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IMPRIMITIVE, IRREDUCIBLE COMPLEX CHARACTERS OF THE ALTERNATING GROUP

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The purpose of this paper is to list all of the characters of A_n, the alternating group, mentioned in the title. The same problem for the symmetric group, S_n, was dealt with by the authors in [**1**]. We show here that, apart from a few exceptions, the imprimitive, irreducible complex characters of A_n fall naturally into two infinite families. (Throughout this paper characters are taken over the complex numbers.)

We recall that a character of a group G is said to be *imprimitive* if it is induced from a character of a proper subgroup of G, and *monomial* if it is induced from a character of degree 1 of any subgroup of G. We denote by T_n the set of all triples (A_n, G, σ), where G is a proper subgroup of A_n, σ an irreducible character of G such that the induced character $\sigma \uparrow A_n$ is also irreducible. We will determine all such triples in this paper. For a subgroup $G \subset S_n$ we shall denote by G^0 the group $G \cap A_n$, and point out that $G = G^0$ or else $[G : G^0] = 2$. We shall refer to G^0 as the *even subgroup* of G.

A major tool we employ is Mackey's criterion for irreducibility, [**9**, p. II-11], which is as follows:

MACKEY'S CRITERION. *Let G be a finite group, H a subgroup of G, and σ an irreducible character of H. Then the induced character $\sigma \uparrow G$ is irreducible if, and only if, for each $t \in G - H$ the restrictions $\sigma \downarrow H \cap H^t$ and $\sigma^t \downarrow H \cap H^t$ are disjoint.*

Here and throughout the paper, $H^t = tHt^{-1}$, and σ^t is the character of H^t defined by $\sigma^t(x) = \sigma(t^{-1}xt)$.

We say that two triples (A_n, G, σ) and (A_n, G', σ') of T_n are *equivalent* if there exists $t \in A_n$ such that $G' = G^t$ and $\sigma' = \sigma^t$. If they are equivalent then $\sigma \uparrow A_n = \sigma' \uparrow A_n$. It suffices to determine the triples of T_n up to equivalence.

For each Young diagram Y we shall denote by Y' the conjugate of Y, and by $[Y]$ the associated irreducible character of S_n. It is well known [**3**] that the restriction $[Y] \downarrow A_n = (Y)$ is irreducible if $Y \neq Y'$, and splits into two components $(Y) = (Y)^+ + (Y)^-$ if $Y = Y'$ and $n > 1$. In the latter case we have also $((Y)^+)^t = (Y)^-$ where t is any odd permutation, and $(Y)^+ \neq (Y)^-$. All irreducible characters of A_n are obtained in this way.

For every divisor m of n such that $1 < m < n$ we denote by $H_{n,m}$ a maximal

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imprimitive subgroup of S_n which has blocks of size m. Thus $H_{n,m} = S_m \wr S_k$ (the *wreath product* of S_m and S_k) where $n = mk$ and the order of $H_{n,m}$ is $k!(m!)^k$. We note that as a permutation group, $H_{2m,m}$ has a unique subgroup $S_m \times S_m$.

We are now in a position to describe the two infinite families of imprimitive irreducible characters of the alternating groups mentioned above.

Family 1. (A_n, A_{n-1}, σ) where $n = m^2 + 1$, $m \geqq 2$, and $\sigma = (Y)^+$ or $(Y)^-$, with Y the square diagram (m^m). We have $\sigma \uparrow A_n = (W)$ where W is the diagram $(m + 1, m^{m-1})$.

Family 2. $(A_{2m}, H_{2m,m}{}^0, \sigma)$ where, according to our notation, $H_{2m,m}{}^0 = H_{2m,m} \cap A_{2m}$. Here σ is either of the two characters of $H_{2m,m}{}^0$ of degree 1 in which $\sigma(x) = 1$ for $x \in A_m \times A_m$ and $\sigma(x) = -1$ for $x \in (S_m \times S_m)^0 - A_m \times A_m$. In this case $\sigma \uparrow A_{2m} = (Y)$ where Y is the diagram $(m + 1, 1^{m-1})$ (or its conjugate).

In the statement of the theorem below we shall refer to the triples in these two families as *standard* triples, and shall refer to all others as *exceptional*.

THEOREM. *For the alternating groups A_n the triples $(A_n, G, \sigma) \in T_n$ are the standard triples described above or else are equivalent to one of the following exceptional triples:*

(i) (A_8, G, σ) *where G is the holomorph of $C_2 \times C_2 \times C_2$, and σ is either of the two complex conjugate, irreducible characters of degree* 3 *of G. Here $\sigma \uparrow A_8 = (Y)^+$ or $(Y)^-$ where $Y = (4, 2, 1^2)$.*

(ii) (A_8, G, σ) *where $G \subset H_{8,4}{}^0$, is the semidirect product $C_2 \cdot (A_4 \times A_4) = A_4 \wr S_2$ with a generator for C_2 of the form* (15)(26)(37)(48), *acting on blocks* $\{1, 2, 3, 4\}$ *and* $\{5, 6, 7, 8\}$. *Here σ is any of the two characters of degree* 1 *of G which when considered as characters of $G/G' = C_6$ are faithful. We have $\sigma \uparrow A_8 = (Y)$ where $Y = (4, 3, 1)$.*

(iii) $(A_8, H_{8,4}{}^0, \gamma)$ *where $\gamma = \sigma \uparrow H_{8,4}{}^0$ is the character induced from the characters σ in* (ii). *Again $\gamma \uparrow A_8 = (Y)$ where $Y = (4, 3, 1)$.*

(iv) (A_9, G, σ) *where G is the primitive group of order* 1512, *and σ is one of the* 2 *complex conjugate non-real characters of degree* 1 *of G. Here $\sigma \uparrow A_9 = (Y)$ where $Y = (5, 2^2)$.*

In the course of the proof we shall need the following results concerning some special types of primitive permutation groups. In the lemmas below we use some old terminology and say that a permutation group has *class* k if k is the minimal number of letters moved by a non-identity permutation in this group.

LEMMA 1. *There exist precisely* 6 *primitive permutation groups of class* 4. *These are:*

(i) *The subgroup of S_5 of order* 20 *generated by* (1, 2, 3, 4, 5) *and* (2, 3, 5, 4), *or its subgroup of order* 10.

(ii) $PGL_2(5)$, *of order* 120, *and* $PSL_2(5)$, *of order* 60, *as subgroups of* S_6.
(iii) $PSL_2(7)$, *of order* 168, *as a subgroup of* S_7.
(iv) *The holomorph of* $C_2 \times C_2 \times C_2$ *as a subgroup of* S_8, *of order* 1344.

This lemma was proved by G. A. Miller in [**7**].

LEMMA 2. *There is only one primitive permutation group of class* 5. *This is the cyclic group* C_5 *in* S_5.

LEMMA 3. *There are eight primitive permutation groups of class* 6 *containing a permutation of type* (123) (456), *five of them with even permutations only. These are:*

(i) *The holomorph of* C_7 *of order* 42 *in* S_7, *and its even subgroup of order* 21.

(ii) $PGL_2(7)$ *and* $PSL_2(7)$ *as subgroups of* S_8. *These have orders* 336 *and* 168, *respectively.*

(iii) *The group of all permutations of the* $GF(8)$ *of the form* $x \rightarrow ax^{\alpha} + b$ *where* $a, b \in GF(8)$, $a \neq 0$, *and* α *is an automorphism of the* $GF(8)$. *This group has order* 168.

(iv) *The holomorph of* $C_3 \times C_3$, *of order* 432, *and its even subgroup of order* 216.

(v) *The group of all permutations of* $GF(8) \cup \{\infty\}$ *of the form*

$$x \rightarrow \frac{ax^{\alpha} + b}{cx^{\alpha} + d}$$

with $ad - bc \neq 0$, *and with* α *as in* (iii). *This group has order* 1512.

This was proved in [**6**]. We may now proceed with the proof of the theorem.

Proof of theorem. Let $(A_n, G, \sigma) \in T_n$. We have to show that (A_n, G, σ) is either standard or equivalent to one of the exceptional triples listed in the theorem. We consider three cases:

Case 1. G is intransitive. There exists a subgroup G_1 of A_n containing G such that G_1 has two orbits, and is maximal subject to these conditions. We let $\sigma_1 = \sigma \uparrow G_1$. We have again $(A_n, G_1, \sigma_1) \in T_n$.

Suppose first that each of the orbits of G_1 has at least 2 letters. We may assume that 1, 2 are in the first orbit, and 3, 4 in the second. Let $t = (13)(24)$, and take $H = G_1 \cap G_1{}^t$. We claim that H is the subgroup of G_1 which stabilizes the sets $\{1, 2\}$ and $\{3, 4\}$. For if $s \in H$, then $(s^{-1}ts)t \in G_1$, and $s^{-1}ts$ has the form $(i, j)(r, k)$ where i, r are in the first orbit, j, k are in the second. Since $(i, j)(r, k)(13)(24) \in G_1$ we must have $\{i, r\} = \{1, 2\}$ and $\{j, k\} = \{3, 4\}$, and this implies that s stabilizes $\{1, 2\}$ and $\{3, 4\}$. It is clear that $z = (12)(34) \in Z(H)$, the centre of H. Let τ be an irreducible component of $\sigma_1 \downarrow H$. Because $z \in Z(H)$, and τ is irreducible, it follows that z must be represented by $\pm I$ in the representation ρ corresponding to τ. We claim that this must, in fact, be $-I$. Assume not. Since $s^{-1}tst = 1$ or z for every $s \in H$ we would have $\rho(s^{-1}tst) = I$ and $\rho(tst) = \rho(s)$ for all $s \in H$. But then, however, $\tau^t = \tau$, and $\sigma_1 \downarrow H$ and $\sigma_1{}^t \downarrow H$ have components in common, contrary to Mackey's criterion. We conclude that $\rho(z) = -I$ in each irreducible component of $\sigma_1 \downarrow H$ and hence that z is represented by $-I$ in the representation corresponding to the character σ_1. But these remarks hold for any permutation $(i, r)(j, s)$ with i and r in the first orbit, j and s in the second. Since these generate G_1, and are all represented by $-I$, σ_1, to be irreducible, must have degree 1. $\sigma_1(x) = 1$ whenever the restriction of x to an orbit is even, and equals -1 otherwise.

Now let $t_1 = (134)$. By a computation similar to that above we see that $H_1 = G_1 \cap G_1^{t_1}$ is the subgroup of G_1 which fixes the letters 1 and 3 since, for $s \in H_1$, the commutator $s^{-1}t_1^{-1}st_1$ is either the identity or a three-cycle in the second orbit. To see this, observe that $s^{-1}t_1^{-1}s$ must be of the form $(1, i, 3)$, where i is in the second orbit. But now $\sigma_1 \downarrow H_1 = \sigma_1^{t_1} \downarrow H_1$ because $\sigma_1(s^{-1}t_1^{-1}st_1) = 1$ for all $s \in H_1$ and so $\sigma_1(t_1^{-1}st_1) = \sigma_1(s)$ for all $s \in H_1$. This again contradicts Mackey's criterion, and it remains to consider the case when, say, the second orbit has only one letter. In this event $G_1 = A_{n-1}$.

Consider the diagram

$$\begin{array}{ccc} S_{n-1} & \longrightarrow & S_n \\ \uparrow & & \uparrow \\ A_{n-1} & \longrightarrow & A_n \end{array}$$

The Littlewood-Richardson rule [**8**, p. 61] enables one to compute the irreducible components of any $\lambda \uparrow S_n$ where λ is an irreducible character of $S_k \times S_{n-k}$. In particular, $\lambda \uparrow S_n$ is never irreducible. Hence, if a character λ of S_{n-1} is not irreducible, $\lambda \uparrow S_n$ has at least four irreducible components. Now since $\sigma_1 \uparrow A_n$ is irreducible, $\sigma_1 \uparrow S_{n-1}$ is irreducible, and σ_1 corresponds to a symmetric Young diagram, Y. We have $\sigma_1 \uparrow S_{n-1} = [Y]$ and $[Y] \uparrow S_n$ has one or two irreducible components. It cannot have one, and so has exactly two. Using, again, the Littlewood-Richardson rule, we see that Y must be rectangular. The requirement $Y = Y'$ forces Y square, and we see that our first family of standard triples is the only possibility in this case. It is easy to check that the induced representations $(Y)^+ \uparrow A_n$ and $(Y)^- \uparrow A_n$ are, in fact, irreducible.

Suppose now that $(A_n, G, \sigma) \in T_n$, with $G \subset A_{n-1}$. By a previous paper [**1**], the character (Y) of A_{n-1} is not imprimitive, and $G = A_{n-1}$. This completes this case.

Case 2. G is transitive, but imprimitive. Let G_1 be a maximal imprimitive subgroup of A_n containing G, and let $\sigma_1 = \sigma \uparrow G_1$. Again, $(A_n, G_1, \sigma_1) \in T_n$. Using our previous notation $G_1 = H_{n,m}{}^0$, where $m|n$. We assume first that $n \geqq 3m$; i.e. there are at least three blocks. Assume, too, that $m \geqq 3$. Take 1, 2, 3 to be in different blocks and let $t = (123)$. In this case, using arguments similar to those used before, we see that t centralizes $H = G_1 \cap G_1{}^t$, and so $\sigma_1 \downarrow H = \sigma_1{}^t \downarrow H$, contradicting Mackey's criterion.

Suppose now that $m = 2$, and suppose further that $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ are three of the blocks. Let $t = (135)(264)$. As before we have $s^{-1}t^{-1}st \in G_1$ for each $s \in H = G_1 \cap G_1^t$. Now $t_1 = s^{-1}t^{-1}s$ is of the form $(i, j, k)(u, v, w)$ where $\{i, u\}$, $\{j, w\}$, $\{k, v\}$ are blocks. Then $t_1t \in G_1$ implies that these blocks are, in fact, $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ in some order. Moreover, we must have $t_1t = 1$; i.e. t commutes with s and consequently t centralizes H. We reach a contradiction as before.

Suppose, finally, that $n = 2m$; i.e. there are only two blocks. Let, first, $m \geqq 5$. Let 1, 2 be in the first block, 3, 4 in the second, and let $t = (13)(24)$. In this case $H = G_1 \cap G_1^t$ is the subgroup of G_1 stabilizing $\{1, 2, 3, 4\}$. Indeed, if $s \in H$ then $s^{-1}tst \in G_1$ and $s^{-1}ts = (ij)(uv)$ with i, u in the first and j, v in the second block. This implies that $\{i, j, u, v\} = \{1, 2, 3, 4\}$ because $m \geqq 5$. Hence, s stabilizes $\{1, 2, 3, 4\}$. Conversely, if $s \in G_1$ stabilizes $\{1, 2, 3, 4\}$ then it is immediate that $s \in H$.

Moreover we see that for $s \in H$ we have $s^{-1}tst = 1$ or z where $z = (12)(34)$. Again z is in the centre of H and in the same way as in Case 1 we infer that $\deg \sigma_1 = 1$ and $\sigma_1(x) = 1$ for $x \in A_m \times A_m$, $\sigma_1(x) = -1$ for $x \in (S_m \times S_m)^0 - (A_m \times A_m)$. Note that $A_m \times A_m \lhd G_1 = H_{2m,m}{}^0$ and $G_1/(A_m \times A_m)$ is the four-group. Therefore there are precisely two characters σ_1 of G_1 of degree 1 having the properties established above. Since $\deg \sigma_1 = 1$ we must have $G = G_1$ and $\sigma = \sigma_1$. In other words, (A_n, G, σ) is a standard triple from the second family. It remains to verify that $\sigma \uparrow A_n$ is indeed irreducible. For this we do not need the hypothesis that $m \geqq 5$.

We must show that $\sigma_1 \uparrow A_n$ is irreducible. Consider the diagram

$$\begin{array}{ccc} A_{2m} & \longrightarrow & S_{2m} \\ \uparrow & & \uparrow \\ H_{2m,m}{}^0 & & \\ \uparrow & & \\ (S_m \times S_m)^0 & \longrightarrow & S_m \times S_m \end{array}$$

Let $\tau = \sigma \downarrow (S_m \times S_m)^0$. We find

$$\tau \uparrow S_m \times S_m = \alpha \otimes 1 + 1 \otimes \alpha$$

where α is the alternating character of S_m. By applying the Littlewood-Richardson rule we get

$$\tau \uparrow S_{2m} = 2(\rho + \rho')$$

where $\rho = [Y]$, $\rho' = [Y']$ and $Y = (m + 1, 1^{m-1})$. Since $(\tau \uparrow A_{2m}) \uparrow S_{2m} = 2(\rho + \rho')$ we must have

$$\tau \uparrow A_{2m} = 2(Y)$$

and consequently $\sigma \uparrow A_{2m} = (Y)$ is irreducible.

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Now we have to look at the cases $m = 2, 3, 4$. One can easily verify, by using published character tables of $H_{2m,m}$ and S_{2m} [**3, 4**], that the only exceptional triples that arise are (ii) and (iii) in the theorem.

Case 3. G is primitive. If the class of G is $\geqq 7$ then taking $t = (123)$ we get $s^{-1}t^{-1}st \in G$ for every $s \in H = G \cap G^t$ and hence $st = ts$. This contradicts Mackey's criterion. Hence the class of G must be 4, 5 or 6 and if it is 6 then G contains a permutation of the type $(123)(456)$. Thus G is one of the groups listed in Lemmas 1, 2, 3. It is a matter of straightforward computation to check that again the triples that arise from these groups are the exceptional triples (i) and (iv) in the theorem. The character table for the group of order 1512 is given in [**5**].

The following corollaries are immediate.

COROLLARY 1. *The only imprimitive irreducible characters of the alternating groups are those associated with the Young diagrams* $(m + 1, m^{m-1})$, $m \geqq 2$; $(m + 1, 1^{m-1})$, $m \geqq 2$; $(4, 3, 1)$, $(4, 2, 1^2)$ *and* $(5, 2^2)$.

COROLLARY 2. *The monomial irreducible characters of the alternating groups are those associated with the diagrams* $(m + 1, 1^{m-1})$, $m \geqq 2$; (m), $m \geqq 2$; $(2, 1)^{\pm}$; $(2^2)^{\pm}$; $(4, 3, 1)$ *and* $(5, 2^2)$.

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