Canad. Math. Bull. Vol. **27** (2), 1984

GELFAND–KIRILLOV DIMENSION IS EXACT FOR NOETHERIAN PI ALGEBRAS

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ABSTRACT. If $0 \to A \to C \to B \to 0$ is a short exact sequence of finitely generated modules over a Noetherian PI-algebra then we show that $\mathrm{GK}(C) = \max\{\mathrm{GK}(A), GK(B)\}$.

The Gelfand–Kirillov dimension has proved to be a very useful tool in the study of enveloping algebras of finite dimensional Lie algebras. It might be hoped that the dimension could be used in studying other kinds of algebras. One obvious place to try would be in PI algebras. Here, there has been some small success, e.g. [L], [S], but, unfortunately, many of the examples that show that GK dimension can have weird properties arise from PI rings. For example, Borho and Kraft, [BK], exhibit an algebra that has a homomorphic image of GK dimension α for all real numbers α between two and three. This algebra is an affine PI algebra. Also, Bergman, [B], has constructed finitely generated modules A and B and an extension C of A by B such that $\mathrm{GK}(C) > \max\{\mathrm{GK}(A), \mathrm{GK}(B)\}$. The ring involved here is again an affine PI algebra.

However, in a recent paper, [LS], Lorenz and Small have shown that if R is a Noetherian PI algebra then $\mathrm{GK}(R) = \mathrm{GK}(R/P)$, for some prime ideal P of R. As a consequence, if $\mathrm{GK}(R) < \infty$ then $\mathrm{GK}(R)$ is an integer. In this note we show that it follows easily from [LS] that GK dimension behaves well on exact sequences of modules.

The main theorem of [LS] depends on a lemma that has a long involved combinatorial proof. As an appendix to this note we provide a somewhat simplified proof of this lemma.

For standard results concerning GK dimension, we refer to [BK].

Throughout the note, but not the appendix, R denotes a Noetherian k-algebra that satisfies a polynomial identity over the field k, and if M is an R-module then $\mathrm{GK}(M)$ is the Gelfand–Kirillov dimension of M_R over k.

LEMMA 1. *If M is a finitely generated right R-module and $A = \mathrm{ann}_R (M)$ then* $\mathrm{GK}(M) = \mathrm{GK}(R/A)$.

Received by the editors June 3, 1983.
AMS classifications: 16A33, 16A38, 16A55.

Proof. Since M is an R/A-module, $\mathrm{GK}(M) \leq \mathrm{GK}(R/A)$. Now Cauchon has shown that R is an H-ring; that is, there exist elements $m_1, \ldots, m_n \in M$ such that $A = \mathrm{ann}_R(m_1, \ldots, m_n)$, see, for example [CH, Theorem 7.8]. Thus, there is an embedding

$$R/A \to m_1 R \oplus \cdots \oplus m_n R \subseteq M \oplus \cdots \oplus M;$$

so that $\mathrm{GK}(R/A) \leq \mathrm{GK}(M \oplus \cdots \oplus M) = \mathrm{GK}(M)$.

It follows from this lemma and the Lorenz–Small result that $\mathrm{GK}(M)$ is either infinite or an integer.

LEMMA 2. *If I, J are ideals of R such that $IJ = 0$ then* $\mathrm{GK}(R) = \max\{\mathrm{GK}(R/I), \mathrm{GK}(R/J)\}$.

Proof. By [LS], there is a prime ideal P of R such that $\mathrm{GK}(R) = \mathrm{GK}(R/P)$. Now $IJ = 0 \subseteq P$, so that either $I \subseteq P$ or $J \subseteq P$, suppose the former. Then $\max\{\mathrm{GK}(R/I), \mathrm{GK}(R/J)\} \geq \mathrm{GK}(R/I) \geqq \mathrm{GK}(R/P) = \mathrm{GK}(R)$. The reverse inequality is obvious.

PROPOSITION 3. *If $0 \to A \to C \to B \to 0$ is a short exact sequence of finitely generated R-modules then* $\mathrm{GK}(C) = \max\{\mathrm{GK}(A), \mathrm{GK}(B)\}$.

Proof. By passing to the factor ring $R/\mathrm{ann}_R(C)$ and using Lemma 1, we may assume that C is a faithful R-module and that $\mathrm{GK}(C) = \mathrm{GK}(R)$. Let $I = \mathrm{ann}_R(B)$ and $J = \mathrm{ann}_R(A)$. Then $CIJ = 0$, so that $IJ = 0$. Hence,

$$\begin{aligned} \mathrm{GK}(C) = \mathrm{GK}(R) &= \max\{\mathrm{GK}(R/I), \mathrm{GK}(R/J)\} \\ &= \max\{\mathrm{GK}(B), \mathrm{GK}(A)\}, \end{aligned}$$

by Lemma 2 and Lemma 1.

As a consequence of Proposition 3, one can see that for Noetherian PI algebras the GK dimension shares many of the formal properties of non-commutative Krull dimension. For example, if n is any integer and M is a finitely generated R-module, then there exists a unique largest submodule N of M such that $\mathrm{GK}(N) \leq n$; in this case, M/N has no nonzero submodules with GK dimension $\leq n$.

If M is an R-module then $\rho_R(M)$ denotes the reduced rank of R; see [CH] for details concerning reduced rank.

LEMMA 4. *If M is a finitely generated R-module such that $\rho_R(M) = 0$ then* $\mathrm{GK}(M) < \mathrm{GK}(R)$.

Proof. By Noetherian induction, assume that the result is true in all proper factors of M. If N denotes the nilpotent radical of R then $X = \mathrm{ann}_M(N) \neq 0$. Now $\rho_R(X) = 0$; so if $0 \neq x \in X$ then there exists an element $c \in R$ that is a nonzero divisor modulo N such that $xc = 0$, [CH, Theorem 2.2]. Hence

$x(cR+N)=0$ and $\mathrm{GK}(xR)\le\mathrm{GK}(R/cR+N)<\mathrm{GK}(R/N)=\mathrm{GK}(R)$. However, $\mathrm{GK}(M/xR)<\mathrm{GK}(R)$, by induction; so $\mathrm{GK}(M)<\mathrm{GK}(R)$, by Proposition 3.

COROLLARY 5. *Let M be a finitely generated R-module with $\mathrm{GK}(M)=n$. If $M>M_1>M_2>\cdots$ is any descending chain of submodules then only finitely many of the factors M_i/M_{i+1} can have* GK *dimension n. In fact, there are at most $\rho_{R/\mathrm{ann}(M)}(M)$ such factors.*

Proof. $\mathrm{GK}(R/\mathrm{ann}(M))=\mathrm{GK}(M)$, by Lemma 1. If $\mathrm{GK}(M_i/M_{i+1})=n$ then $\rho_{R/\mathrm{ann}(M)}(M_i/M_{i+1})\geqslant 1$, by Lemma 4. The result follows since reduced rank is additive on short exact sequences. [CH, Theorem 2.2]

Given this corollary, we can make a comparision between GK dimension and the noncommutative Krull dimension.

COROLLARY 6. *If s denotes the minimal possible* GK *dimension of any simple R-module then, for any finitely generated R-module M,*

$$K.\dim(M)\le\mathrm{GK}(M)-s$$

Proof. cf. [Sm, Lemma 2.3]

APPENDIX

The main result in [LS] depends on the following lemma. We present a proof that has fewer combinatorial manipulations and so may be easier to follow.

LEMMA. *Let R be a k-algebra and let N be a nilpotent ideal of R that is finitely generated as a right ideal of R. If R/N is finitely generated module over a commutative subalgebra S/N, then* $\mathrm{GK}(R)=\mathrm{GK}(R/N)$.

Proof. It is only necessary to show that $\mathrm{GK}(R)\le\mathrm{GK}(R/N)$: so assume that $\mathrm{GK}(R/N)=d<\infty$. We argue by induction on the index of nilpotency, ρ, of N. If $\rho=1$ then $N=0$ and the result is true. Otherwise, assume that $\mathrm{GK}(R/N^{\rho-1})=d$; so that $\mathrm{GK}(N_R)=\mathrm{GK}(N_{R/N^{\rho-1}})\le d$.

Notice that R is a finitely generated S-module, so that $\mathrm{GK}(R)=\mathrm{GK}(S)$, by [LS, Lemma 1i]. Also, N_S is finitely generated; so, by replacing R by S, we may assume that R/N is commutative.

Let V be any finite dimensional k-subspace of R with $1\in V$. Then, by the Noether normalization theorem, the subalgebra $k[\bar V]$ of R/N is a finitely generated module over a polynomial ring $T/N=k[\bar y_1,\ldots,\bar y_r]$, where $y_1,\ldots,y_r$ are suitable elements of R. Now, the ring $k[V]+N$ is a finitely generated module over T, so that $\mathrm{GK}(k[V]+N)=\mathrm{GK}(T)$. If $\mathrm{GK}(T)=s$, then, for any $\varepsilon>0$, there exists a real number $\alpha=\alpha(\varepsilon)$ such that $\dim(V^n)\le\alpha n^{s+\varepsilon}$. Consequently, $\dim(V^n)\le\alpha n^{d+\varepsilon}$, provided that we can show that $\mathrm{GK}(T)\le d$. Since V is an arbitrary finite dimensional k-subspace of R, this shows that $\mathrm{GK}(R)\le d$.

So, let T be such a subring and let $y_1, \ldots, y_r$ be elements of T such that $T/N = k[\bar{y}_1, \ldots, \bar{y}_r]$ and the $\bar{y}_i$ are algebraically independent. Notice that $r \leq d$. Let V be any finite dimensional k-subspace of T of the form $V = Y \oplus Z$, where $Y = k + y_1 k + \cdots y_r k$ and $Z \subseteq N$. Fix a finite generating set Γ of N as an R-module. Choose a finite dimensional k-space W of R such that $V \subseteq W$ and $Z + V\Gamma + [V, V] \subseteq \Gamma W$. Now any monomial u in V^n is either in Y^n, or contains at least one element from $Z \subset \Gamma W$, so that, for some i, we have $u \in V^{i-1}(\Gamma W)V^{n-i} \subseteq \Gamma W^n$. Hence, $V^n \subseteq Y^n + \Gamma W^n$, for all integers $n \geq 1$.

Let $u_1, \ldots, u_n \in Y$. Now $[Y, Y] \subseteq \Gamma W$; so, for $1 \leq i \leq n-1$, we have $u_1 \cdots u_{i-1}(u_i u_{i+1} - u_{i+1} u_i) u_{i+2} \cdots u_n \in Y^{i-1}\Gamma W Y^{n-i-1} \subseteq \Gamma W^{n-1}$. Hence $u_1 \cdots u_n = u_1 \cdots u_{i-1} u_{i+1} u_i u_{i+2} \cdots u_n \bmod(\Gamma W^{n-1})$.

Now let Y_n denote the k-subspace spanned by the elements $y_1^{e(1)} \cdots y_r^{e(r)}$, where $\sum_{i=1}^r e(i) \leq n$. The equation above allows us to rearrange the order of elements of Y^n, provided that we work $\bmod(\Gamma W^{n-1})$. Hence, $Y^n \subseteq Y_n + \Gamma W^{n-1}$.

Thus, $V^n \subseteq Y^n + \Gamma W^n \subseteq Y_n + \Gamma W^n$, so that $\dim(V^n) \leq \dim(Y_n) + \dim(\Gamma W^n)$. However, since $r \leq d$, there exists a real number α such that $\dim(Y_n) \leq \alpha n^d$, and, since $\mathrm{GK}(N) \leq d$, for any $\varepsilon > 0$, there exists a real number $\beta = \beta(\varepsilon)$ such that $\dim(\Gamma W^n) \leq \beta n^{d+\varepsilon}$. Setting $\gamma = \alpha + \beta$, we obtain $\dim(V^n) \leq \gamma n^{d+\varepsilon}$, for any $\varepsilon > 0$.

Finally, if X is an arbitrary finite dimensional k-subspace of T, then, since $T = k[y_1, \ldots, y_r] + N$, there exists an integer a such that $X \subseteq V^a$, for some V of the form treated above. Thus $\dim(X^n) \leq \dim(V^{an}) \leq \gamma(an)^{d+\varepsilon} = (\gamma a^{d+\varepsilon}) n^{d+\varepsilon}$, for any $\varepsilon > 0$, and so $\mathrm{GK}(T) \leq d$.

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