: for $f, g \in \mathcal{L}^1(G, A)$ we have

$$f * g(s) := \int_G f(r)\alpha_r\big(g(r^{-1}s)\big)\, ds, \tag{A.1}$$

and

$$f^*(s) := \Delta(s^{-1})\alpha_s\big(f(s^{-1})^*\big).$$

It takes some work, though, to see that f^* and $f * g$ are well-defined elements of $L^1(G, A)$. The first step is to see that $(r, s) \mapsto f(r)\alpha_r\big(g(r^{-1}s)\big)$ is a measurable function from $G \times G$ to A. This is a bit thorny as there is no *a priori* reason to suspect that $(r, s) \mapsto g(r^{-1}s)$ is measurable if g is merely measurable rather than continuous or Borel. There are a number of finesses for this. Here, we use the following lemma; we assume g is integrable to ensure that we can approximate it globally with functions in $C_c(G, A)$.

Lemma A.1 *Let (A, G, α) be a dynamical system. Suppose that $g \in \mathcal{L}^1(G, A)$ and $h(r, s) := \alpha_r\big(g(r^{-1}s)\big)$. Then $h\colon G \times G \to A$ is measurable.*

[1]This program has now been successfully implemented by the first two authors in [10].

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Proof Since Haar measure is a Radon measure, a subset S of G is measurable if and only if $S \cap C$ is measurable for all compact sets $C \subset G$ [9, Theorem III.11.31]. It follows that h is measurable if and only if $h|_L$ is measurable for each compact set $L \subset G \times G$. Therefore it will suffice to show that $h|_{K\times K}$ is measurable for each compact set $K \subset G$. To do this, we will produce measurable functions h_n such that $h_n \to h$ almost everywhere on $K \times K$.

Since $g \in \mathcal{L}^1(G, A)$, there are $g_n \in C_c(G, A)$ such that $g_n \to g$ in $L^1(G, A)$. Passing to a subsequence and relabeling, we can assume that there is a *Borel* null set N such that $g_n(s) \to g(s)$ for all $s \notin N$. Since g_n is continuous, $h_n(r, s) := \alpha_r\bigl(g_n(r^{-1}s)\bigr)$ defines a measurable function (continuous in fact), and

$$h_n(r, s) \to h(r, s)$$

for all $(r, s) \in K \times K \setminus D$, where $D = \bigl\{(r, s) \in K \times K : r^{-1}s \in N\bigr\}$. Since N is Borel, D is a measurable subset of $K \times K$. Since $K \times K$ has finite product-measure, Tonelli's Theorem, as proved in [9, Theorem III.13.9], implies that

$$\mu \times \mu(D) = \int_G \mu(D_r)\, dr,$$

where $D_r := \bigl\{s : (r, s) \in D\bigr\}$. Since $D_r \subset rN$ and $\mu(rN) = 0$ for all r, it follows that D is a null set. This completes the proof. ■

With Lemma A.1 in hand, the measurability of the function m given by $(r, s) \mapsto f(r)\alpha_r\bigl(g(r^{-1}s)\bigr)$ follows because the product of vector-valued measurable functions is measurable.[2] Since f and g are integrable, m must be supported on a σ-finite set, and Tonelli's Theorem shows that $\|m\|$ is in $\mathcal{L}^1(G \times G)$. Consequently, $m \in \mathcal{L}^1(G \times G, A)$. Now a vector-valued Fubini's Theorem, such as [8, Theorem II.16.3], implies that the right-hand side of (A.1) is defined for almost all s and that $f * g$ is a well-defined element of $L^1(G, A)$. (Since $\|f * g\|_1 \le \|f\|_1\|g\|_1$, it is not hard to see that the class of $f * g$ depends only on the classes of f and g.)

Remark A.2 Similar considerations are often glossed over when it is observed that convolution is associative. For example, if f, g and h are in $\mathcal{L}^1(G, A)$, then Lemma A.1 implies that

$$(r, t, s) \mapsto f(r)\alpha_r\bigl(g(r^{-1}t)\bigr)\,\alpha_t\bigl(h(t^{-1}s)\bigr) \tag{A.2}$$

is measurable, and Tonelli's Theorem implies that (A.2) is integrable on $G \times G \times G$. Then Fubini's Theorem implies that for almost all $s \in G$,

$$(r, t) \mapsto f(r)\alpha_r\bigl(g(r^{-1}t)\bigr)\,\alpha_t\bigl(h(t^{-1}s)\bigr) \tag{A.3}$$

is in $\mathcal{L}^1(G \times G, A)$. This will allow us to apply Fubini's Theorem to double integrals with integrands such as (A.3) in the sequel.[3]

[2] It suffices, for example, to see that a product of measurable simple functions is again a measurable simple function.

[3] It is possible (by approximating by simple functions) to see that (A.3) is measurable without resorting to functions on $G \times G \times G$. However, it is interesting to note that it is not obvious that (A.3) is integrable without appealing to the integrability of (A.2).

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We can view the multiplier algebra $M(A)$ as the C^*-algebra $\mathcal{L}(A_A)$ of adjointable operators on the right Hilbert A-module A_A.

Lemma A.3 *Suppose that (A, G, α) is a dynamical system and that T and S are bounded linear operators on $L^1(G, A)$ such that for all f and h in $L^1(G, A)$ we have*

$$T(f * h) = Tf * h, \quad S(f * h) = Sf * h, \quad \text{and} \quad (Tf)^* * h = f^* * Sh.$$

Then T and S extend to elements of $\mathcal{L}(A \rtimes_\alpha G) = M(A \rtimes_\alpha G)$ satisfying $T^* = S$.

Proof Let $\{e_i\}$ be a bounded approximate identity for $L^1(G, A)$. Then if π is a representation of $A \rtimes_\alpha G$,

$$\begin{aligned} \big\| \pi(Tf) \big\| &= \lim_i \big\| \pi\big(T(e_i * f) \big) \big\| \\ &\leq \limsup_i \big\| \pi\big(Te_i) \big) \big\| \, \big\| \pi(f) \big\| \\ &\leq M \|T\| \big\| \pi(f) \big\|. \end{aligned}$$

It follows that T is bounded with respect to the universal norm on $L^1(G, A) \subset A \rtimes_\alpha G$. Thus T and S extend to operators on $A \rtimes_\alpha G$. Since

$$\langle Tf \, , h \rangle_{A \rtimes_\alpha G} = (Tf)^* h = f^* Sh = \langle f \, , Sh \rangle_{A \rtimes_\alpha G},$$

it follows that T is adjointable with $T^* = S$. ∎

Remark A.4 If B is any C^*-subalgebra of $M(A)$, then we can identify $\mathcal{L}^1(G, B)$ with a subalgebra of $L^1(G, M(A))$. In particular, if $f \in \mathcal{L}^1(G, A)$ and $g \in \mathcal{L}^1(G, B)$, then Lemma A.1 implies that

$$(r, s) \mapsto f(r) \bar{\alpha}_r (g(r^{-1} s))$$

is a measurable function of $G \times G$ into $M(A)$ taking values in A. Thus, it is a measurable function of $G \times G$ into A. A similar statement can be made if $f \in \mathcal{L}^1(G, B)$ and $g \in \mathcal{L}^1(G, A)$.

Proof of Proposition 3.2 The integrand on the right-hand side of (3.4) is measurable in view of Remark A.4 and Lemma A.1. Thus applications of the Tonelli and Fubini Theorems imply that the right-hand side of (3.4) defines an element $T_g f$ in $L^1(G, A)$. Furthermore, $\|T_g\| \leq \|g\|_1$.

If $h \in \mathcal{L}^1(G, A)$, then $f * h$ is too, and by definition, for almost all s,

$$\begin{aligned} T_g(f * h)(s) &= \int_G g(r) \alpha_r \big(f * h(r^{-1} s) \big) \, dr \\ &= \int_G \int_G g(r) \alpha_r \big(f(t) \alpha_t \big(h(t^{-1} r^{-1} s) \big) \big) \, dt dr \\ &= \int_G \int_G g(r) \alpha_r \big(f(r^{-1} t) \big) \, \alpha_t \big(h(t^{-1} s) \big) \, dt dr. \end{aligned}$$

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The integrand $(r,t) \mapsto g(r)\alpha_r(f(r^{-1}t))\alpha_t(h(t^{-1}s))$ is in $\mathcal{L}^1(G\times G, A)$ by Remark A.2, so Fubini's Theorem implies that for almost all s,

$$\begin{aligned} T_g(f*h)(s) &= \int_G T_g f(t)\alpha_t\big(h(t^{-1}s)\big)\,dt \\ &= T_g f * h(s). \end{aligned}$$

Thus $T_g(f*h) = T_g f * h$ in $L^1(G,A)$.

Next we want to show that $(T_g f)^* * h = f^* * (T_{g^*}h)$ in $L^1(G,A)$. But for almost all s,

$$\begin{aligned} (T_g f)^* * h(s) &= \int_G (T_g f)^*(r)\alpha_r\big(h(r^{-1}s)\big)\,dr \\ &= \int_G \alpha_r\big(T_g f(r^{-1})\big)^*\Delta(r^{-1})\alpha_r\big(h(r^{-1}s)\big)\,dr \\ &= \int_G\int_G \alpha_r\Big(g(t)\alpha_t\big(f(t^{-1}r^{-1})\big)\Big)^*\Delta(r^{-1})\alpha_r\big(h(r^{-1}s)\big)\,dtdr \\ &= \int_G\int_G \alpha_{r^{-1}t}\big(f(t^{-1}r)^*\big)\,\alpha_{r^{-1}}\big(g(t)^*\big)\,\alpha_{r^{-1}}\big(h(rs)\big)\,dtdr. \end{aligned}$$

It follows from simple variations on Lemma A.1 that the integrand above is measurable, and (the scalar version of) Tonelli's Theorem implies it is integrable. Hence we can use the vector-valued version of Fubini's Theorem to conclude that, for almost all s,

$$(T_g f)^* * h(s) = \int_G\int_G \alpha_{r^{-1}t}\big(f(t^{-1}r)^*\big)\,\alpha_{r^{-1}}\big(g(t)^*\big)\,\alpha_{r^{-1}}\big(h(rs)\big)\,drdt$$

and, since it now makes sense to send $r \mapsto tr$, this is equal to

$$\int_G\int_G \alpha_{r^{-1}}\big(f(r)^*\big)\,\alpha_{r^{-1}t^{-1}}\big(g(t)^*h(trs)\big)\,drdt,$$

which, after sending $r \mapsto r^{-1}$ and $t \mapsto t^{-1}$, is equal to

$$\int_G\int_G f^*(r)\alpha_r\Big(g^*(t)\alpha_t\big(h(t^{-1}r^{-1}s)\big)\Big)\,drdt.$$

To apply Fubini's Theorem we need to see that the above integrand is a measurable function of r, t and s; as before, this follows from variations on Lemma A.1. Thus for almost all s,

$$\begin{aligned} \text{(A.4)} \qquad (T_g f)^* * h(s) &= \int_G f^*(r)\alpha_r\Big(\int_G g^*(t)\alpha_t\big(h(t^{-1}r^{-1}s)\big)\,dt\Big)\,dr \\ &= \int_G f^*(r)\alpha_r\big(T_{g^*}h(r^{-1}s)\big)\,dr \\ &= f^* * (T_{g^*}h)(s). \end{aligned}$$

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It follows from Lemma A.3 that T_g defines a multiplier with adjoint $T_g^* = T_{g^*}$.

Next we want to establish (3.5). Fubini's Theorem implies that the right-hand side of (3.5) defines a function l in $\mathcal{L}^1(G, A)$. Since $fT_g = (T_{g^*} f^*)^*$ in $A \rtimes_\alpha G$, it follows that $fT_g \in L^1(G, A)$, and we have to show that $l = fT_g$ in $L^1(G, A)$.

Let $h \in \mathcal{L}^1(G, A)$. We can repeat the computation of $(T_g f)^* * h$ above with f replaced by f^* and g replaced by g^*, and use (A.4) to conclude that

$$\begin{aligned}(fT_g) * h(s) = \bigl(T_{g^*} f^*\bigr)^* * h(s) &= \int_G \int_G f(r)\alpha_r\bigl(g(t)\alpha_t\bigl(h(t^{-1}r^{-1}s)\bigr)\bigr)\,dt\,dr \\ &= \int_G \int_G f(r)\bar{\alpha}_r\bigl(g(r^{-1}t)\bigr)\,\alpha_t\bigl(h(t^{-1}s)\bigr)\,dt\,dr,\end{aligned}$$

which by Fubini's Theorem is equal to

$$\begin{aligned}\int_G \Bigl(\int_G f(r)\bar{\alpha}_r\bigl(g(r^{-1}t)\bigr)\,dr\Bigr)\,\alpha_t\bigl(h(t^{-1}s)\bigr)\,dt &= \int_G l(t)\alpha_t\bigl(h(t^{-1}s)\bigr)\,dt \\ &= l * h(s).\end{aligned}$$

It follows that $l = fT_g$ in $L^1(G, A)$, as claimed. ■

B The Proof of Lemma 3.3

To begin with, let C be an arbitrary C^*-algebra. We let H and K be arbitrary locally compact groups with Haar measures μ_H and μ_K, respectively. (For most of what follows, C could be any Banach space and H and K could be arbitrary locally compact spaces equipped with Radon measures μ_H and μ_K, respectively.)

If $f \in L^1(H, C)$, then we will sometimes write

$$\int_H^C f(s)\,ds$$

to emphasize where our integral takes its value.

The key lemma is a very special case of [4, Lemma III.11.17], and, modulo some facts about vector-valued integrals, has a fairly straightforward proof. We include the proof here because the arguments from [4] are difficult and can be substantially simplified in our situation.

Lemma B.1 *Suppose that $g \in \mathcal{L}^1\bigl(K, L^1(H, C)\bigr)$ and that f is a $\mu_K \times \mu_H$-integrable C-valued function on $K \times H$ such that for almost all t, the class of $f(t, \cdot)$ equals $g(t)$. Then*

(a) *for almost all $s \in H$, $f(\cdot, s) \in \mathcal{L}^1(K, C)$,*

(b) *the function*

$$s \mapsto \int_K^C f(t, s)\,dt$$

is in $\mathcal{L}^1(H, C)$, and

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(c) *as elements of* $L^1(H,C)$,

$$\int_K^{L^1(H,C)} g(t)\,dt = \left(s \mapsto \int_K^C f(t,s)\,dt\right);$$

that is, for almost all s,

$$\int_K^{L^1(H,C)} g(t)\,dt(s) = \int_K^C f(s,t)\,dt.$$

The first two assertions follow immediately from any vector-valued Fubini Theorem worthy of the name. Our proof of the third assertion is straightforward except for the following result which is [4, Lemma III.6.8].

Lemma B.2 *If* $f \in \mathcal{L}^1(H,C)$ *and if*

$$\int_E f(s)\,ds = 0$$

for all measurable subsets $E \subset H$*, then* f *vanishes almost everywhere.*

Although the proof of Lemma B.2 is routine in the scalar-valued case, we see no elementary proof in the vector-valued case. The proof in [4, III §2] goes as follows. Given $h \in \mathcal{L}^1(H,C)$, define a C-valued set function on measurable subsets of H by

$$\lambda(E) := \int_E^C h(s)\,ds.$$

Of course, λ is additive, and has a *total variation* defined by

$$\nu(E) := \sup\Big\{ \sum_i \|\lambda(E_i)\| : E_1, \dots, E_n \text{ is a partition of } E \Big\}.$$

One sees easily that

$$\nu(E) \leq \int_E \|h(s)\|\,ds.$$

But it can be shown [4, Theorem III.2.20] that

$$\nu(E) = \int_E \|h(s)\|\,ds.$$

With these assertions in place, Lemma B.2 is an easy consequence; if $h = f$ as in Lemma B.2, then $\nu(E) = 0$ for all E, and so f is zero almost everywhere.

The only other tool we need for the proof of Lemma B.1 is that bounded linear maps commute with vector-valued integrals. For each measurable subset $E \subset H$, we can define a bounded linear map $\varphi_E \colon L^1(H,C) \to C$ by

$$\varphi_E(h) := \int_E^C h(s)\,ds.$$

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Proof of Lemma B.1 To prove the final assertion, it suffices, in view of Lemma B.2, to show that for all measurable subsets $E \subset H$,

$$\varphi_E\Big(\int_K^{L^1(H,C)} g(t)\,dt\Big) = \varphi_E\big(s \mapsto \int_K^C f(t,s)\,dt\big)\,. \tag{B.1}$$

Since bounded linear maps commute with integrals, the left-hand side of (B.1) is

$$\int_K^C \varphi_E(g(t))\,dt = \int_K^C \int_E^C g(t)(s)\,dsdt$$

which, by assumption, is equal to

$$\int_K^C \int_E^C f(t,s)\,dsdt$$

which, since having $f \in \mathcal{L}^1(K \times E, C)$ allows us to apply Fubini's Theorem, is equal to

$$\int_E^C \int_K^C f(t,s)\,dtds = \varphi_E\big(s \mapsto \int_K^C f(t,s)\,dt\big)\,.$$

This establishes (B.1) and completes the proof. ∎

Example B.3 Now suppose that (C, H, α) is a dynamical system, and that h and k are in $\mathcal{L}^1(H, C)$. Then

$$h * k = \int_H^{L^1(H,C)} h(r) i_H(r)(k)\,dr.$$

Proof of the Example We want to apply Lemma B.1 with

$$g(r) := h(r) i_H(r)(k) \quad \text{and} \quad f(r,s) := h(r) i_H(r)(k)(s).$$

In order to do so, we have to check that g and f are integrable. However, $f(r,s) = h(r)\alpha_r\big(k(r^{-1}s)\big)$, and f is known to be in $\mathcal{L}^1(K \times H, C)$ by standard arguments.

Note that $r \mapsto i_H(r)(k)$ is continuous from H to $L^1(H, C)$. It follows that $r \mapsto h(r) i_H(r)(k)$ is measurable from H to $L^1(H, C)$ and has σ-finite support. Since

$$\|h(r) i_H(r)(k)\| \le \|k\|_1 \|h(r)\|,$$

it follows from Tonelli that $g \in \mathcal{L}^1(K, L^1(H, C))$.

Now the result follows immediately from Lemma B.1: for almost all s,

$$\int_H^{L^1(H,C)} g(r)\,dr(s) = \int_H^C f(r,s)\,dr = h * k(s). \qquad ∎$$

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Remark B.4 In the statement of Lemma 3.3, we view the e_i as elements of $C \rtimes_{\tau,r} H$, and consequently we write their product as $e_1 e_2$. In the proof however, we will want to use that each e_i is in $L^1(H, C)$, and that the product is given by convolution. Thus it will be a bit clearer to use the usual notation $e_1 * e_2$ for their product.

Proof of Lemma 3.3 Let

$$h(r, t, s) := e_1(r) i_H(r)\big((\sigma \rtimes \mathrm{id})_t(e_2)\big)(s). \tag{B.2}$$

Our assumptions imply that h is integrable on $H \times H \times K$: for example if $e_1(r) = \Delta_H(r)^{-\frac{1}{2}} u\tau_r(v^*)$ and $e_2(r) = \Delta_H(r)^{-\frac{1}{2}} w^*\tau_r(x)$, then

$$\begin{aligned} h(r, t, s) &= \Delta_H(r)^{-\frac{1}{2}} u\tau_r(v^*)\tau_r\big((\sigma \rtimes \mathrm{id})_t(e_2)\big)(r^{-1}s) \\ &= \Delta_H(r)^{-\frac{1}{2}} u\tau_r(v^*)\tau_r\big(\sigma_t(w^*\tau_{r^{-1}s}(x))\big)\,\Delta_H(r^{-1}s)^{-\frac{1}{2}} \\ &= u\tau_r(v^*)\sigma_t\big(\tau_r(w^*)\tau_s(x)\big)\,\Delta_H(s)^{-\frac{1}{2}}, \end{aligned}$$

which is integrable by assumption. In general, the integrability of h follows as e_1 and e_2 are sums of functions of the form given above.

Now Fubini's Theorem implies that $r \mapsto h(r, s, t)$ is integrable for almost all (t, s), and that

$$f(t, s) := \int_H^C h(r, t, s)\, dr \tag{B.3}$$

defines an integrable function f on $K \times H$.[4] Furthermore, it follows from Example B.3 that for almost all t and s,

$$f(t, s) = e_1 * (\sigma \rtimes \mathrm{id})_t(e_2)(s).$$

Next we define $g(t) := e_1 * (\sigma \rtimes \mathrm{id})_t(e_2)$. Then g is a function from K to $\mathcal{L}^1(H, C)$ and $g(t) = f(t, \cdot\,)$ for almost all t. We want to see that $g \in \mathcal{L}^1(K, L^1(H, C))$. However, $t \mapsto (\sigma \rtimes \mathrm{id})_t(e_2)$ is continuous from K to $L^1(H, C)$. Therefore g itself is continuous, hence measurable, and it suffices to see that $\|g\|$ is integrable. But

$$\begin{aligned} \int_K \|g(t)\|_1\, dt &= \int_K \int_H \|g(t)(s)\|\, ds dt \\ &= \int_K \int_H \|f(t, s)\|\, ds dt \\ &= \|f\|_1 < \infty. \end{aligned}$$

Therefore $g \in \mathcal{L}^1(K, L^1(H, C))$. Now an application of Lemma B.1 implies that (3.8) is a representative for

$$\int_K^{L^1(H,C)} g(t)\, dt.$$

[4] Of course, (B.3) only defines f almost everywhere. As is standard practice, we assume the convention that $f(t, s) := 0$ when $r \mapsto h(r, t, s)$ is not integrable.

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Since inclusion is a bounded linear map of $L^1(H, C)$ into $C \rtimes_{\tau,r} H$, it follows that (3.8) is also a representative for

$$\int_K^{C \rtimes_{\tau,r} H} g(t)\, dt,$$

which is what we wanted to prove. ∎

C The Proof of Lemma 4.1

Lemma C.1 (Rieffel) *Suppose G acts freely and properly on a locally compact Hausdorff space P, there is a nondegenerate homomorphism $\theta\colon C_0(P) \to M(A)$, and an action α of G on A such that $\alpha_s(\theta(f)a) = \theta(\mathrm{lt}_s(f))\alpha_s(a)$. Then the proper action α of G on A is saturated with respect to $A_0 := \theta(C_c(P))A\theta(C_c(P))$.*

For the proof, we need to know that the action of G on $C_0(P)$ is saturated in the sense of [24]; this is proved, for example, in [23].

Let $E_0 = {}_E\langle A_0\,, A_0\rangle \subset L^1(G, A)$ where E is the closure of E_0 in $A \rtimes_{\alpha,r} G$. It suffices to see that E_0 is dense in $L^1(G, A)$ in the inductive limit topology. To see this, it suffices to see that functions of the form $s \mapsto (\varphi \otimes a)(s) := \varphi(s)a$, where $\varphi \in C_c(G)$ and a is in any dense subset of A, belong to E. Since A_0^2 is dense in A, we can fix $f, p, k \in C_c(P)$, $a, b \in A_0$ and $\varphi \in C_c(G)$ and let

$$F(s) := \varphi(s)\theta(f)a\theta(p)b^*\theta(\bar{k}), \tag{C.1}$$

and it will suffice to see that F can be approximated in the inductive limit topology by elements of E_0.

First we prove a related statement: if φ, a, b, f, p and k are as above then

$$s \mapsto \varphi(s)\theta(f)a\theta(p)\alpha_s(b^*)\theta\big(\mathrm{lt}_s(\bar{k})\big)$$

belongs to E. Let W be a compact neighborhood of $\operatorname{supp}\varphi$. Because the action of G on $C_0(P)$ is proper and saturated with respect to $C_c(P)$, given $\epsilon > 0$, we can find $g_i, h_i \in C_c(P)$ such that

$$\Big\| \varphi(s)p - \Delta(s)^{-1/2} \sum_i g_i\, \mathrm{lt}_s(\bar{h}_i) \Big\|_\infty < \frac{\epsilon}{\|f\|_\infty \|a\| \|b\| \|k\|_\infty} \quad \text{and}$$

$$\operatorname{supp}\Big(s \mapsto \Delta(s)^{-1/2} \sum_i g_i\, \mathrm{lt}_s(\bar{h}_i) \Big) \subset W.$$

Now let

$$a_i := \theta(f)a\theta(g_i) \quad \text{and} \quad b_i := \theta(k)b\theta(h_i).$$

Then

$$\begin{aligned} \sum_i {}_E\langle a_i\,, b_i\rangle(s) &= \sum_i \Delta(s)^{-1/2} a_i \alpha_s(b_i^*) \\ &= \theta(f)a\theta\Big(\Delta(s)^{-1/2} \sum_i g_i\, \mathrm{lt}_s(\bar{h}_i) \Big)\, \alpha_s(b^*)\theta\big(\mathrm{lt}_s(\bar{k})\big)\,. \end{aligned}$$

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Now
$$\Big\| \theta\Big(\Delta(s)^{-\frac{1}{2}} \sum_i g_i \operatorname{lt}_s(\bar{h}_i) \Big) - \varphi(s)\theta(p) \Big\| < \frac{\epsilon}{\|f\|_\infty \|a\| \|b\| \|k\|_\infty}.$$

Thus
$$\Big\| \sum_i {}_E\langle a_i\,, b_i\rangle(s) - \varphi(s)\theta(f)a\theta(p)\alpha_s(b^*)\theta\big(\operatorname{lt}_s(\bar{k})\big) \Big\| < \epsilon.$$

Since the neighborhood W does not depend on ϵ, and ϵ is arbitrary, it follows that the function $s \mapsto \varphi(s)\theta(f)a\theta(p)\alpha_s(b^*)\theta\big(\operatorname{lt}_s(\bar{k})\big)$ is in E.

Now let N be a neighborhood of e in G such that $s \in N$ implies that
$$\|b^*\theta(\bar{k}) - \alpha_s(b^*)\theta\big(\operatorname{lt}_s(\bar{k})\big)\| = \|b^*\theta(\bar{k}) - \alpha_s\big(b^*\theta(\bar{k})\big)\| < \frac{\epsilon}{\|\varphi\|_\infty \|f\|_\infty \|a\| \|p\|_\infty}.$$

Choose $r_1, \ldots, r_n \in G$ such that $\operatorname{supp}\varphi \subset \bigcup N r_i$. Let $\{\varphi_i\} \subset C_c^+(G)$ be such that $\operatorname{supp}\varphi_i \subset N r_i$ and $\sum_i \varphi_i \equiv 1$ on $\operatorname{supp}\varphi$ and dominated by 1 elsewhere. We showed above that
$$F_i(s) := \varphi(s)\varphi_i(s)\theta(f)a\theta(p)\alpha_{sr_i^{-1}}(b^*)\theta\big(\operatorname{lt}_{sr_i^{-1}}(\bar{k})\big)$$
defines an element of E; if
$$F(s) := \varphi(s)\theta(f)a\theta(p)b^*\theta(\bar{k}),$$
then
$$\begin{aligned}\Big\| F(s) - \sum F_i(s) \Big\| &= \Big\| \sum \varphi_i(s)F(s) - F_i(s) \Big\| \\ &= \Big\| \sum \varphi(s)\varphi_i(s)\theta(f)a\theta(p)\big(b^*\theta(\bar{k}) - \alpha_{sr_i^{-1}}(b^*\bar{k})\big) \Big\| \\ &\le \sum \varphi_i(s)\|\varphi\|_\infty \|f\|_\infty \|a\| \|p\|_\infty \|b^*\theta(\bar{k}) - \alpha_{sr_i^{-1}}\big(b^*\theta(\bar{k})\big)\|\end{aligned}$$
which, since we may assume $s \in N r_i$, is
$$\le \epsilon \sum \varphi_i(s) \le \epsilon.$$

Since $\sum_i F_i \in E$ and both $\operatorname{supp} F_i$ and $\operatorname{supp} F \subset \operatorname{supp}\varphi$, the result follows.

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https://doi.org/10.4153/CJM-2005-038-2 Published online by Cambridge University Press