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PERIODIC SOLUTIONS OF QUASILINEAR NON-AUTONOMOUS SYSTEMS WITH IMPULSES

S.G. Hristova and D.D. Bainov

The paper considers a system of differential equations with impulse perturbations at fixed moments in time of the form

$$\dot{x}(t) = A(t)x + f(t) + \varepsilon X(t, x, \varepsilon) , \quad t \neq t_i ,$$

$$\Delta x|_{t=t_i} = B_i x(t_i) + a_i + \varepsilon I_i(x(t_i), \varepsilon) ,$$

where $x \in R^n$, ε is a small parameter,

$$\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0) .$$

Sufficient conditions are found for the existence of the periodic solution of the given system in the critical and non-critical cases.

Systems with impulses find a growing application in mathematical modelling related to control theory, radiophysics, pharmacokinetics, biology and so on. Hence the necessity to organize a mathematical theory of systems with impulses.

The first papers in this theory [2], [3] are related to the names of Millman and Mishkis. A growing interest in this theme followed their initial works. A number of papers have been published, as for instance,

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[4], [5], [6], [8], [9], [10], [11], as well as the monographs [1] and [7]. We could note, however, that the results of these contributions are only of quantitative nature. There is need to carry out investigations, along with the qualitative study of the solutions of impulsively perturbed systems, concerning approximate analytic and numeric methods helping to find these solutions, since the integration of systems with impulses in closed form is possible only in exceptional cases.

The present paper deals with the problem of existence and approximate determination of the periodic solutions of a system of differential equations with impulses of the form

(1)
$$\dot{x}(t) = A(t)x + f(t) + \varepsilon X(t, x, \varepsilon) \ , \quad t \neq t_i \ ,$$
$$\Delta x|_{t=t_i} = B_i x(t_i-0) + a_i + \varepsilon I_i\left(x(t_i-0), \varepsilon\right) \ ,$$

where $x \in R^n$, $f : R \to R^n$, $X : R \times \Omega \to R^n$, $I_i : \Omega \to R^n$ $(i \in Z)$, $A(t)$ and $B(t)$ are $n\times n$-dimensional matrices, $t_i \in R$ $(i \in Z)$ are fixed points for which $t_{i+1} > t_i$ and $\lim_{i\to\pm\infty} t_i = \pm\infty$, $\Delta x|_{t=t_i} = x(t_i+0) - x(t_i-0)$, $\Omega = \{(x, \varepsilon) \in R^n \times R : \|x\| \le c, |\varepsilon| \le \varepsilon^*\}$, $c = \text{const} > 0$, $\varepsilon^* > 0$, Z is the set of non-zero integers.

By P denote the points with coordinates $(t, x(t))$ where $x(t)$ is the solution of equation (1).

The motion of the point P may be described in the following way. The point starts its motion at the point (t_0, x_0) and moves along the integral curve $(t, x(t))$ of the system of ordinary differential equations

(2)
$$\dot{x} = A(t)x + f(t) + \varepsilon X(t, x, \varepsilon)$$

till the moment $t_1 > t_0$ when the point P "instantly" jumps from position $(t_1, x(t_1-0))$ into the position $(t_1, x(t_1+0))$, where $x(t_1+0) = x(t_1-0) + B_1 x(t_1-0) + a_1 + \varepsilon I_1\left(x(t_1-0), \varepsilon\right)$. Further the point P moves along the integral curve $(t, x(t))$ of system (2) with initial condition $x(t_1) = x(t_1+0)$ till the moment $t_2 > t_1$ and so on.

Therefore the solution of system (1) is a piecewise continuous function $x(t)$ with first order discontinuity at the points t_i, which for $t \in (t_i, t_{i+1})$, $i \in Z$, satisfies equation (2), while for $t = t_i$ it satisfies the jump condition

$$x(t_i+0) - x(t_i-0) = B_i x(t_i-0) + a_i + \varepsilon I_i(x(t_i-0), \varepsilon) .$$

Besides, the function $x(t)$ will be viewed as continuous from the left at the jump points $t = t_i$, $i \in Z$, that is

$$x(t_i) = x(t_i-0) = \lim_{\varepsilon \uparrow 0} x(t_i-\varepsilon) .$$

1. Basic assumptions and definitions

Consider the homogeneous linear system with impulses

$$\text{(3)} \qquad \dot{x} = A(t)x , \quad t \neq t_i , \qquad \Delta x|_{t=t_i} = B_i x(t_i-0) .$$

DEFINITION 1. We will say that the system (3) is non-critical with respect to T if it does not have a nontrivial periodic solution with period T.

DEFINITION 2. We will say that the system (3) is critical with respect to T if it has at least one nontrivial solution which is periodic with period T.

DEFINITION 3. The following system with impulses will be called a generating system of system (1):

$$\text{(4)} \qquad \dot{x} = A(t)x + f(t) , \quad t \neq t_i , \qquad \Delta x|_{t=t_i} = B_i x + a_i .$$

We will say that the set of conditions (A) holds provided the following conditions are satisfied:

A1. the function $f : R \to R^n$ is periodic in t with period T, and it is piecewise continuous with discontinuity points of first order at the points t_i, $i \in Z$;

A2. the function $X : R \times \Omega \to R^n$ is piecewise continuous in t with discontinuity points of first order at the points t_i, $i \in Z$, it is periodic in t with period T and satisfies the Lipschitz condition

$$\|X(t, x, \varepsilon)-X(t, y, \varepsilon)\| \le K\|x-y\| , \quad (t, x, \varepsilon), (t, y, \varepsilon) \in R \times \Omega ;$$

A3. the matrix $A(t)$ is periodic with period T and piecewise continuous with discontinuity points of first order at the points t_i, $i \in Z$;

A4. the matrices $(E+B_i)$ are non-singular $(i \in Z)$;

A5. the functions $I_i : \Omega \to R^n$ are continuous and satisfy the Lipschitz condition in their first argument, $\|I_i(x, \varepsilon)-I_i(y, \varepsilon)\| \le K_1\|x-y\|$ uniformly in i, when $(x, \varepsilon), (y, \varepsilon) \in \Omega$;

A6. a natural number p exists such that

$$t_{i+p} = t_i + T , \quad B_{i+p} = B_i , \quad a_{i+p} = a_i ,$$

$$I_{i+p} = I_i , \quad i \in Z .$$

2. The non-critical case

Consider the case when the system (3) is non-critical with respect to T. Then, in view of [12], the system (4) has a unique T-periodic solution $\varphi(t)$ given by the formula

$$\varphi(t) = \int_0^T G(t, \tau)f(\tau)d\tau + \sum_{0<t_i<t} G(t, t_i)(E+B_i)^{-1}a_i , \tag{5}$$

where

$$G(t, \tau) = \begin{cases} X(t)[E-X(T)]^{-1}X^{-1}(\tau) , & 0 \le \tau < t \le T , \\ X(t)[E-X(T)]^{-1}X(T)X^{-1}(\tau) , & 0 \le t \le \tau \le T , \end{cases} \tag{6}$$

$X(t)$ is the fundamental matrix of solutions of the system (3) for which $X(0) = E$.

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To find the T-periodic solution of the system (1) we will employ the method of subsequent approximations.

Set

$$x^{(0)}(t) = \varphi(t) . \tag{7}$$

Construct a sequence of functions $x^{(k)}(t)$, $k = 1, 2, \ldots$, where the function $x^{(k)}(t)$ is a T-periodic solution of the system

$$\begin{aligned} \dot{x}_k^{(k)}(t) &= A(t)x^{(k)}(t) + f(t) + \varepsilon X\left(t, x^{(k-1)}(t), \varepsilon\right) , \quad t \neq t_i , \\ \Delta x^{(k)}\big|_{t=t_i} &= B_i x^{(k)}\left(t_i\right) + a_i + \varepsilon I_i\left(x^{(k-1)}\left(t_i\right), \varepsilon\right) . \end{aligned} \tag{8}$$

The fact that the system (3) is non-critical with respect to T implies that the system (8), for $k = 1$, has a unique T-periodic solution $x^{(1)}(t)$ given by the formula

$$x^{(1)}(t) = \int_0^T G(t, \tau)\left[f(\tau)+\varepsilon X\left(\tau, \varphi(\tau), \varepsilon\right)\right]d\tau + \sum_{0<t_i<T} G\left(t, t_i\right)\left(E+B_i\right)^{-1}\left[a_i+\varepsilon I_i\left(\varphi\left(t_i\right), \varepsilon\right)\right] ,$$

where the function $G(t, \tau)$ is defined by means of the equality (5).

Analogously, for every $k \geq 2$, the system (8) has a unique T-periodic solution written in the form

$$x^{(k)}(t) = \int_0^T G(t, \tau)\left[f(\tau)+\varepsilon X\left(\tau, x^{(k-1)}(\tau), \varepsilon\right)\right]d\tau + \sum_{0<t_i<T} G\left(t, t_i\right)\left(E+B_i\right)^{-1}\left[a_i+\varepsilon I_i\left(x^{(k-1)}\left(t_i\right), \varepsilon\right)\right] . \tag{9}$$

Taking into consideration the representation (5) of the function $\varphi(t)$ and the equality (9), we get

$$x^{(k)}(t) = \varphi(t) + \varepsilon\left[\int_0^T G(t, \tau)X\left(\tau, x^{(k-1)}(\tau), \varepsilon\right)d\tau + \sum_{0<t_i<T} G\left(t, t_i\right)\left(E+B_i\right)^{-1}I_i\left(x^{(k-1)}\left(t_i\right), \varepsilon\right)\right] . \tag{10}$$

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Conditions A2, A5 and the equality (10) imply that a constant $\bar{K} > 0$ exists such that

(11) $$\|x^{(k)}(t)-x^{(k-1)}(t)\| \le \bar{K}\varepsilon\|x^{(k-1)}(t)-x^{(k-2)}(t)\| .$$

Therefore, for $|\varepsilon| < \min(\varepsilon^*, 1/\bar{K})$ and $t \in [0, T]$, the sequence of functions $\{x^{(k)}(t)\}_0^\infty$ is uniformly convergent. The results obtained above may be used to state the following theorem.

THEOREM 1. *Let the following conditions hold:*

(1) *the set of conditions* (A) *holds;*

(2) *the system* (3) *is non-critical with respect to* T .

Then a number $\varepsilon_1 > 0$ *exists such that for* $|\varepsilon| < \varepsilon_1$ *the system* (1) *has a unique* T*-periodic solution* $x(t, \varepsilon)$ *which, for* $\varepsilon = 0$, *coincides with the solution of the generating system* (4).

3. The critical case

Consider the case when the system (3) is critical, that is it has m linearly independent T-periodic solutions $\varphi^{(1)}(t), \varphi^{(2)}(t), \ldots, \varphi^{(m)}(t)$.

Consider the system

(12) $$\begin{aligned} \dot{y} &= -A^T y , \quad t \ne t_i , \\ \Delta y|_{t=t_i} &= -\left(E+B_i^T\right)B_i^T y(t_i-0) , \end{aligned}$$

conjugate to the system (3), where A^T and B^T denote the transposes of A and B .

In view of [12], the system (12) has m linearly independent T-periodic solutions $\psi^{(1)}(t), \psi^{(2)}(t), \ldots, \psi^{(m)}(t)$.

Moreover, the system (4) has a T-periodic solution if and only if the following equality holds:

(13) $$\int_0^T \left\langle f(t), \psi^{(j)}(t)\right\rangle dt + \sum_{0<t_i<T} \left\langle \psi^{(j)}(t_i), (E+B_i)^{-1}a_i\right\rangle = 0 ,$$

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where the symbol $\langle \cdot, \cdot \rangle$ denotes scalar product.

Consider the auxiliary system

$$\dot{x}(t) = A(t)x + f(t) + \varepsilon X\big(t, x(t), \varepsilon\big) + \sum_{i=1}^{m} w_i \varphi^{(i)}(t) , \quad t \neq t_i ,$$

(14)

$$\Delta x|_{t=t_i} = B_i x\big(t_i\big) + a_i + \varepsilon I_i\big(x\big(t_i\big), \varepsilon\big) ,$$

where w_i $(i = 1, \ldots, m)$ are constants to be determined.

The constants w_i $(i = 1, \ldots, m)$ are defined as a solution of the non-homogeneous system of linear algebraic equations:

$$(15) \quad \sum_{i=1}^{m} d_{ij} w_i + \varepsilon \int_0^T \langle X\big(t, x(t), \varepsilon\big), \psi^{(j)}(t) \rangle dt$$
$$+ \varepsilon \sum_{0<t_i<T} \left\langle \psi^{(j)}\big(t_i\big), \big(E+B_i\big)^{-1} I_i\big(x\big(t_i\big), \varepsilon\big) \right\rangle = 0 ,$$

$j = 1, 2, \ldots, n$, where

$$d_{ij} = \int_0^T \langle \varphi^{(i)}(t), \psi^{(j)}(t) \rangle dt .$$

Let $x^*(t)$ be a partial T-periodic solution of the generating system (4). Then every T-periodic solution of (4) has the form

$$x(t) = M_1^0 \varphi^{(1)}(t) + M_2^0 \varphi^{(2)}(t) + \ldots + M_m^0 \varphi^{(m)}(t) + x^*(t) ,$$

where $M_i^0 = \text{const}$, $i = 1, \ldots, m$.

Consider an arbitrary fixed point $M^0 \in R^m$, $M^0 = \big(M_1^0, M_2^0, \ldots, M_m^0\big)$ and denote by Ω_{M^0} a neighbourhood of the point M^0 .

Then the following theorem holds.

THEOREM 2. *Let the following conditions hold:*

(1) *the set of conditions* (A) *holds;*

(2) *the system* (3) *has* m *linearly independent* T*-periodic solutions* $\varphi^{(1)}(t), \varphi^{(2)}(t), \ldots, \varphi^{(m)}(t)$;

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(3) *equality* (13) *holds, where* $\psi^{(1)}(t), \psi^{(2)}(t), \ldots, \psi^{(m)}(t)$ *are* m *linearly independent* T*-periodic solutions of the system* (12).

Then the system (14) *has an* m'*-parametric family of* T*-periodic solutions of the form*

$$x^*(t, M, \varepsilon) = M_1\varphi^{(1)}(t) + \ldots + M_m\varphi^{(m)}(t) + x^*(t) + \varepsilon\tilde{x}(t, M, \varepsilon) , \tag{16}$$

where $M = (M_1, M_2, \ldots, M_m)$ *, the function* $\tilde{x}(t, M, \varepsilon)$ *is defined for* $t \in R$ *,* $M \in \Omega_{M^0}$*,* $|\varepsilon| \le \varepsilon^*$ *, it is continuous in the parameters* $M_1, \ldots, M_m$ *, it is periodic in* t *with period* T *and has discontinuities of first order at the points* t_i .

Proof. The proof of the theorem will be accomplished by help of the method of successive approximations.

As an initial approximation $x^{(1)}(t)$ we choose the T-periodic solution of the system (14) for $\varepsilon = 0$, that is, the T-periodic solution of the system

$$\begin{aligned} &\dot{x}(t) = A(t)x + f(t) + \sum_{i=1}^{m} W_i^{(1)}\varphi^{(i)}(t) , \quad t \ne t_i , \\ &\Delta x|_{t=t_i} = B_i x + a_i , \end{aligned} \tag{17}$$

where the ordered m-tuple $\left(W_1^{(1)}, W_2^{(1)}, \ldots, W_m^{(1)}\right)$ is a solution of the system (15) for $\varepsilon = 0$. Since $\det(d_{ij})_1^m \ne 0$, then $W_i^{(1)} \equiv 0$, $i = 1, \ldots, m$, and the system (17) coincides with the generating system (4). Therefore, constants $M_1, M_2, \ldots, M_m$ exist such that

$$x^{(1)}(t) = M_1\varphi^{(1)}(t) + M_2\varphi^{(2)}(t) + \ldots + M_m\varphi^{(m)}(t) + x^*(t) .$$

The approximation $x^{(k)}(t)$ is defined as a T-periodic solution of the system with impulses

(18)
$$\dot{x}^{(k)}(t) = A(t)x^{(k)}(t) + f(t) + \varepsilon X\left(t, x^{(k-1)}(t), \varepsilon\right) + \sum_{i=1}^{m} W_i^{(k)}\varphi^{(i)}(t) , \quad t \neq t_i ,$$
$$\Delta x^{(k)}\big|_{t=t_i} = B_i x^{(k)}(t_i) + a_i + \varepsilon I_i\left[x^{(k-1)}(t_i), \varepsilon\right] ,$$

where the constants $W_i^{(k)}$, $i = 1, \ldots, m$ are a solution of the linear system of algebraic equations

(19)
$$\sum_{i=1}^{m} d_{ij}W_i^{(k)} + \varepsilon \int_0^T \left\langle X\left(t, x^{(k-1)}(t), \varepsilon\right), \psi^{(j)}(t)\right\rangle dt + \varepsilon \sum_{0<t_i<T} \left\langle \psi^{(j)}(t_i), (E+B_i)^{-1}I_i\left[x^{(k-1)}(t_i), \varepsilon\right]\right\rangle = 0 ,$$
$$j = 1, \ldots, m .$$

The system (19) has a unique solution since $\det\left(d_{ij}\right)_1^m \neq 0$.

Condition (3) of Theorem 2 implies that the equality

(20)
$$\int_0^T \left\langle f(t)+\varepsilon X\left(t, x^{(k-1)}(t), \varepsilon\right) + \sum_{i=1}^{m} W_i^{(k)}\varphi^{(i)}(t), \psi^{(j)}(t)\right\rangle dt + \sum_{0<t_i<T} \left\langle \psi^{(j)}(t_i), (E+B_i)^{-1}\left[a_i+\varepsilon I_i\left[x^{(k-1)}(t_i), \varepsilon\right]\right]\right\rangle = 0 ,$$

is fulfilled.

The equality (20) yields that the system (18) has a T-periodic solution $x^{(k)}(t)$, $k \geq 2$.

In view of conditions A2 and A5 the sequence of T-periodic functions $\{x^{(k)}(t)\}$ is uniformly convergent for sufficiently small ε and tends to the T-periodic solution $x^*(t, M, \varepsilon)$ of the system (14).

Moreover, the representation (16) holds for the function $x^*(t, M, \varepsilon)$.

Thus Theorem 2 is proved.

Let the system (1) have a T-periodic solution $x(t, M, \varepsilon)$, which, for $\varepsilon = 0$, coincides with the solution $x(t)$ of the generating system

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(4),

(21) $$x(t) = M_1^0\varphi^{(1)}(t) + \ldots + M_m^0\varphi^{(m)}(t) + x^*(t) .$$

Since $x(t, M, \varepsilon)$ is a periodic solution of the system (1), then, in view of [12], the equality

(22) $$P_j(M, \varepsilon) = \int_0^T \langle X(t, x(t, M, \varepsilon), \varepsilon), \psi^{(j)}(t)\rangle dt + \sum_{0<t_i<T} \left\langle \psi^{(j)}(t_i), (E+B_i)^{-1} I_i(x(t_i, M, \varepsilon), \varepsilon)\right\rangle = 0$$

holds.

Eqaulity (22) for $\varepsilon = 0$ implies the necessary condition for the existence of the periodic solution of the system (1) which, for $\varepsilon = 0$, coincides with the function $x(t)$ defined by the equality (21). These conditions have the form

(23) $$P_j(M^0, 0) = \int_0^T \langle X(t, x(t, M^0, 0), 0), \psi^{(j)}(t)\rangle dt + \sum_{0<t_i<T} \left\langle \psi^{(j)}(t_i), (E+B_i)^{-1} I_i\left(x\left(t_i, M^0, 0\right), 0\right)\right\rangle = 0 .$$

Therefore, the constants M_j^0 in equality (21) should satisfy the system (23).

The results obtained give us grounds to formulate the following theorem.

THEOREM 3. *Let conditions* (A) *hold, and let the system* (3) *be critical with respect to* T . *Then the necessary condition for the system* (1) *to have a* T*-periodic solution which would coincide with the solution* $x(t)$ *of the generating system* (4) *for* $\varepsilon = 0$, *the function* $x(t)$ *being defined by the equality* (21), *is that the constants* M_j^0 , $j = 1, \ldots, m$, *should satisfy the system* (23).

We will give the necessary and sufficient conditions for existence of the periodic solution of quasilinear systems having impulse perturbation in the critical case.

Let the auxiliary system (14) have a T-periodic solution $x^*(t, M, \varepsilon)$ for which

$$(24) \qquad x^*(t, M, 0) = M_1\varphi^{(1)}(t) + \ldots + M_m\varphi^{(m)}(t) + x^*(t) .$$

Introduce the notation

$$(25) \quad \omega_j = \varepsilon P_j^0(M, \varepsilon) = \varepsilon\left[\int_0^T \left\langle X\left(t, x^*(t, M, \varepsilon), \varepsilon\right), \psi^{(j)}(t)\right\rangle dt + \sum_{0<t_i<T} \left\langle \psi^{(j)}(t_i), (E+B_i)^{-1} I_i\left(x^*(t_i, M, \varepsilon), \varepsilon\right)\right\rangle\right], \quad j = 1, \ldots, m .$$

THEOREM 4. *Let the conditions of Theorem 2 hold. Then the necessary and sufficient condition for the system (1) to have T-periodic solution $x(t, \varepsilon)$ that would coincide, for $\varepsilon = 0$, with the solution $x(t)$ of the generating system (4) defined by the equality (21), is that the system of equations*

$$(26) \qquad P_j^0(M, \varepsilon) = 0 , \quad j = 1, \ldots, m ,$$

have solution $M_i(\varepsilon)$, $i = 1, \ldots, m$, for $|\varepsilon| \le \varepsilon_1 < \varepsilon^*$, for which

$$(27) \qquad M_i(0) = M_i^0 , \quad i = 1, \ldots, m .$$

Proof. Necessity. Let the system (1) have a T-periodic solution $x(t, \varepsilon)$. Substitute it in the auxiliary system (14) and we get the following identities:

$$(28) \qquad \sum_{i=1}^{m} \omega_i\varphi^{(i)}(t) \equiv 0 .$$

Since $\varphi^{(i)}(t)$, $i = 1, \ldots, m$, are linearly independent, then the equalities (28) imply that $W_i = 0$, $i = 1, \ldots, m$.

Sufficiency. In view of Theorem 2 the system (14) has a T-periodic solution $x^*(t, M, \varepsilon)$ of the form (16). Let the system (26) have a solution $M_i(\varepsilon)$ satisfying the equalities (27). Substitute $M_i(\varepsilon)$ in equality (24) and we obtain $\omega_i = 0$, $i = 1, \ldots, m$. Therefore, in this case the system (14) coincides with the system (1) and the function

$x^*(t, M, \varepsilon)$ defined by the equality (16), for $M = (M_1(\varepsilon), M_2(\varepsilon), \ldots, M_m(\varepsilon))$ will be a T-periodic solution of the system (1) that coincides with the function $x(t)$ for $\varepsilon = 0$, the function being defined by the equalities (21).

Thus Theorem 4 is proved.

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23 Oborishte Str.,
1504 Sofia,
Bulgaria.