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# ON THE EQUATION $x^y \pm y^x = \prod n_i!$

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**Abstract.** In this note we investigate the diophantine equation

$$x^y \pm y^x = \prod n_i!$$

where $x$ and $y$ are odd and greater than 1. We prove that this equation has no integer solutions.

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**0. Introduction.** We study the diophantine equation

$$x^y \pm y^x = \prod n_i!, \tag{1}$$

when $x$ and $y$ are odd. The case $xy$ even is less interesting since then both $x$ and $y$ are even and the terms $x^y$ and $y^x$ have a large greatest common divisor.

The main tool of our study is a result on linear forms in two 2–adic logarithms, due to M. Laurent and Y. Bugeaud. This result enables us to show that equation (1) has only a finite number of solutions. More precisely, we first get reasonable bounds on $x$ and $y$ and then we have to "fill the gap".

To solve the remaining computational problem was not at all trivial. For this purpose, the elementary Proposition 1 below played an essential role in the sense that it replaced a problem of quadratic cost by one of linear cost. Thus, the verification took a reasonable time.

Before proving Proposition 1 and using linear forms in 2–adic logarithms, we gather a few elementary facts on factorials.

**1. Preliminary results.**

Lemma 1. *For each positive integer $n$ and any prime number $p$, we have*

$$\frac{n}{p-1} - \frac{\log(n+1)}{\log p} \le v_p(n!) \le \frac{n}{p-1}.$$

*Proof*. See [**1**, Lemma 1].

COROLLARY 1. *For each positive integer $n \geq 2$, we have*

$$\frac{n}{3} \leq v_2(n!) \leq n.$$

*Proof*. Notice first that the function $x \mapsto \frac{\log(x+1)}{x}$ is non-increasing for $x \geq 2$. Then, by Lemma 1 above, we see that the result is certainly true for $n \geq 3$. The inequalities claimed are obvious for $n = 2$.

LEMMA 2. *Suppose that $x$ and $y$ are rational integers with $1 < x < y$. Let $h = y - x$. Then, for $x = 2$ one has*

$$2^y - y^2 = 2^y\big(1 - (y2^{-y/2})^2\big) \geq \tfrac{7}{32} \cdot 2^y \quad \text{for} \quad y \geq 5,$$

*while for $x \geq 3$ one has*

$$x^y - y^x > x^y\big(1 - (e/x)^h\big).$$

*Proof*. For $x = 2$, the result follows from the fact that the function $y \mapsto y2^{-y/2}$ is non-increasing for $y \geq 3$. For $x \geq 3$, we have

$$x^y - y^x = x^y\big(1 - x^{-h}(y/x)^x\big)$$

and

$$(y/x)^x = \exp\big(x \log(1 + h/x)\big) < e^h.$$

LEMMA 3. *For any rational integer $n \geq 2$, we have*

$$3.69\,(n/e)^n \leq n! \leq 3.77\,(n/2.5)^n$$

*and*

$$n! \leq 2.83\,(n/e)^{(n+1)} \qquad \textit{when} \quad n \geq 6.$$

*Proof*. We prove only the first two inequalities. The proof of the last one is similar. The proof follows from Stirling's formula

$$n! = \sqrt{2\pi n}\, e^{\theta/n}\,(n/e)^n \qquad \textit{with} \quad 0 < \theta < 1/6.$$

More precisely, the left inequality is a direct implication of this formula (for $n \geq 3$ and it can be directly verified for $n = 2$) while the right inequality is implied by it for $n \geq 8$ and an elementary verification can be used for the remaining values of $n$. One may notice that the minimum of the constant appearing on the left is reached for $n = 2$, while the maximum of the constant appearing on the right is obtained for $n = 6$.

PROPOSITION 1. *Let $a$ and $b$ be odd integers and let $n \geq 1$. Then, the equation $ax^y + by^x \equiv 0 \pmod{2^n}$ with $x$ and $y$ odd in $\mathbb{Z}/2^n\mathbb{Z}$ has exactly $2^{n-1}$ solutions.*

*Proof*. We proceed by induction on $n$. When $n = 1$, the result is trivial.

Suppose that the result is true for some $n \geq 1$ and consider an odd solution $(x, y)$ of the equation $ax^y + by^x \equiv 0 \pmod{2^n}$. Let us search for the solutions $(x', y')$ $(\text{mod } 2^{n+1})$ with $x' \equiv x \pmod{2^n}$ and $y' \equiv y \pmod{2^n}$. Equivalently, $x' = x + \alpha t$ and $y' = y + \beta t$ with $\alpha, \beta \in \{0, 1\}$ and $t = 2^n = \varphi(2^{n+1})$. From Euler's theorem and from the fact that both $x$ and $y$ are odd, it follows that

$$ax'^{y'} + by'^{x'} \equiv ax'^{y} + by'^{x} \pmod{2^{n+1}}.$$

It now follows, by the binomial formula, that

$$\begin{aligned} ax'^{y} + by'^{x} = a(x+\alpha t)^y + b(y+\beta t)^x &\equiv ax^y + by^x + txy\big(a\alpha x^{y-2} + b\beta y^{x-2}\big) \\ &\equiv ax^y + by^x + t(a\alpha + b\beta) \pmod{2^{n+1}}. \end{aligned}$$

If we put $ax^y + by^x = ut$, we then get the congruence

$$u + a\alpha + b\beta \equiv 0 \pmod 2$$

which has, obviously, exactly two solutions.

Corollary 2. *Let $n \geq 1$. Then the equation $x^y - y^x \equiv 0 \pmod{2^n}$ with $x$ and $y$ odd in $\mathbb{Z}/2^n\mathbb{Z}$ has only the solutions $(x, x)$ with $x$ odd in $\mathbb{Z}/2^n\mathbb{Z}$.*

*Proof*. The $2^{n-1}$ pairs $(x, x)$ are trivial solutions and, since the number of solutions is exactly $2^{n-1}$, it follows that there can be no other ones.

Remarks.

(1) The above proposition (as well as its corollary) is true for some other moduli. For example, it is true modulo $3 \cdot 2^n$ when $x$ and $y$ are subject to the condition $\gcd(xy, 6) = 1$.

(2) The proof of Proposition 1 can be adapted to imply the following stronger result: *Let $a$ and $b$ be odd integers and let $c$ be an even integer. Then, for any positive integer $n$, the equation $ax^y + by^x \equiv c \pmod{2^n}$ with $x$ and $y$ odd in $\mathbb{Z}/2^n\mathbb{Z}$ has exactly $2^{n-1}$ solutions.*

**2. Application of 2-adic linear forms in two logarithms.** Suppose that for two odd integers $x$ and $y$ with $y > x > 1$ we have

$$\Lambda := x^y \pm y^x = \pm \prod_{i=1}^{k} n_i!.$$

From equation (1) and Lemma 1, we get that $v_2(x^y \pm y^x) \geq N/3$, where $N = \sum n_i$. We now apply [**1**, Theorem 1]. With their notations, we have

$$p = 2, \quad D = e = g = t = 1, \quad v_p(\Lambda) \geq N/3$$

and

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$$\alpha_1 = y, \quad \alpha_2 = x, \quad b_1 = x, \quad b_2 = y.$$

We take

$$A_1 = y, \quad A_2 = x.$$

From [**1**, formula (2)], we have

$$v_p(\Lambda) \leq 2(KL - 1/2) = 2KL - 1 \tag{2}$$

whenever $K \geq 3$ and $L \geq 2$ are integers such that

$$2K(L-1)\log 2 > 3\log(KL) + (K-1)\log b + 2L\Big(\frac{1}{2} - \frac{KL}{6RS}\Big)(R\log y + S\log x) \tag{3}$$

where

$$b = \frac{(R-1)y + (S-1)x}{2}\Big(\prod_{k=1}^{K-1} k!\Big)^{-2/(K^2-K)}$$

and $R$ and $S$ are two positive integers such that $K, L, R, S$ satisfy [**1**, inequalities (1)].

We distinguish two cases.

*Case 1.* $x$ and $y$ are multiplicatively independent.
We employ the method described in [**1**, Section 5.1]. Let

$$a_1 = \frac{\log y}{\log 2}, \quad a_2 = \frac{\log x}{\log 2}.$$

We choose $K = \lfloor kLa_1a_2 \rfloor + 1$ where $k$ is a positive parameter. From [**1**, Lemma 13], we know that

$$\log b \leq \log\Big(\frac{b_1}{a_2} + \frac{b_2}{a_1}\Big) - \frac{1}{2}\log k - \log 2 + \frac{3}{2} + \log\frac{(1+\sqrt{k-1})\sqrt{k}}{k-1}. \tag{4}$$

Using [**1**, Lemma 12], one may easily show that inequality (3) holds whenever

$$\begin{aligned} kL(L-1)a_1a_2 >& 3\frac{\log(kL^2a_1a_2 + L)}{2\log 2} + \frac{kLa_1a_2\log b}{2\log 2} + \frac{1}{3}\sqrt{k}L^2a_1a_2 \\ &+ \frac{2L^{3/2}\sqrt{a_1a_2}}{3} + \frac{L(a_1+a_2)}{3}. \end{aligned} \tag{5}$$

(see [**3**] for detailed proof of the fact that inequality (5) implies inequality (3)).
In conclusion, from inequality (2), it follows that

$$N \leq 3(2\lfloor kLa_1a_2 \rfloor L + 2L - 1) \tag{6}$$

whenever $k$ and $L$ are such that inequalities (4) and (5) are satisfied.

From an elementary argument and from Lemma 3, we get

$$\prod n_i! \leq N! \leq 2.83\,(N/e)^{(N+1)}.$$

Moreover, it follows, by Lemma 2, that $|\Lambda| \geq x^y\big(1-(e/3)^2\big)$. Hence, $x^y \leq 5.59 \cdot |\Lambda|$ and

$$y \leq \frac{\log\big(15.82\,(N/e)^{(N+1)}\big)}{\log x}. \tag{7}$$

We first use the above inequality and [**2**, Corollary 3] to get a rough upper bound on $y$, namely $y < 10^7$. We then refine this estimate by using the full machinery of [**2**, Theorem 1] to obtain

$$y < 3 \cdot 10^6.$$

More precisely, we choose suitable values of $k \in [0.8, 1.2]$ and $L \in \{25, 26, 27, 28\}$ and we solve inequalities (4), (5), (6) and (7).

*Case 2*. $x$ and $y$ are not multiplicatively independent.

Write $x^a = y^b$ for some coprime positive integers $a$ and $b$. At least one of the integers $a$ and $b$, say $a$, is odd. Now computing the order at which 2 appears in $\Lambda$ is the same as computing the order at which 2 appears in

$$(x^y)^a \pm (y^x)^a = (x^a)^y \pm y^{ax} = y^{by} \pm y^{ax} = z(y^{\epsilon(by-ax)} \pm 1),$$

where $z = y^{ax}$ or $z = -y^{by}$ according to whether $\epsilon = 1$ or $\epsilon = -1$. It follows that

$$\begin{aligned} v_2(\Lambda) &\leq \max(v_2(y+1), v_2(y-1)) + v_2(|ax-by|) \\ &< \log_2(y+1) + \log_2(y) + \log_2(\max(a,b)). \end{aligned}$$

It now suffices to notice that $\max(a,b)$ is precisely the largest exponent at which some prime number appears in the prime factor decomposition of either $x$ or $y$. In particular, $\max(a,b) \leq \log_3(y)$. Hence,

$$v_2(\Lambda) < \log_2(y+1) + \log_2(y) + \log_2(\log_3(y)) < 3\log_2(y+1).$$

It follows, by Corollary 1, that $N < 9\log_2(y+1)$. Combining this last inequality with inequality (7) we get $y < 211$.

**3. The computer verification.**

(1) *The "+" case*.

We first consider the equation

$$x^y + y^x \equiv 0 \pmod{2^k}$$

when $x$ and $y$ are odd, $1 < x < y < 3 \cdot 10^6$ and $k$ is a large enough integer. More precisely, we used the algorithm described in the proof of Proposition 1 to write a

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C-program which verified, in about 40 minutes, that there are only 3982 pairs $(x, y)$ with $x$ and $y$ odd, $1 < x < y < 3 \cdot 10^6$ which verify the above congruence for $k = 30$. Then, a second program—written in Pari—proved in a few minutes that all these pairs satisfy $x^y + y^x \not\equiv 0 \pmod{2^{40}}$. These computations prove the following proposition.

PROPOSITION $2^+$. *Let $x$, $y$ be odd integers,* $1 < x < y < 3 \cdot 10^6$. *Then*

$$x^y + y^x \not\equiv 0 \pmod{2^{40}}.$$

From the bound $N \leq 3 \cdot v_2(\Lambda)$, we saw that $N \leq 117$. This implied $x^y \leq 117!$; hence, $y \leq 403$. We ran again our C-program which told us that in this range $v_2(\Lambda) \leq 17$. We now got $N \leq 51$ and $y \leq 138$. A second application of the C-program gave $v_2(\Lambda) \leq 14$, which implied that $N \leq 42$ and $y \leq 107$. A third application of the C-program gave $v_2(\Lambda) \leq 13$, which implied that $N \leq 39$ and $y \leq 97$. A fourth application of the C-program gave $v_2(\Lambda) \leq 10$, which implied $N \leq 39$ and $y \leq 75$. Finally, we considered all the pairs $(x, y)$ with $x$ and $y$ odd and $1 < x < y \leq 75$ and we computed $P(x^y + y^x)$ where $P(k)$ denotes the largest prime factor of $k$. It happens that, in this range, $P(x^y + y^x) \geq 239$ (thus $x^y + y^x$ cannot be a product of factorials because $P(\prod n_i!) \leq P(N!) \leq N \leq 39$), except for the pair $(x, y) = (3, 9)$. However, this last pair gives $x^y + y^x = 2^2 \times 3^6 \times 7$ which is, certainly, not a product of factorials.

(2) *The "$-$" case.*
We now consider the equation

$$x^y - y^x \equiv 0 \pmod{2^k}.$$

In this case, thanks to the Corollary of Proposition 1, we need no computation and we get at once the following result.

PROPOSITION $2^-$. *Let $x$, $y$ be odd integers,* $1 < x < y < 3 \cdot 10^6$. *Then*

$$x^y - y^x \not\equiv 0 \pmod{2^{22}}.$$

By an argument similar to the one employed in the "$+$" case, we get $N \leq 3 \times 21$. Thus, $y \leq \lfloor \log(5.59 \cdot 63!)/\log 3 \rfloor = 184$. Now the Corollary of Proposition 1 implies $x^y - y^x \not\equiv 0 \pmod{2^8}$. Hence, $N \leq 21$ and $y \leq 42$. A further application of this argument gives $N \leq 15$ and $y \leq 27$. Then, a trivial verification achieves the goal: except for the pair $(x, y) = (3, 9)$ we have $P(x^y - y^x) > 24$ whenever $x$ and $y$ are odd and $1 < x < y \leq 27$. Since $3^9 - 9^3 = 2 \times 3^6 \times 13$ it follows, as in the previous case, that this number is not a product of factorials.

(3) *Conclusion.*
The above arguments prove the following result.

THEOREM. *The diophantine equation*

$$x^y \pm y^x = \pm \prod n_i!$$

*has no odd solutions $x$ and $y$ with* $\min(x, y) > 1$.

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