ON MORITA DUALITY

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1. Introduction. A contravariant category-equivalence between categories $\mathfrak{A}$, $\mathfrak{B}$ of right R-modules and left S-modules (all rings have units, all modules are unitary) that contain R_R, ${}_SS$ and are closed under submodules and factor modules, is naturally equivalent to a functor Hom $(-, U)$ with a bimodule ${}_SU_R$ such that ${}_SU$, U_R are injective cogenerators with $S = \text{End } U_R$ and $R = \text{End }{}_SU$, and all modules in $\mathfrak{A}$, $\mathfrak{B}$ are U-reflexive. Conversely, for any ${}_SU_R$, Hom$(-, U)$ is a contravariant category equivalence between the categories of U-reflexive modules, and if U has the properties just stated, then these categories are closed under submodules, factor modules, and finite direct sums and contain R_R, U_R, ${}_SS$, and ${}_SU$. Such a functor will be called a (Morita) duality between R and S induced by U (see (**5**)).

The following question naturally arises: Which rings R possess a duality? Osofsky (**7**) has shown that if R has duality, it is semi-perfect. Then U_R will be a finite direct sum of all the isomorphism types of injective hulls of simple right R-modules (Lemma 1), and $S = \text{End } U_R$. We call $\langle R, U_R, S\rangle$ "standard" if R is a semi-perfect ring, U_R is the minimal (injective) cogenerator, and $S = \text{End } U_R$. Here, a duality exists if and only if:

(1) ${}_SU$ is an injective cogenerator, and

(2) $R = R'$, the second commutator End ${}_SU$ of U_R.

We would like to replace these two conditions by more explicit ones like those known in the following two cases: If R is right-Artinian, then duality exists if and only if U_R has finite length (**5**; **1**). If R is commutative-Noetherian, then duality exists if and only if R is complete (**4**).

We shall show that condition (2) ($R = R'$) holds if and only if R is complete in that uniformity for which the completely meet-irreducible right ideals form a subbase for the neighbourhoods of 0. Then we investigate rings R for which the intersection of the powers of the Jacobson radical is zero; such a ring turns out to have duality if and only if it is right-Noetherian, complete (in the topology defined by the powers of the radical), and if U_R is Artinian.

2. Prerequisites. A semi-perfect ring possesses a basic ring which is similar to it; hence we may (and will to simplify some formulations) assume that R and S are self-basic rings.

LEMMA 1. *If we have duality between* (*self-basic*) *rings R and S, then* ${}_SU$ *and*

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U_R are the minimal cogenerators (the direct sums of the injective hulls of the simple modules).

Proof. Every cogenerator contains the minimal cogenerator. Since ${}_SS$ is finitely generated and the lattices of submodules of ${}_SS$ and U_R are anti-isomorphic, U_R is co-finitely generated, hence essential over a finite socle, therefore a finite direct sum of injective hulls of all types of simple R-modules. Since $S = \text{End } U_R$ is self-basic, these summands are mutually non-isomorphic.

Next we collect a number of essentially known facts when $\langle R, U_R, S\rangle$ is standard. S is self-basic semi-perfect (cf. **3**, § 4.4); the radical factor rings $R/\text{rad } R$ and $S/\text{rad } S$ are isomorphic under $s \leftrightarrow x$, given by $su = ux$ for all u in the socle Soc U_R. Since Soc U_R is isomorphic to $R/\text{rad } R_R$, it is actually an S-R-submodule of U; it coincides with the S-socle of U and as such is isomorphic to ${}_SS/\text{rad } S$. ${}_SU$ is essential over its socle. The maps $I \to \text{Ann}_U I$ and $W \to \text{Ann}_S W$ onto the annihilators in U and S are order-inverting one-to-one maps of the sets of submodules of R_R and U_R into the sets of submodules of ${}_SU$ and ${}_SS$; actually, $\text{Ann}_R \text{Ann}_U I = I$ and $\text{Ann}_U \text{Ann}_S W = W$. Simple right R-modules and left S-modules are U-reflexive; semi-simple modules are reflexive if and only if they are of finite length; submodules of reflexive R-modules are reflexive. ${}_SU$ is an injective cogenerator if it is only injective or a cogenerator. If U_R is of finite length, then ${}_SU$ is injective (cf., e.g., **1**).

3. A topology on rings. For any faithful module X_R over an arbitrary ring R, the second commutator R' is a topological ring under the finite topology which is Hausdorff and actually defines a uniform structure, in a natural way. R is embedded into R' and is hence topologized by the relative topology which will be called the X-topology (uniformity). The Hausdorff completion $\hat{R}$ of R with respect to this uniformity operates on X (a generalized Cauchy sequence $\hat{r} = (r_\alpha)$ satisfies $xr_\alpha = xr_\beta$ for large α, β and fixed $x \in X$; define this value to be $x\hat{r}$) and we obtain an embedding $R \subseteq \hat{R} \subseteq R'$ since for $s \in \text{End } X_R$, $x \in X$, $\hat{r} = (r_\alpha) \in \hat{R}$, and sufficiently large α, we have $s(x\hat{r}) = s(xr_\alpha) = (sx)r_\alpha = (sx)\hat{r}$. Consequently, X-completeness of R is necessary for the second commutator relation to hold for X_R.

LEMMA 2. *If X_R is a cogenerator, then R is dense in R'.*

Proof. Let $x_1, \ldots, x_n \in X$ and $\rho \in R'$ be given. Set

$$Y = \bigoplus_{i=1}^{n} X, \qquad y = (x_1, \ldots, x_n),$$

and observe that the second commutator of Y is R'. Then for all $s \in \text{End } Y_R$ with $sy = 0$ we obtain $0 = (sy)\rho = s(y\rho)$, hence

$$y\rho \in \bigcap\{\text{Ker } s \mid s \in \text{End } Y_R, sy = 0\} = K.$$

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Consider any $y' \in Y$, $y' \notin yR$; then we have a maximal submodule

$$y'R + yR \supseteq M \supseteq yR$$

hence a map $Y \to Y$ which vanishes on yR but not on y', since Y is a cogenerator. Consequently, $y' \notin K$ and $K = yR$; therefore $y\rho = yr$ for a suitable $r \in R$, which means that $x_i\rho = x_ir$, $i = 1, \ldots, n$.

Observing that $\hat{R}$ is the closure of R in R', we see that for a cogenerator X, X-completeness of R is necessary and sufficient for the second commutator relation to hold. In particular, this is valid for the minimal cogenerator U. We characterize the U-topology intrinsically.

LEMMA 3. *The completely meet-irreducible right ideals of R form a subbase for the neighbourhoods of* 0 *in the U-topology.*

Proof. The right ideals $\mathrm{Ann}_R\, u$, where u is any element of an injective hull of any simple right R-module T, form a subbase. We have $uR \cong R/\mathrm{Ann}_R\, u$; and a cyclic module R/I is isomorphic to some uR if and only if it is essential over a simple submodule isomorphic to T. This means that the right ideals properly containing I all contain one right ideal $L \supseteq I$ with $L/I \cong T$; hence I is completely meet-irreducible.

THEOREM 4. *The following are equivalent for any ring R:*

(1) *R is complete in the completely meet-irreducible uniformity;*

(2) *The minimal right cogenerator U satisfies the second commutator relation;*

(3) *Every right cogenerator satisfies the second commutator relation;* cf. (**4**, Theorem 16.2).

Proof. It remains to show that (2) implies (3). If X is a cogenerator, it contains the injective hulls of all simple R-modules, as direct summands; hence $U \oplus P \cong \oplus X = Y$. The second commutator of X is the same as that of Y, and the latter is mapped, by restriction to U, into the second commutator of U. This map is onto since the second commutator of U is R, by assumption. For any $\rho \neq 0$ in the second commutator of Y and $y \in Y$ with $y\rho \neq 0$ we obtain a map σ from Y to U that does not vanish on $y\rho$, since U is a cogenerator. Since $\sigma: Y \to U \subseteq Y$ is in $\mathrm{End}\, Y_R$, we have $0 \neq \sigma(y\rho) = (\sigma y)\rho \in U\rho$, hence $\rho | U \neq 0$ and the restriction map is one-to-one; consequently, the second commutators of X, Y, and U all agree, and the last one is R.

Examples. (1) The following statements are equivalent:

(i) The X-topology is discrete (and therefore R is X-complete);

(ii) There exist elements $x_1, \ldots, x_n \in X$ with $\bigcap_{i=1}^n \mathrm{Ann}_R\, x_i = 0$;

(iii) R_R is embeddable in a finite direct sum of copies of X_R.

In case $\langle R, U_R, S\rangle$ is standard and $X = U_R$, then (i)–(iii) are equivalent to: R_R is essential over a finite socle.

(2) Consequently, a right-Artinian ring R is discrete in the X-topology for every faithful X_R; in particular, every cogenerator over an Artinian ring satisfies the second commutator relation.

(3) Let $\langle R, U_R, S\rangle$ be standard; then $R = R'$ means that R_R is reflexive, hence all submodules are reflexive and Soc R_R is of finite length. Since every right module over a left-perfect ring is essential over its socle, a left-perfect ring is complete in the completely meet-irreducible topology if and only if Soc R_R is of finite length.

(4) Considering right-Artinian rings whose left socle is infinite, we obtain examples of rings that are discrete and complete in the (right-) completely meet-irreducible topology, but neither discrete nor complete in the analogously defined left topology.

(5) If R is a commutative Noetherian ring, the Artin-Rees Lemma implies that the radical topology is finer than the completely meet-irreducible topology. Conversely, if R is also semi-local, then $R/\text{rad } R^n$ is Artinian, hence discrete, which shows that rad R^n is open in the completely meet-irreducible topology, and that both topologies agree. If R is not semi-local, this will no longer hold, in general: e.g., for a Dedekind domain R, the open ideals in the completely meet-irreducible topology are precisely the non-zero ideals, while rad $R = 0$ if R is not semi-local.

4. Rings with $\bigcap_{n=0}^{\infty} \text{rad } R^n = 0$.

LEMMA 5. *If* $\bigcap_{n=0}^{\infty} \text{rad } R^n = 0$, *if* R *is complete in the radical topology, and if* X *is a right ideal such that* $X/X \text{ rad } R$ *is finitely generated, then* X *is finitely generated.*

Proof. By assumption, we have $X = \sum_{i=1}^{n} x_i R + XJ$, where $J = \text{rad } R$, which implies that $XJ^n = \sum_{i=1}^{n} x_i J^n + XJ^{n+1}$. Therefore, any $x \in X$ may be written as

$$x = \sum_{i=1}^{n} x_i r_i^{(0)} + \ldots + \sum_{i=1}^{n} x_i r_i^{(n)} + x^{(n+1)}, \quad r_i^{(k)} \in J^k, x^{(n+1)} \in XJ^{n+1} \subseteq J^{n+1};$$

hence $x = \sum_{i=1}^{n} x_i (r_i^{(0)} + \ldots + r_i^{(n)}) + x^{(n+1)}$. The sequences

$$r_i^{(0)} + \ldots + r_i^{(n)}$$

converge to limits $\hat{r}_i$, and we obtain $x = \sum_{i=1}^{n} x_i \hat{r}_i$, hence $X = \sum_{i=1}^{n} x_i R$.

LEMMA 6. *If* $\langle R, U_R, S\rangle$ *is standard and if* U_R *is Artinian, then*

$$\text{Ann}_S \text{Ann}_U \text{rad } R^n = \text{rad } S^n.$$

Proof. It is well known that $\text{rad } S = \text{Ann}_S \text{Soc } U_R = \text{Ann}_S \text{Soc}_S U = \text{Ann}_S \text{Ann}_U \text{rad } R$, since U_R is the injective hull of Soc U_R (cf. 3, § 4.4). Suppose that the statement is true for n, and consider $s \in \text{rad } S$, $t \in \text{rad } S^n$. With $J = \text{rad } R$ and $U_k = \text{Ann}_U J^k$, we obtain $sU_{n+1}J^n \subseteq sU_1 = 0$ since $U_{n+1}J^nJ = 0$; therefore $sU_{n+1} \subseteq U_n$ and $tsU_{n+1} \subseteq tU_n = 0$, and consequently $\text{rad } S^{n+1} U_{n+1} = 0$ and $\text{rad } S^{n+1} \subseteq \text{Ann}_S \text{Ann}_U \text{rad } R^{n+1}$; observe that this inclusion does not require the assumption that U_R is Artinian.

Now consider any $s \in \operatorname{Ann}_S U_{n+1}$. Since U_R is a cogenerator, we obtain $\bigcap\{\operatorname{Ker} t | \, t \in S, \, tU_n = 0\} = U_n$, and since U is Artinian, this reduces to a finite intersection $\bigcap_{i=1}^m \operatorname{Ker} t_i = U_n$. Define $t_0 = \bigoplus_{i=1}^m t_i$: $U \to \bigoplus_{i=1}^m U$, let H be an injective hull of the image $\operatorname{Im} t_0$ of t_0 in $\oplus U$; then $\oplus U = H \oplus H'$. The map st_0^{-1}: $\operatorname{Im} t_0 \to U$ is well-defined since $\operatorname{Ker} s \supseteq U_{n+1} \supseteq U_n = \operatorname{Ker} t_0$; it is extendable to H by the injectivity of U_R, and to $\oplus U$ by setting it equal to zero on H'; call the resulting map $b = \oplus b_i$: $\oplus U \to U$. We have

$$s = bt_0 = \sum_{i=1}^m b_i t_i \quad \text{and} \quad t_i \in \operatorname{Ann}_S U_n = \operatorname{rad} S^n,$$

by hypothesis of induction. The socle of $\operatorname{Im} t_0$ is annihilated by $\operatorname{rad} R$, hence contained in $t_0(U_{n+1})$, and therefore $0 = sU_{n+1} = bt_0 U_{n+1} \supseteq b(\operatorname{Soc} \operatorname{Im} t_0) = b(\operatorname{Soc} H)$; on the other hand, $b(\operatorname{Soc} H') \subseteq b(H') = 0$ and consequently $b(\operatorname{Soc} \oplus U) = 0$ hence $b_i(\operatorname{Soc} U) = 0$ and $b_i \in \operatorname{rad} S$. This proves that

$$s = \sum_{i=1}^m b_i t_i \in \operatorname{rad} S^{n+1}.$$

THEOREM 7. *Suppose that* $\bigcap_{n=0}^\infty \operatorname{rad} R^n = 0$, *and that R has duality with a ring S, induced by a bimodule U. Then R is right-Noetherian, S is left-Noetherian,* $\bigcap_{n=0}^\infty \operatorname{rad} S^n = 0$, *$R$ and S are complete in the radical topology, ${}_SU$ and U_R are Artinian,* $\operatorname{Ann}_U \operatorname{rad} R^n = \operatorname{Ann}_U \operatorname{rad} S^n$ *for all n, and* $U = \bigcup_{n=0}^\infty \operatorname{Ann}_U \operatorname{rad} R^n$.

Proof. The existence of the duality implies that for every right ideal X of R, $X/X \operatorname{rad} R$ is reflexive and semi-simple, hence of finite length. Furthermore, we have order anti-isomorphisms of the lattices of submodules of ${}_SS$ and R_R with the lattices of submodules of U_R and ${}_SU$ given by the annihilators. Therefore, $\bigcap \operatorname{rad} R^n = 0$ implies $\bigcup \operatorname{Ann}_U \operatorname{rad} R^n = U$. Next we show that the U-topology and the radical topology on R agree: For every $u \in U$ we obtain $u \in \operatorname{Ann}_U \operatorname{rad} R^n$ for some n, hence $\operatorname{rad} R^n \subseteq \operatorname{Ann}_U u$ and the radical topology is finer than the U-topology. Conversely, for each n, $R/\operatorname{rad} R^n$ is semi-primary and $\operatorname{rad} R/\operatorname{rad} R_R^2$ is of finite length, hence $R/\operatorname{rad} R^n$ is right-Artinian (**7**, Lemma 11). This implies that $R/\operatorname{rad} R^n$ is discrete in the completely meet-irreducible topology; in other words, $\operatorname{rad} R^n$ is open in this topology. Since the duality guarantees the U-completeness of R, we obtain the radical completeness of R. Then by Lemma 5, R is right-Noetherian, and therefore ${}_SU$ is Artinian.

From the proof of Lemma 6 we know that $\operatorname{rad} S^n \, U_n = 0$, and since we have already proved that $U = \bigcup U_n$, we obtain $\bigcup \operatorname{Ann}_U \operatorname{rad} S^n = U$ and by the duality, $\bigcap \operatorname{rad} S^n = 0$. All the remaining statements except for $\operatorname{Ann}_U \operatorname{rad} R^n = \operatorname{Ann}_U \operatorname{rad} S^n$ follow now from symmetry, and this equality from Lemma 6.

Remark. Theorem 7 yields a new result even in the commutative case: A commutative ring with $\bigcap \operatorname{rad} R^n = 0$ which has duality is Noetherian.

THEOREM 8. *Let $\langle R, U_R, S\rangle$ be standard; suppose that* $\bigcap_{n=0}^\infty \operatorname{rad} R^n = 0$ *and that*

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(1) *R is complete in the radical topology,*
(2) U_R *is Artinian, and*
(3) rad $R/\text{rad } R_R^2$ *is of finite length.*
Then R has duality.

Proof. By (**9**, Theorem 1.1) and since U_R is Artinian, we have

$$\text{Ann}_S \text{Ann}_U L = L$$

for all left ideals L of S, and from (**9**, Lemma 3.1) we obtain $\text{Ann}_U \text{Ann}_S W = W$ for all submodules W of U_R. Hence the lattices of submodules of ${}_SS$ and U_R are anti-isomorphic, and S is left-Noetherian.

The finite length of J/J_R^2, $J = \text{rad } R$, implies that the semi-primary ring R/J^{n+1} is right-Artinian; hence J^n/J_R^{n+1} is of finite length for all n. From Lemma 5, J_R^n is finitely generated; hence uJ^n is finitely generated for any $u \in U$; consequently, by Nakayama's lemma, uJ^n contains uJ^{n+1} properly if $uJ^n \neq 0$. Therefore the descending sequence $uR \supseteq uJ \supseteq uJ^2 \supseteq \ldots$ terminates with $uJ^n = 0$ for some n, whence $u \in U_n = \text{Ann}_U J^n$ and $\bigcup U_n = U$. As in the proof of Theorem 7, we see that the radical topology and the completely meet-irreducible topology agree; consequently, U_R satisfies the second commutator relation.

$U_n = \text{Ann}_U J^n$ is the minimal cogenerator over R/J^n; for it is essential over Soc U_R $(= U_1)$ hence contains all simple modules, and it is (R/J^n)-injective: An (R/J^n)-map from a right ideal I/J^n of R/J^n into U_n can be extended to an R-map from R/J^n to U, but such a map is always into U_n and hence an (R/J^n)-map. The endomorphism ring of $U_{n,\, R/J^n}$ is $S/\text{Ann}_S U_n = S/\text{rad } S^n$ (by Lemma 6); it is left-Noetherian and semi-primary, hence left-Artinian. Since the left ideals of $S/\text{rad } S^n$ correspond to the submodules of $U_{n,R}$, this module is of finite length. Observing finally that R/J^n is right-Artinian (since it is semi-primary and since J/J_R^2 is of finite length), we see that R/J^n has duality with $S/\text{rad } S^n$ induced by U_n; and consequently, ${}_SU_n$ has finite length and is $(S/\text{rad } S^n)$-injective.

Now consider any S-map f of a left ideal L of S into U. ${}_SL$ is finitely generated and $U = \bigcup U_n$, hence $f(L) \subseteq U_m$ for some m. Therefore $L/\text{Ker } f$ is of finite length, and this implies that $\text{Ann}_U \text{Ker} f/\text{Ann}_U L$ is of finite length (by lattice anti-isomorphism). It follows that $\text{Ann}_U \text{Ker} f \subseteq \text{Ann}_U L + U_n$ for some n, and again by anti-isomorphism, $\text{Ker} f \supseteq L \cap \text{Ann}_S U_n = L \cap \text{rad } S^n$. Therefore f induces an S-map $\bar{f}$: $(L + \text{rad } S^n)/\text{rad } S^n \cong L/(L \cap \text{rad } S^n) \to U$ which is actually into $\text{Ann}_U \text{rad } S^n = U_n$ and an $(S/\text{rad } S^n)$-map. Consequently, $\bar{f}$ extends to $S/\text{rad } S^n$:

$$f(s) = \bar{f}(s + \text{rad } S^n) = (s + \text{rad } S^n)u = su \quad \text{for some } u \in U_n \subseteq U;$$

and ${}_SU$ is injective, and we obtain duality.

Remark. One may ask if conditions (1), (2), and (3) of the theorem are independent. Taking as R the localization of the integers at a prime p and

$U = Z_{p^\infty}$ we obtain an example satisfying all assumptions of the theorem except for (1). Taking for R any right-Artinian ring without duality, we see that all assumptions other than (2) are fulfilled. However, for commutative R, (2) is implied by the other conditions; for (1) and (3) yield rad R finitely generated by Lemma 5, which implies that R is Noetherian (**6**, 31.7) and has duality (**4**).

We do not know if (3) may be derived from the other assumptions; however, we prove the following result.

LEMMA 9. *The following statements are equivalent:*

(1) *U_R Artinian implies* rad $R/\text{rad}\, R_R^2$ *of finite length, for any semi-perfect ring R;*

(2) *U_R of finite length implies R right-Artinian, for every semi-primary ring R with* rad $R^2 = 0$;

(3) *U_R of finite length implies the existence of a duality, for every ring R;*

(4) *Vector space dimension* $[\text{Hom}_T(X, T){:}D] < \infty$ *implies* $[X{:}T] < \infty$, *for every bimodule ${}_DX_T$ over division rings D and T.*

Remark. Statement (4) obviously follows from (**7**, p. 385, conjecture (P)) which may be phrased as: $[X{:}T] = \aleph$ implies $[\text{Hom}_T(X, T){:}D] > \aleph$ for every bimodule ${}_DX_T$ over division rings D and T.

Proof of Lemma 9. (2) is a special case of (1) since rad $R/\text{rad}\, R_R^2$ of finite length implies R right-Artinian for every semi-primary ring R.

If (2) holds and U_R is of finite length, then R is semi-primary (**8**, Theorem 4). The minimal cogenerator for $R/\text{rad}\, R^2$ is $U_2 = \text{Ann}_U \text{rad}\, R^2$ which is of finite length, hence $R/\text{rad}\, R^2$ is right Artinian by (2); consequently, rad $R/\text{rad}\, R_R^2$ is of finite length and R is right-Artinian. Then duality exists.

Consider a bimodule ${}_DX_T$ over division rings such that $[\text{Hom}_T(X, T){:}D] < \infty$, and define

$$R = \begin{pmatrix} D & X \\ 0 & T \end{pmatrix}.$$

Then the minimal cogenerator U_R is of finite length since

$$\text{Hom}(J/J_R^2, R/J_R)_R = \text{Hom}_T(X, T)_D \quad (\text{rad}\, R = J\ (\mathbf{7}, \text{Theorem 1})).$$

Therefore (3) yields duality, thus R is right-Artinian, and consequently $[X{:}T] < \infty$.

Now assume (4) and consider a semi-perfect ring R with Artinian minimal cogenerator U_R. As in the proof of Theorem 8, we see that the minimal cogenerator over R/J^2 is $U_2 = \text{Ann}_U J^2$ and has finite length; hence

$$\text{Hom}(J/J_{R/J}^2, R/J_{R/J})_{R/J}$$

has finite length. R/J is semi-simple, hence a direct sum of simple Artinian rings $K_1 \oplus \ldots \oplus K_s$, and the bimodule ${}_{R/J}J/J^2{}_{R/J}$ decomposes into

$$\sum \bigoplus_{i,j=1}^{s} {}_{K_i}X_{ij,K_j}$$

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and $\mathrm{Hom}(J/J_{R/J}^2, R/J_{R/J})_{R/J}$ into $\sum \oplus_{i,j} \mathrm{Hom}(X_{ij,K_j}, K_{j,K_j})_{K_i}$, hence all these summands have finite length. Now $K_i = D_n$ and $K_j = T_m$ are full matrix rings over division rings D and T, and the finite length of $\mathrm{Hom}(X_{T_m}, T_{m,T_m})_{D_m}$ implies the same for

$$\mathrm{Hom}(X \otimes_{T_m} T_T{}^m, T_m \otimes_{T_m} T_T{}^m) \otimes_{D_n} D_D{}^n$$
$$\cong \mathrm{Hom}(\mathrm{Hom}({}_{D_n}D^n, {}_{D_n}X \otimes_{T_m} T^m)_T, T_T{}^m)_D$$
$$\cong \oplus_1{}^m \mathrm{Hom}(Y_T, T_T)_D$$

with the bimodule ${}_DY_T = \mathrm{Hom}({}_{D_n}D^n, {}_{D_n}X \otimes_{T_m} T^m)$ (where D^n is the D_n-D-bimodule $\oplus_1{}^n D$). Consequently, (4) yields $[Y:T] < \infty$ which implies that $[X \otimes_{T_m} T^m:T] < \infty$ and that X_{T_m} is of finite length; and we have the finite length of $J/J_{R/J}{}^2$.

5. Ring extensions. If a ring R has duality with a ring S induced by U, and if I is any two-sided ideal of R, it is rather immediate that $\mathrm{Ann}_U I$ induces a duality between R/I and $S/\mathrm{Ann}_S \mathrm{Ann}_U I$. Conversely, we discuss the simplest type of ring extensions, namely a split extension of R by a bimodule ${}_RX_R$ with $X^2 = 0$. The elements of such an extension $R + X$ are pairs (r, x) with multiplication given by $(r, x)(r', x') = (rr', rx' + xr')$.

THEOREM 10. *If U induces a duality between R and S, then the minimal cogenerator of $R + X$ is $Hom_R(X, U) + U$, with multiplication given by $(f, u)(r', x') = (fr', fx' + ur')$, and its endomorphism ring is*

$$S + \mathrm{Hom}_R(\mathrm{Hom}_R(X, U), U).$$

We have a duality between these rings if and only if both X_R and $\mathrm{Hom}_R(X, U)_R$ *are U-reflexive.*

Proof. X is a nilpotent ideal in $R + X = T$ hence in the radical, and $\mathrm{rad}(R + X) = \mathrm{rad}\, R + X$. Consequently, R and $R + X$ have the same simple modules, and the injective module $\mathrm{Hom}_R(T, U)_T$ contains

$$\mathrm{Hom}_R(T, \mathrm{Soc}\, U)_T \supseteq \mathrm{Hom}_T(T, \mathrm{Soc}\, U)_T \cong \mathrm{Soc}\, U_T$$

hence also contains all simple T-modules. Then it is the minimal cogenerator over T since it is essential over $\mathrm{Hom}_T(T, \mathrm{Soc}\, U)_T$: For any non-zero map $f \in \mathrm{Hom}_R(T, U)$, either $f(0, x_0) \neq 0$ for some $x_0 \in X$, then $f(0, x_0r_0) \neq 0$, $f(0, x_0r_0) \in \mathrm{Soc}\, U$ since U_R is essential over its socle, and $f \cdot (0, x_0r_0)$ is non-zero and in $\mathrm{Hom}_T(T, \mathrm{Soc}\, U)$; or $f(0, x) = 0$ for all x, then $f(1, 0) \neq 0$ and $f(r_1, 0) \neq 0$, $f(r_1, 0) \in \mathrm{Soc}\, U$; therefore $f \cdot (r_1, 0)$ is non-zero and in $\mathrm{Hom}_T(T, \mathrm{Soc}\, U)$. Now

$$\mathrm{Hom}_R(T, U) \cong \mathrm{Hom}_R(X \oplus R, U) \cong \mathrm{Hom}_R(X, U) \oplus U$$

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as additive groups, and it is easily checked that $R + X$ operates as indicated. Next

$$\operatorname{End}(\operatorname{Hom}_R(T, U)_T) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(T, U) \otimes_T T, U)$$
$$\cong \operatorname{Hom}_R(\operatorname{Hom}_R(R + X, U), U) \cong S + \operatorname{Hom}_R(\operatorname{Hom}_R(X, U), U)$$

and we check that the multiplication is given by $(s, h)(s', h') = (ss', sh' + hs')$; hence we have a ring extension of S by the S-S-module

$$H = \operatorname{Hom}_R(\operatorname{Hom}_R(X, U), U)$$

of the same type as $R + X$, and it operates on $\operatorname{Hom}_R(X, U) + U$ by $(s, h)(f, u) = (sf, su + hf)$.

Therefore the minimal cogenerator for $S + H$ is $\operatorname{Hom}_S(H, U) + U$ and $\operatorname{Hom}_R(X, U) + U$ is an injective cogenerator as an $(S + H)$ left module if and only if it coincides with this module, in other words if

$$\operatorname{Hom}_R(X, U) = \operatorname{Hom}_S(\operatorname{Hom}_R(\operatorname{Hom}_R(X, U), U), U)$$

or $\operatorname{Hom}_R(X, U)_R$ is U-reflexive. If that is the case, we may compute the second commutator of $\operatorname{Hom}_R(X, U) + U_{R+X}$ as the endomorphism ring of ${}_{S+H}\operatorname{Hom}_S(H, U) + U$ and we obtain

$$R + \operatorname{Hom}_S(\operatorname{Hom}_S(H, U), U) = R + \operatorname{Hom}_S(\operatorname{Hom}_R(X, U), U);$$

therefore the second commutator relation holds if and only if X_R is U-reflexive.

Remark. The theorem may be used to obtain numerous examples. An interesting case arises if we further assume that $R = S$ and take the bimodule ${}_RX_R = {}_SU_R$. Then U_R and $\operatorname{Hom}(U_R, U_R)_R = S_R = R_R$ are reflexive; and $\operatorname{Hom}_R(X, U) + U = R + U = R + X$; therefore the ring $R + X$ is an injective cogenerator on both sides, a so-called generalized quasi-Frobenius ring. Osofsky (**7**) has given an example of such a ring which is not quasi-Frobenius; her example is obtained in our context by choosing for R the ring of p-adic integers. An example of a non-commutative generalized quasi-Frobenius ring (there seems to be none in the literature) is obtained as follows: Take $R = K[[x, y]]$, the power series ring in two indeterminates over a field, then it has duality with itself. Choose $X = R$, where the R-module structure on one side is modified by the automorphism of R that interchanges x and y; then $T = R + X$ is non-commutative and has duality with itself. Our construction is applied again to T, to obtain a generalized quasi-Frobenius ring that is neither commutative nor Artinian.

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