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ARTICLE

Comparing the sets of volume polynomials and Lorentzian polynomials

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Abstract

Given n convex bodies in the Euclidean space $\mathbb{R}^d$, we can find their volume polynomial which is a homogeneous polynomial of degree d in n variables. We consider the set of homogeneous polynomials of degree d in n variables that can be represented as the volume polynomial of any such given convex bodies. This set is a subset of the set of Lorentzian polynomials. Using our knowledge of operations that preserve the Lorentzian property, we give a complete classification of the cases for (n, d) when the two sets are equal.

Keywords: Lorentzian polynomial; volume polynomial; mixed volume; convex body

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Introduction

For n convex bodies $\mathcal{K} = (K_1, \ldots, K_n)$ in $\mathbb{R}^d$ and non-negative $x_1, \ldots, x_n \in \mathbb{R}$ their linear combination, also called the Minkowski sum, is defined as the set

$$x_1K_1 + \ldots + x_nK_n := \left\{ x_1k_1 + \ldots + x_nk_n \in \mathbb{R}^d \mid k_1 \in K_1, \ldots, k_n \in K_n \right\}.$$

This is again a convex body and Minkowski [16] proved that the volume of this linear combination is a homogeneous polynomial

$$\mathrm{vol}(x_1K_1 + \ldots + x_nK_n) = \sum_{\alpha \in \Delta_n^d} \frac{d!}{\alpha!} V_\alpha(\mathcal{K}) x^\alpha$$

of degree d, where the coefficients $V_\alpha(\mathcal{K})$ for $\alpha \in \Delta_n^d$ are called the mixed volumes of $\mathcal{K}$. The volume here refers to the restriction of the Hausdorff measure on $\mathbb{R}^d$ to the set of convex bodies in $\mathbb{R}^d$. We specifically note that the Hausdorff measure coincides with the Lebesgue measure for Borel subsets of $\mathbb{R}^d$ and thus, for convex bodies [17, Kor. 2.8].

This leads us to the problem whether a given homogeneous polynomial in n variables of degree d and with non-negative coefficients can be represented as the volume polynomial of n convex bodies in the Euclidean space $\mathbb{R}^d$. Over the years, there have been several advances in relation to answering this question. Most famously, Alexandrov and Fenchel independently from each other noticed that the coefficients satisfy the *Alexandrov-Fenchel inequality*

$$V_\alpha(\mathcal{K})^2 \geq V_{\alpha - e_i + e_j}(\mathcal{K}) V_{\alpha - e_j + e_i}(\mathcal{K})$$

for every $\alpha \in \Delta_n^d$ with $\alpha_i, \alpha_j > 0$ (see [1, 10]). This inequality started a whole line of further inequalities that could be deduced using the Alexandrov-Fenchel inequality and that the coefficients of volume polynomials satisfy. These are sometimes loosely referred to as the *known inequalities*. In 1938, Heine [13] managed to show that these inequalities describe the set of volume polynomials completely in the case $(n, d) = (3, 2)$. He further proved that they are not enough to classify the set of volume polynomials in the case $(n, d) = (4, 2)$. This was generalised by Shephard [19] who constructed an example of a homogeneous polynomial in $d + 2$ many variables for degree d whose coefficients satisfy all known inequalities but which cannot be represented as the volume polynomial of any $d + 2$ many convex bodies. Further, he proved that the known inequalities fully describe the set of volume polynomials in two variables of any degree.

Generalising the known inequalities, Gurvits [11] introduced the set of strongly log-concave polynomials and showed that it contains the set of volume polynomials. Furthermore, he conjectured that the sets are equal in the case of three variables. This was disproved by Brändén and Huh (see [7, 14]) who used the *reverse Khovanskii-Teissier inequality* [15, Theorem 5.7]

$$\binom{d}{i} V_{(d-i)e_1+ie_2}(\mathcal{K}) V_{ie_1+(d-i)e_3}(\mathcal{K}) \geq V_{de_1}(\mathcal{K}) V_{ie_2+(d-i)e_3}(\mathcal{K})$$

to construct an example of a strongly log-concave polynomial that cannot be a volume polynomial. They also introduced the set of Lorentzian polynomials which equals the set of strongly log-concave polynomials in the homogeneous case (see [7, Theorem 2.30]). Almost simultaneously, Anari, Liu, Oveis Gharan, and Vinzant (see [2–5]) introduced completely log-concave polynomials which also equal Lorentzian ones for homogeneous polynomials as was proven by Brändén and Huh (see [7, Theorem 2.30]).

Working with Lorentzian polynomials and particularly operations that preserve the Lorentzian property, Brändén and Huh answered a question of Gurvits and proved that the product of two Lorentzian polynomials is again Lorentzian (see [7, Corollary 2.32]). This statement was also proven by Anari, Oveis Gharan, and Vinzant independently from Brändén and Huh (see [5, Proposition 2.2]). On the other hand, polynomial factors of Lorentzian polynomials generally do not have to be Lorentzian. But as there are certain cases when we can deduce that the factors are Lorentzian polynomials, we can ask if the same is true for volume polynomials. Using our results for this problem, we can generalise the polynomials constructed by Shephard [19] and Brändén and Huh [7, 14] which are examples for Lorentzian polynomials that cannot be volume polynomials. Thus we can fully classify the cases in which the set of volume polynomials equals the set of Lorentzian polynomials. Particularly, this fully settles the question, whether the Alexandrov-Fenchel inequality including its corollarys are enough to classify mixed volumes.

Our main findings can be summarised as follows.

1. If a Lorentzian polynomial can be factorised into polynomials with disjoint sets of variables, these factors are again Lorentzian. (Theorem 2.1)
2. If a volume polynomial can be factorised into polynomials with disjoint sets of variables, these factors are again volume polynomials. (Theorem 2.5)
3. If a Lorentzian polynomial can be written in the form $g = x_1^d f$ with $\deg_{x_1}(f) = 1$, the factors are Lorentzian. (Theorem 2.2)
4. If a volume polynomial can be written in the form $g = x_1^d f$ with $\deg_{x_1}(f) = 1$, the factors are volume polynomials. (Theorem 2.6)
5. The set of volume polynomials equals the set of Lorentzian polynomials if and only if $n \leq 2$, $d = 1$ or $(n, d) = (3, 2)$. (Theorem 3.2)

The paper is structured as follows. Section 1 is devoted to preliminaries; we recall basic definitions and properties of Lorentzian polynomials as well as volume polynomials. Section 2 focuses on the factors of Lorentzian (resp. volume) polynomials and the question whether they are again

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Lorentzian (resp. volume) polynomials. Finally, in Section 3 we use our prior findings to fully classify when the set of volume polynomials equals the set of Lorentzian polynomials and proof our main Theorem 3.2 that the sets V_n^d and L_n^d coincide if and only if $n = 2$ or $(d, n) = (2, 3)$ for $d, n \geq 2$.

1. Preliminaries

The paper is set in $\mathbb{R}^d$ with the standard Euclidean topology. Particularly, we denote the Euclidean norm of a vector $v \in \mathbb{R}^d$ by

$$\|v\| := \sqrt{v \cdot v},$$

where $\cdot : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denotes the dot product in $\mathbb{R}^d$. By a *convex body* K in the Euclidean space $\mathbb{R}^d$, we refer to a non-empty compact convex set. Particularly, the convex bodies we consider do not need to have non-empty interior and thus can be less than full-dimensional. When we talk about the dimension of a convex body, we refer to the dimension of the smallest affine space containing the convex body.

We fix some notation and terminology concerning convex bodies and their volume polynomials. As a general reference, we suggest the monograph of Schneider [18, Chapter 1]. Let n and d be positive integers. We write $[n] := \{1, \ldots, n\}$ and

$$\Delta_n^d := \{\alpha \in \mathbb{N}_0^n \mid \sum_{i=1}^n \alpha_i = d\}$$

for the d-th discrete simplex. The space of homogeneous polynomials of degree d in n variables over $\mathbb{R}$ is denoted by H_n^d. For a polynomial $f \in H_n^d$, we denote its degree in the variable x_i by $\deg_i(f)$.

Let $\mathcal{K} := (K_1, \ldots, K_n)$ be convex bodies in $\mathbb{R}^d$. Their *volume polynomial* is the homogeneous polynomial

$$\mathrm{vol}_{\mathcal{K}}(x) := \mathrm{vol}(x_1 K_1 + \cdots + x_n K_n) := \sum_{\alpha \in \Delta_n^d} \frac{d!}{\alpha!} V_\alpha(\mathcal{K}) x^\alpha$$

for non-negative $x_1, \ldots, x_n$ and $x := (x_1, \ldots, x_n)$ and we call $V_\alpha(\mathcal{K})$ for $\alpha \in \Delta_n^d$ the *mixed volumes* of $\mathcal{K}$. For a multi-index $\alpha \in \Delta_n^d$ we write

$$\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_n! \qquad \text{and} \qquad x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}.$$

We generally assume $\mathcal{K}$ to be *full-dimensional*, meaning that the affine dimension of the convex body

$$\sum \mathcal{K} := \sum_{i=1}^n K_i$$

equals d. This guarantees that the volume polynomial is non-zero. We further use the notation

$$V_\alpha(\mathcal{K}) = V(\mathcal{K}^\alpha) = V(K_1^{\alpha_1}, \ldots, K_n^{\alpha_n})$$

to refer to the mixed volume of the convex bodies $\mathcal{K} = (K_1, \ldots, K_n)$. The set of all volume polynomials is denoted by V_n^d.

The mixed volumes satisfy several useful properties of which we will only list a few here. For a more thorough understanding, we refer to the monograph of Schneider [18, Chapter 5].

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Proposition 1.1.

(a) *The mixed volumes are non-negative, symmetric in the convex bodies and they satisfy*

$$V(aK_1 + b\tilde{K_1}, K_2, \ldots, K_d) = aV(K_1, \ldots, K_d) + bV(\tilde{K_1}, K_2, \ldots, K_d)$$

for non-negative $a, b \in \mathbb{R}_{\geq 0}$ *and convex bodies* $K_1, \ldots, K_d, \tilde{K_1} \subseteq \mathbb{R}^d$ *(see Schneider [18, Chapter 5.1]).*

(b) *For* $a_1, \ldots, a_n \in \mathbb{R}^d$ *and a regular matrix* $A \in \mathbb{R}^{d\times d}$*, we have*

$$\mathrm{vol}(x_1 T_{a_1,A}(K_1) + \ldots + x_n T_{a_n,A}(K_n)) = |\det(A)|\mathrm{vol}(x_1 K_1 + \ldots + x_n K_n),$$

whereat $T_{a_i,A}$ *denotes the affine transformation* $\mathbb{R}^d \to \mathbb{R}^d, x \mapsto Ax + a_i$ *(see Shephard [19, p. 126]).*

(c) *For convex bodies* $K_1, \ldots, K_d$ *in* $\mathbb{R}^d$ *the mixed volume* $V(K_1, \ldots, K_d)$ *is positive if and only if there exist line segments* $L_i \subseteq K_i$ *with linearly independent directions for all* $i \in [d]$ *(see Schneider [18, Theorem 5.1.8]).*

(d) *For* $i \in [n]$*, the volume polynomial and mixed volumes carry the information* $\deg_i(\mathrm{vol}_{\mathcal{K}}) = \dim(K_i)$ *and* $\mathrm{vol}(K_i) = V_{de_i}(\mathcal{K})$ *(see Gurvits [12, Fact A.7]).*

Due to part (b) of the above Proposition 1.1, we may always assume that the considered convex bodies contain the origin, which often times simplifies our settings and calculation. For a k-dimensional linear subspace $E \subset \mathbb{R}^d$, we denote by vol_E (resp. V_E) the volume (resp. the mixed volume) in the space E. We omit the subspace in the notation if it can be deduced from the context. By $K|E$, we denote the orthogonal projection of a convex body $K \subset \mathbb{R}^d$ onto the space E. If some of the convex bodies we are considering lie in a common linear subspace, we can use this to represent the mixed volume in $\mathbb{R}^d$ as a product of the mixed volumes in the smaller subspace and its orthogonal complement.

Proposition 1.2. [18, Theorem 5.3.1] *Let* E *be a* k*-dimensional linear subspace of* $\mathbb{R}^d$ *and let* $L_1, \ldots, L_k \subset E$ *as well as* $K_1, \ldots, K_{d-k} \subset \mathbb{R}^d$ *be convex bodies. We have*

$$\binom{d}{k} V(L_1, \ldots, L_k, K_1, \ldots, K_{d-k}) = V_E(L_1, \ldots, L_k) V_{E^\perp}(K_1|E^\perp, \ldots, K_{d-k}|E^\perp),$$

where $E^\perp$ *refers to the orthogonal space of* E.

The mixed volumes satisfy several useful inequalities, the most famous being the *Alexandrov–Fenchel inequality* (see [1, 10])

$$V(K_1, \ldots, K_n)^2 \geq V(K_1^2, K_3, \ldots, K_n) V(K_2^2, K_3, \ldots, K_n).$$

A generalisation of polynomials with coefficients satisfying this inequality leads us to the set of Lorentzian polynomials.

Definition 1.3. (see [7]) *A subset* $J \subseteq \mathbb{N}^n$ *is called M-convex if for any* $\alpha, \beta \in J$ *and any index* $i \in [n]$ *with* $\alpha_i > \beta_i$*, there exists an index* $j \in J$ *with* $\alpha_j < \beta_j$ *and* $\alpha - e_i + e_j,\ \beta - e_j + e_i \in J$*. We denote by* M_n^d *the set of all polynomials in* H_n^d *with non-negative coefficients and M-convex support. We further define the set of Lorentzian polynomials as* $L_n^1 := M_n^1$ *and for* $d \geq 2$ *as*

$$L_n^d := \{f \in M_n^d \mid \text{for all } \alpha \in \Delta_n^{d-2}\colon \mathcal{H}_{\partial^\alpha f} \text{ has at most one positive eigenvalue}\},$$

where $\mathcal{H}_f$ *refers to the Hessian of a polynomial* $f \in H_n^d$.

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The conditions for the Hessian matrices lead to the fact that the coefficients of Lorentzian polynomials always satisfy the Alexandrov-Fenchel inequality. In fact, we have the inclusion $V_n^d \subseteq L_n^d$.

Theorem 1.4. [7, Theorem 4.1] *Every volume polynomial is Lorentzian.*

As Lorentzian polynomials have been thoroughly studied, especially concerning operations that preserve their properties, we will focus on some basic notions here and refer the reader to the work of Brändén and Huh [7] for a broader understanding.

Proposition 1.5.

(a) The product of Lorentzian polynomials is Lorentzian (see [7, Corollary 2.32]).
(b) Let $A \in \mathbb{R}_{\geq 0}^{n\times m}$ be a $(n \times m)$-matrix with non-negative entries. For a Lorentzian polynomial $f \in L_n^d$ and $x := (x_1, \ldots, x_m)^\top$, we have $f(Ax) \in L_m^d$ (see [7, Theorem 2.10]).

Both of these properties can be transferred to volume polynomials. To do so in the case of products of volume polynomials, it is enough to first consider two polynomials in distinct variables in $V_{n_1}^{d_1}$ (resp. in $V_{n_2}^{d_2}$) and then embed the associated convex bodies in $\mathbb{R}^{d_1}$ (resp. $\mathbb{R}^{d_2}$) in the Euclidean space $\mathbb{R}^{d_1+d_2}$. The volume polynomial of the resulting convex bodies is exactly the product of the two polynomials. The general case follows immediately from an appropriate substitution of the variables.

Remark 1.6.

(a) The product of volume polynomials is a volume polynomial.
(b) Let $A \in \mathbb{R}_{\geq 0}^{n\times m}$ be a $(n \times m)$-matrix with non-negative entries. For a volume polynomial $f \in V_n^d$ and $x := (x_1, \ldots, x_m)^\top$, we have $f(Ax) \in V_m^d$ (see [11, Example 1.2]).

Generally, the set of Lorentzian polynomials allows more operations that preserve it than the set of volume polynomials. One such operation is the derivative: Whereas the definition of Lorentzian polynomials immediately shows that the derivative of any polynomial $f \in L_n^d$ is again Lorentzian, the same is not true for volume polynomials. This can be seen by considering the Lorentzian polynomial

$$f := \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_1^2x_3 + \frac{1}{2}x_1^2x_4 + 2x_1x_2x_3 + 2x_1x_2x_4 + \frac{1}{2}x_1x_3x_4 + x_2x_3x_4 \in L_4^3,$$

which is the volume polynomial of the four convex bodies

$$K_1 := \operatorname{conv}(0, e_1, e_2), \quad K_2 := \operatorname{conv}(0, e_3),$$

$$K_3 := \operatorname{conv}(0, 2e_1 + e_3) \quad \text{and} \quad K_4 := \operatorname{conv}\left(0, \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}\right)$$

in $\mathbb{R}^3$. The derivative

$$\partial_1 f = x_1x_2 + x_1x_3 + x_1x_4 + 2x_2x_3 + 2x_2x_4 + \frac{1}{2}x_3x_4$$

on the other hand cannot be a volume polynomial due to the findings of Heine (see [13, p. 119]). In Section 3, we will go into more depth as to why the polynomial $\partial_1 f$ cannot be represented as the volume polynomial of any four convex bodies in $\mathbb{R}^2$.

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The derivative is just one example of such an operation. Another one is provided by the next Proposition 1.7.

Proposition 1.7. [8, Lemma 4.4] *Let $f \in L_n^d$ be a Lorentzian polynomial and let us write*

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{d} x_n^{d-i} f_i(x_1, \ldots, x_{n-1}).$$

Then f_i is a Lorentzian polynomial of degree i for every $i \in [d]$.

In contrast to Proposition 1.5 and Remark 1.6, an easy example shows that Proposition 1.7 is not necessarily true for volume polynomials. Let us consider the polynomial

$$\begin{aligned} f := {} & x_5^3 + x_5^2(x_1 + x_2 + x_3 + \tfrac{3}{2}x_4) + x_5(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\ & + x_1x_2x_3 + \tfrac{1}{2}x_1x_2x_4 + \tfrac{1}{2}x_1x_3x_4 + \tfrac{1}{2}x_2x_3x_4, \end{aligned}$$

which is the volume polynomial of the convex bodies

$$\begin{aligned} K_1 &:= \operatorname{conv}(0, e_1), \\ K_2 &:= \operatorname{conv}(0, e_2), \\ K_3 &:= \operatorname{conv}(0, e_3), \\ K_4 &:= \operatorname{conv}(0, \frac{1}{2}(e_1 + e_2 + e_3)), \end{aligned}$$

and the unit cube

$$K_5 := \operatorname{conv}(0, e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3).$$

Just as above, according to Heine (see [13, p. 119]), the elementary symmetric polynomial in four variables of degree two, and thus f_2, cannot be represented as a volume polynomial.

2. Operations

We have noted that the product of Lorentzian (resp. volume) polynomials is again a Lorentzian (resp. volume) polynomial. Generally, the factors of Lorentzian or volume polynomials do not have to be either. For example, the polynomial

$$f := x^3 + 3x^2y + 3xy^2 = x(x^2 + 3xy + 3y^2)$$

is Lorentzian as the Hessian matrices of $\partial_x f$ and $\partial_y f$ have exactly one positive eigenvalue. Because of the fact that f is bivariate, it is due to Shephard [19, Theorem 4] that f is a volume polynomial. On the other hand, the factor $x^2 + 3xy + 3y^2$ is not Lorentzian due to its Hessian matrix having two positive eigenvalues and thus, it cannot be a volume polynomial (Theorem 1.4).

Nevertheless, there are certain cases where the factors of Lorentzian (resp. volume) polynomials are Lorentzian (resp. volume) polynomials.

Theorem 2.1. *Let $f := gh \in L_{n_1+n_2}^{d_1+d_2}$ be a Lorentzian polynomial with factors $g \in H_{n_1}^{d_1}$ and $h \in H_{n_2}^{d_2}$ with non-negative coefficients and in distinct variables $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$. Then both factors are again Lorentzian.*

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Proof. Let $f := gh \in L^{d_1+d_2}_{n_1+n_2}$ be a Lorentzian polynomial with g and h having distinct variables. We define the $(n_1 + n_2) \times (n_2 + 1)$-matrix

$$A := \begin{pmatrix} \frac{1}{|g|_1} & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{|g|_1} & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where $|g|_1$ defines the sum of the (non-negative) coefficients of g. Taking this matrix we know that the polynomial

$$f\left(A\begin{pmatrix} x \\ y_1 \\ \vdots \\ y_{n_2} \end{pmatrix}\right) = x^{d_1} h$$

is Lorentzian by Proposition 1.5. Essentially, we plug in the variable x for any of the n_1 variables that appear in the polynomial g, while simultaneously scaling the polynomial. Now, as the derivative of any Lorentzian polynomial is again Lorentzian, the polynomial h, which equals exactly the derivative

$$\partial_x^{d_1} x^{d_1} h = d_1! h$$

of the above polynomial up to scaling, is a Lorentzian polynomial and analogously, the same is true for g. □

The next proposition shows that we can skip the restriction of distinct variables if we restrict the degree of the (only) common variable of the two factors.

Theorem 2.2. *Let $f := x_1^{d_1} g \in L^{d_1+d_2}_n$ be a Lorentzian polynomial with a polynomial $g \in H^{d_2}_n$ such that* $\deg_1(g) \leq 1$. *Then the polynomial g is also Lorentzian.*

Proof. As the derivative of any Lorentzian polynomial is again Lorentzian, we can gather that the polynomial

$$\partial_1^{d_1} f = d_1!(1 + d_1 x_1 \partial_1) g = d_1! g((1 + d_1)x_1, x_2, \dots, x_n)$$

is a Lorentzian polynomial. Proposition 1.5 then allows us to conclude $g \in L^{d_2}_n$ by scaling the variable x_1. □

Using the same technique of transforming the variables as in the proof of Proposition 2.1, we can deduce the following corollary.

Corollary 2.3. *Let $f := gh \in L^{d_1+d_2}_{n_1+n_2-1}$ be a Lorentzian polynomial, such that $g \in H^{d_1}_{n_1}$ and $h \in H^{d_2}_{n_2}$ share (exactly) one variable x_1 and the polynomial h has at most degree 1 in x_1. Then h is Lorentzian.*

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Proof. Let the polynomial g be in the variables $x_1, \ldots, x_{n_1}$ and h be in the variables $x_1, y_2, \ldots, y_{n_2}$. We set $x_1 = \ldots = x_{n_1}$ in f and scale the resulting polynomial by $\frac{1}{|g|_1}$ which preserves the Lorentzian property by Proposition 1.5. The resulting polynomial is of the form

$$x_1^{d_1} h$$

with $\deg_1(h) \leq 1$. Thus, it satisfies the conditions of Proposition 2.2 and h is Lorentzian. □

Remark 2.4. Originally, we looked at the special case of multiaffine factors and proved that for a Lorentzian polynomial $f := gh \in L^{d_1+d_2}_{n_1+n_2-1}$ with multiaffine factorsg $\in H^{d_1}_{n_1}$ and $h \in H^{d_2}_{n_2}$ sharing one variable, both factors are again Lorentzian. Motivated by this, it was possible to generalise the techniques we used and thus, come to the proof of Proposition 2.2 and Corollary 2.3.

In order to conclude similar results for volume polynomials, we need to use different techniques as the derivative of a volume polynomial is generally not a volume polynomial anymore, which renders most of the previously used techniques useless for volume polynomials. Instead, we can use the geometric aspects of the given convex bodies to transfer the results to volume polynomials.

Theorem 2.5. *Let $f := gh \in V^{d_1+d_2}_{n_1+n_2}$ be a volume polynomial with factors $g \in H^{d_1}_{n_1}$ and $h \in H^{d_2}_{n_2}$ with non-negative coefficients and in distinct variables $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$. Then both factors are again volume polynomials.*

Proof. We proceed similar to the the case of Lorentzian polynomials and first assume that $f := x_{n+1}^{d_1} g \in V^{d_1+d_2}_{n+1}$ is a volume polynomial with $g \in H^{d_2}_n$ being a polynomial in the variables $x_1, \ldots, x_n$. Let $\mathcal{K} := (K_1, \ldots, K_{n+1})$ be the convex bodies in $\mathbb{R}^{d_1+d_2}$ associated to the corresponding variables $x_1, \ldots, x_{n+1}$. As we have $\deg_{n+1}(f) = d_1$, the convex body K_{n+1} must lie in a d_1-dimensional linear subspace $E \subseteq \mathbb{R}^{d_1+d_2}$ due to Proposition 1.1(d). Due to the factorisation of f, it must be

$$V(K_1^{\alpha_1}, \ldots, K_n^{\alpha_n}, K_{n+1}^{d_1-1}) = 0$$

for all $\alpha \in \Delta_n^{d_2+1}$. With Proposition 1.1(c) this allows us to assume $K_1, \ldots, K_n \subseteq E^\perp$ as it means, that we cannot expand any system of $d_1 - 1$ linearly independent vectors from E with $d_2 + 1$ linearly independent vectors coming from line segments in $K_1, \ldots, K_n$ to get a basis of $\mathbb{R}^{d_1+d_2}$. Now we take $\alpha \in \Delta_n^{d_2}$ and we have

$$\begin{aligned} V_{(\alpha,d_1)}(K) &= \binom{d_1 + d_2}{d_1}^{-1} V(K_{n+1}^{d_1}) V(K_1^{\alpha_1}, \ldots, K_n^{\alpha_n}) \\ &= \frac{d_1! d_2!}{(d_1 + d_2)!} V(K_{n+1}^{d_1}) V(K_1^{\alpha_1}, \ldots, K_n^{\alpha_n}) \end{aligned}$$

by Proposition 1.2. For the volume polynomial f, this leads to

$$\begin{aligned} f &= \sum_{\alpha \in \Delta_n^{d_2}} \frac{(d_1 + d_2)!}{\alpha! d_1!} V_{(\alpha,d_1)}(\mathcal{K}) x^\alpha x_{n+1}^{d_1} \\ &= x_{n+1}^{d_1} \sum_{\alpha \in \Delta_n^{d_2}} \frac{d_2!}{\alpha!} \mathrm{vol}_{d_1}(K_{n+1}) V(K_1^{\alpha_1}, \ldots, K_n^{\alpha_n}) x^\alpha \\ &= x_{n+1}^{d_1} \mathrm{vol}_{d_1}(K_{n+1}) \mathrm{vol}(x_1 K_1 + \cdots + x_n K_n), \end{aligned}$$

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where vol_{d_1} refers to the d_1-dimensional volume in the subspace E of $\mathbb{R}^{d_1+d_2}$. Hence, g is a volume polynomial.

For the general case, we use a transformation of the variables as before and obtain the result. $\square$

Theorem 2.6. *Let $f := x_1^{d_1} g \in V_n^{d_1+d_2}$ be a volume polynomial with a polynomial $g \in H_n^{d_2}$ such that $\deg_1(g) \leq 1$. Then the polynomial g is also a volume polynomial.*

Proof. Let f be the volume polynomial of the convex bodies $\mathcal{K} := (K_1, \ldots, K_n)$ in $\mathbb{R}^{d_1+d_2}$, so that we have

$$\begin{aligned} f &= \sum_{\alpha \in \Delta_n^{d_1+d_2}} \frac{(d_1+d_2)!}{\alpha!} V_\alpha(\mathcal{K}) x^\alpha \\ &= \sum_{\substack{\alpha \in \Delta_n^{d_1+d_2} \\ \alpha_1 = d_1}} \frac{(d_1+d_2)!}{\alpha!} V_\alpha(\mathcal{K}) x^\alpha + \sum_{\substack{\alpha \in \Delta_n^{d_1+d_2} \\ \alpha_1 = d_1+1}} \frac{(d_1+d_2)!}{\alpha!} V_\alpha(\mathcal{K}) x^\alpha \\ &= x_1^{d_1} \left(\sum_{\substack{\alpha \in \Delta_n^{d_2} \\ \alpha_1 = 0}} \frac{(d_1+d_2)!}{(\alpha + d_1 e_1)!} V_{\alpha + d_1 e_1}(\mathcal{K}) x^\alpha + \sum_{\substack{\alpha \in \Delta_n^{d_2} \\ \alpha_1 = 1}} \frac{(d_1+d_2)!}{(\alpha + d_1 e_1)!} V_{\alpha + d_1 e_1}(\mathcal{K}) x^\alpha \right) \end{aligned}$$

If we have $\deg_1(g) = 0$, the statement follows immediately from Proposition 2.6. Thus, we only consider the case that $\deg_1(g) = 1$. With Proposition 1.1(d), we gather $\dim(K_1) = d_1 + 1$ and due to Proposition 1.1(c), we can assume $K_2, \ldots, K_n \subseteq V$ for a d_2-dimensional linear subspace $V \subseteq \mathbb{R}^{d_1+d_2}$. We denote by U_1 the (d_1+1)-dimensional linear subspace containing K_1 and get $U_1 \cap V = \mathbb{R}v$ for a vector $v \in \mathbb{R}^{d_1+d_2}$. We can now write $U_1 = U + \mathbb{R}v$ for a d_1-dimensional subspace $U \subseteq \mathbb{R}^{d_1+d_2}$ and by a change of basis and Proposition 1.1(b), we assume $U = V^\perp$ without loss of generality, particularly $v \in U^\perp$. We write $C_1 := K_1 | U$ and chose the length of v such that we get

$$\mathrm{vol}_{d_1+1}(K_1) = \mathrm{vol}_{d_1+1}(C_1 + \mathrm{conv}(0, v)) = \|v\| \mathrm{vol}_{d_1}(C_1).$$

For an $\alpha \in \Delta_n^{d_2}$ with $\alpha_1 = 0$, we have $\alpha + d_1 e_1 \in \Delta_n^{d_1+d_2}$ and with Proposition 1.2, we can rewrite the mixed volume

$$\frac{(d_1+d_2)!}{d_1!\alpha!} V(K_1^{d_1}, K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}) = \frac{d_2!}{\alpha!} V_U(C_1) V_V(K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}).$$

On the other hand, for an $\alpha \in \Delta_n^{d_2}$ with $\alpha_1 = 1$, we get with Proposition 1.2

$$\begin{aligned} &\frac{(d_1+d_2)!}{(d_1+1)!\hat{\alpha}!} V(K_1^{d_1+1}, K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}) \\ &= \frac{(d_2-1)!}{\hat{\alpha}!} \mathrm{vol}_{d_1+1}(K_1) V_{U_1^\perp}\left((K_2 \mid U_1^\perp)^{\alpha_2}, \ldots, (K_n \mid U_1^\perp)^{\alpha_n} \right), \end{aligned}$$

where $\hat{\alpha}$ refers to $(\alpha_2, \ldots, \alpha_n)$. Our choice of v allows us to further rewrite the above mixed volume, so that we gather

$$\begin{aligned} &\frac{(d_2-1)!}{\hat{\alpha}!} \mathrm{vol}_{d_1+1}(K_1) V_{U_1^\perp}\left((K_2 \mid U_1^\perp)^{\alpha_2}, \ldots, (K_n \mid U_1^\perp)^{\alpha_n} \right) \\ &= \frac{(d_2-1)!}{\hat{\alpha}!} \mathrm{vol}_{d_1+1}(C_1 + \mathrm{conv}(0, v)) V_{U_1^\perp}\left((K_2 \mid U_1^\perp)^{\alpha_2}, \ldots, (K_n \mid U_1^\perp)^{\alpha_n} \right) \\ &= \frac{(d_1+d_2)!}{(d_1+1)!\hat{\alpha}!} V((C_1 + \mathrm{conv}(0, v))^{d_1+1}, K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}). \end{aligned}$$

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By multilinearity in the first argument due to Proposition 1.1(a), we can split the convex body $C_1 + \mathrm{conv}(0, v)$, which allows us to then use Proposition 1.1(c) and finally Proposition 1.2 to conclude

$$\begin{aligned}
&\frac{(d_1+d_2)!}{(d_1+1)!\hat{\alpha}!} V((C_1+\mathrm{conv}(0,v))^{d_1+1}, K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}) \\
=&\frac{(d_1+d_2)!}{(d_1+1)!\hat{\alpha}!} \sum_{i=0}^{d_1+1} \binom{d_1+1}{i} V(C_1^{d_1+1-i}, \mathrm{conv}(0,v)^i, K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}) \\
=&\frac{d_2!}{\hat{\alpha}!} \binom{d_1+d_2}{d_1} V(C_1^{d_1}, \mathrm{conv}(0,v), K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}) \\
=&\frac{d_2!}{\alpha!} V_U(C_1) V_V(\mathrm{conv}(0,v), K_2^{\alpha_2}, \ldots, K_n^{\alpha_n}).
\end{aligned}$$

Inserting both cases into our polynomial f, we get

$$f = x_1^{d_1} \mathrm{vol}_{d_1}(C_1) \mathrm{vol}_{d_2}(x_1 \mathrm{conv}(0,v) + x_2 K_2 + \ldots + x_n K_n)$$

and thus, g is a volume polynomial. □

As in the case of Corollary 2.3 for Lorentzian polynomials, we can now deduce the following Corollary by referring to Remark 1.6 and then using the above Proposition 2.6.

Corollary 2.7. *Let $f := gh \in V_{n_1+n_2-1}^{d_1+d_2}$ be a volume polynomial such that $g \in H_{n_1}^{d_1}$ and $h \in H_{n_2}^{d_2}$ only share one variable x_1 and the polynomial h has at most degree 1 in x_1. Then h is a volume polynomial.*

Remark 2.8. Similarly to the corresponding results for Lorentzian polynomials, Proposition 2.6 and Corollary 2.7 were also motivated by the special case of multiaffine factors. For volume polynomials multiaffine polynomials allow an explicit description of the corresponding convex bodies as they all have to be line segments due to Proposition 1.1(d). This allows a straight forward approach for the proof of Proposition 2.6, as we can use the directions of the line segments to define the different linear subspaces explicitly. Further, we know by Proposition 1.1(c), that the directions of the considered line segments are linearly independent if and only if the mixed volumes are positive, so that respective directions form a basis of the appropriate linear subspace. The generalisation of this technique can be seen above, where we do not rely on an explicit description of the different subspaces.

The above results illustrate how we can use our knowledge of Lorentzian polynomials to obtain new information on volume polynomials as we have first studied the factors of Lorentzian polynomials and then transferred this knowledge to volume polynomials appropriately. But as mentioned before, the vast majority of results and operations for Lorentzian polynomials are not transferable to volume polynomials. Instead, we often need further restrictions or some adjusting of the results to be able to transfer the operations preserving the Lorentzian property to volume polynomials. We have seen one such example of a non-transferable result in Proposition 1.7. As we explicitly do not require the convex bodies to have non-empty interior, we can transfer at least parts of Proposition 1.7 to volume polynomials.

Proposition 2.9. *Let $f \in V_n^d$ be the volume polynomial of n convex bodies $\mathcal{K} := (K_1, \ldots, K_n)$ in $\mathbb{R}^d$ and let us write*

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{d} x_n^{d-i} f_i(x_1, \ldots, x_{n-1}).$$

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Then f_d is a volume polynomial of degree d and f_{d-m} for $m := \dim(K_n)$ is a volume polynomial of degree $d-m$.

Proof. We have $f_d = f(x_1, \ldots, x_{n-1}, 0) = \mathrm{vol}(x_1K_1 + \ldots + x_{n-1}K_{n-1})$. Let $U \subseteq \mathbb{R}^d$ be the m-dimensional linear subspace with $K_n \subseteq U$. We have

$$\begin{aligned} f_{d-m} &= \sum_{\alpha\in\Delta_{n-1}^{d-m}} \frac{d!}{m!\alpha!} V(K^\alpha, K_n^m)x^\alpha \\ &= \sum_{\alpha\in\Delta_{n-1}^{d-m}} \frac{(d-m)!}{\alpha!} V_U(K_n^m)V_{U^\perp}((K|U^\perp)^\alpha)x^\alpha \\ &= \mathrm{vol}_m(K_1)\mathrm{vol}(x_1(K_1|U^\perp) + \ldots + x_{n-1}(K_{n-1}|U^\perp)). \end{aligned}$$

□

3. Volume polnomials as a subset of Lorentzian polynomials

The *Alexandrov-Fenchel inequality* (see [1, 10]), being the first major restriction for sequences that can be realised as a sequence of coefficients of a volume polynomial, started a long line of further inequalities that can be deduced from it. The set of homogeneous polynomials with coefficients satisfying these inequalities contains the set of Lorentzian polynomials ([11, Example 1.2(3)] and [7, Proposition 4.4]) which allows us to solely focus on this smaller set as Brändén and Huh found that every volume polynomial is Lorentzian [7, Theorem 4.1].

We denote by AF_n^d the set of homogeneous polynomials in n variables of degree d with non-negative coefficients satisfying the *Alexandrov-Fenchel inequality* as well as the resulting inequalities [19, p. 132]

$$V(K^\alpha, K_i^{r-1}, K_j)V(K^\alpha, K_i, K_j^{r-1}) \geq V(K^\alpha, K_i^r)V(K^\alpha, K_j^r)$$

for $\alpha \in \Delta_n^{d-r}$ and

$$(-1)^r \det\left(\left(V(K^\beta, K_i, K_j)\right)_{i,j\in[r]}\right) \leq 0$$

for $\beta \in \Delta_n^{d-2}$ and $r \leq n$. Considering the polynomial

$$g := c_{111}x_1^3 + 3c_{223}x_2^2x_3 + 3c_{233}x_2x_3^2$$

with $c_{111}, c_{223}, c_{233} > 0$ which lies in AF_3^3 but not in L_3^3, one sees that focusing on the set L_n^d instead of AF_n^d already reduces the number of polynomials due to the additional condition of the M-convexity of the support of polynomials in L_n^d. Thus going forward, we regard the set V_n^d as a subset of L_n^d instead of a subset of the bigger set AF_n^d.

Shephard proved [19, Theorem 4] that for any degree $d \in \mathbb{N}$, we have

$$V_2^d = L_2^d$$

and he further proved [19, Theorem 5] that for $(d+2)$-many variables, the inclusion

$$V_{d+2}^d \subsetneq L_{d+2}^d$$

is strict. This generalised a result of Heine [13, p. 119] for polynomials in four variables and of degree two. To illustrate the idea behind the proof, we will mention Heine's example here.

Example 3.1. [13, p. 119] *The elementary symmetric polynomial in four variables of degree two*

$$f := x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

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is Lorentzian as can be seen straight forwardly by computing the Hessian matrix. If it were the volume polynomial of convex bodies $K_1, \dots, K_4 \subseteq \mathbb{R}^2$*, these would have to be line segments by Remark 1.1. Without loss of generality, we assume* $K_i = conv(0, e_i)$ *for* $i = 1, 2$ *and* $K_3 = conv(0, a)$, $K_4 = conv(0, b)$ *for* $a, b \in \mathbb{R}^2$*. Computing the mixed volumes of these convex bodies leads to*

$$1 = \pm a_i = \pm b_i = \pm(a_1 b_2 - a_2 b_1)$$

for $i = 1, 2$ *and thus to a contradiction.*

This example also illustrates why it is often useful to first refer to multiaffine polynomials as they allow an easy computation of the mixed volumes, which would otherwise be more difficult (see [6, 9]).

In the case of three variables, Heine [13, p. 118] proved

$$V_3^2 = L_3^2.$$

Later, Gurvits [11, Conjecture 5.1] conjectured that this might be true for all degrees. This was disproved by Brändén and Huh ([7, Footnote 15] and [14, Example 14]), who constructed the Lorentzian polynomial

$$f := 14x_1^3 + 6x_1^2x_2 + 24x_1^2x_3 + 12x_1x_2x_3 + 6x_1x_3^2 + 3x_2x_3^2,$$

which cannot be a volume polynomial as the coefficients do not satisfy the *reverse Khovanskii-Teissier inequality* [15, Theorem 5.7]. This inequality states that for three convex bodies K_1, K_2, K_3 in $\mathbb{R}^d$, the mixed volumes satisfy

$$\binom{d}{k} V(K_1^{d-k}, K_2^k) V(K_1^k, K_3^{d-k}) \geq V(K_1^d) V(K_2^k, K_3^{d-k})$$

for all non-negative integers $k \leq d$.

To give some geometric motivation for the inequality, we assume that we have three convex bodies $K_1, K_2, K_3 \subseteq \mathbb{R}^d$ with $\dim(K_2) = k \leq d$ and $\dim(K_3) = d - k$. Let $U \subseteq \mathbb{R}^d$ be the k-dimensional linear subspace with $K_2 \subseteq U$. As the inequality is trivial when the right hand side equals zero, we can assume $K_3 \subseteq U^\perp$ due to Proposition 1.1(c). With this in mind, Proposition 1.2 leads us to

$$\begin{aligned}
\binom{d}{k} V(K_2^k, K_3^{d-k}) &= \mathrm{vol}_k(K_2)\mathrm{vol}_{d-k}(K_3),\\
\binom{d}{k} V(K_1^{d-k}, K_2^k) &= \mathrm{vol}_{d-k}(K_1|U^\perp)\mathrm{vol}_k(K_2),\\
\binom{d}{k} V(K_1^k, K_3^{d-k}) &= \mathrm{vol}_k(K_1|U)\mathrm{vol}_{d-k}(K_3).
\end{aligned}$$

By approximating the volume of K_1, we get

$$\begin{aligned}
V(K_1^d) &\leq \mathrm{vol}_k(K_1|U)\mathrm{vol}_{d-k}(K_1|U^\perp)\\
&\leq \binom{d}{k} \frac{V(K_1^k, K_3^{d-k}) V(K_1^{d-k}, K_2^k)}{V(K_2^k, K_3^{d-k})}.
\end{aligned}$$

As was communicated by Ivan Soprunov, this shows that the above example of a polynomial in $L_3^3 \setminus V_3^3$ by Brändén and Huh cannot be a volume polynomial (without using Hodge theory, as in their proof). In the general case, when the convex bodies $K_1, K_2, K_3 \subseteq \mathbb{R}^d$ have dimension greater than k or $d - k$, one cannot use the above technique to see that the mixed volumes satisfy the *reverse Khovanskii–Teissier inequality*.

Using the above polynomials and our prior results, we are now in the position to prove our main theorem and thus to fully classify when the inclusion $V_n^d \subseteq L_n^d$ is strict. First, the case $n = 1$ obviously leads to $V_1^d \subseteq L_1^d$ for all $d \in \mathbb{N}$. Second, the case $d = 1$ obviously leads to $V_n^1 \subseteq L_n^1$ for all $n \in \mathbb{N}$. The remaining cases are solved in the following.

Theorem 3.2. *Let $d, n \geq 2$. The sets V_n^d and L_n^d coincide if and only if $n = 2$ or $(d, n) = (2, 3)$.*

Proof. Shephard [19, Theorem 4] proved that the sets are equal for $n = 2$ and Heine [13, p. 118] proved the same for $(d, n) = (2, 3)$. We define the polynomial

$$f_k := x_2^k(14x_1^3 + 6x_1^2x_2 + 24x_1^2x_3 + 12x_1x_2x_3 + 6x_1x_3^2 + 3x_2x_3^2)$$

for $k \in \mathbb{N}_0$ which is a Lorentzian polynomial in L_3^{3+k} by Proposition 1.5 as it is the product of two Lorentzian polynomials. By the results of Brändén and Huh ([7, Footnote 15] and [14, Example 14]), the second factor cannot be realised as a volume polynomial. By Proposition 2.6, the polynomial f_k cannot lie in V_3^{3+k} either. Hence, we have $V_3^{3+k} \subsetneq L_3^{3+k}$ for all $k \in \mathbb{N}_0$. The polynomial

$$f := x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

leads to the strict inclusion $V_4^2 \subsetneq L_4^2$ (Shephard [19, Theorem 5] and Heine [13, p. 119]). Given n with $V_n^d \subsetneq L_n^d$, we can deduce $V_{n+1}^d \subsetneq L_{n+1}^d$ by taking a polynomial $g \in L_n^d \setminus V_n^d$. By Proposition 1.5, the polynomial

$$g(x_1, \ldots, x_{n-1}, x_n + x_{n+1})$$

is a Lorentzian polynomial in L_{n+1}^d. If the new polynomial was a volume polynomial, the same would be true for g as setting $x_{n+1} = 0$ preserves volume polynomials due to Remark 1.6. □

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References

[1] Alexandrov, A. D. (1996) *Selected Works. Part I*. Gordon and Breach Publishers, Amsterdam.

[2] Anari, N., Liu, K., Gharan, S. O. and Vinzant, C. (2024) Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid. *Ann. Math.* **199**(1) 259–299.

[3] Anari, N., Liu, K., Gharan, S. O. and Vinzant, C. (2024) Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids. *Proc. Amer. Math. Soc.* **152**(5) 1969–1981.

[4] Anari, N., Liu, K., Gharan, S. O., Vinzant, C. and Vuong, T. D. (2021) Log-concave polynomials IV: approximate exchange, tight mixing times, and near-optimal sampling of forests. In STOC '21 – Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, pp. 408–420.

[5] Anari, N., Gharan, S. O. and Vinzant, C. (2021) Log-concave polynomials, I: entropy and a deterministic approximation algorithm for counting bases of matroids. *Duke Math. J.* **170**(16) 3459–3504.

[6] Bárány, I. and Füredi, Z. (1987) Computing the volume is difficult. *Discrete Comput. Geom.* **2**(4) 319–326.

https://doi.org/10.1017/S0963548325100151 Published online by Cambridge University Press

[7] Brändén, P. and Huh, J. (2020) Lorentzian polynomials. *Ann. Math.* **192**(3) 821–891.
[8] Brändén, P., Leake, J. and Pak, I. (2023) Lower bounds for contingency tables via Lorentzian polynomials. *Israel J. Math.* **253**(1) 43–90.
[9] Dyer, M., Gritzmann, P. and Hufnagel, A. (1998) On the complexity of computing mixed volumes. *SIAM J. Comput.* **27**(2) 356–400.
[10] Fenchel, W. (1936) Inégalités quadratiques entre les volumes mixtes des corps convexes. *C. R. Acad. Sci., Paris* **203** 647–650.
[11] Gurvits, L. (2009) On multivariate Newton-like inequalities. *Adv. Comb. Math.* **61-78**.
[12] Gurvits, L. (2009) A polynomial time algorithm to approximate the mixed volume within a simply exponential factor. *Discrete Comput. Geom.* **41** 533–555.
[13] Heine, R. (1938) Der Wertvorrat der gemischten Inhalte von zwei, drei und vier ebenen Eibereichen. *Math. Ann.* **115**(1) 115–129.
[14] Huh, J. (2022) Combinatorics and Hodge theory. *Proc. Int. Congress Math.* **1**.
[15] Lehmann, B. and Xiao, J. (2017) Correspondences between convex geometry and complex geometry. *EPIGA* **1**, 6,29.
[16] Minkowski, H. (1903) Volumen und Oberfläche. *Math. Ann.* **57**(4) 447–495.
[17] Morgan, F. (2016) *Geometric Measure Theory*. Elsevier/Academic Press, Amsterdam.
[18] Schneider, R. (2014) *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge University Press, Cambridge.
[19] Shephard, G. (1960) Inequalities between mixed volumes of convex sets. *Mathematika* **7** 125–138.

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