A PROPERTY OF TWO CHORDS WHICH DIVIDE A CONVEX CURVE INTO FOUR ARCS OF EQUAL LENGTH

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Introduction. It will be shown in this paper that if two chords of a closed plane convex curve θ divide θ into four arcs of equal length and intersect inside the domain bounded by θ, then the sum of the lengths of the two chords is at least equal to $(\sqrt{5} - 2)^{\frac{1}{2}}$ times the length of θ. We shall show firstly that we need only consider the case when θ is a convex (possibly degenerate) quadrilateral and then prove the result in this case.

This result is related to a conjecture of P. Ungar in which the chords are assumed to be perpendicular and the factor $(\sqrt{5} - 2)^{\frac{1}{2}}$ is replaced by $\frac{1}{2}$. But Ungar's conjecture is neither proved nor disproved by this result. Another related paper "An extremal problem for plane convexities" by Chandler Davis has been published in the *Proceedings of the Symposium on Convexity* (1961). In this the author solves an analogous problem involving areas instead of arc lengths. His method is different from that employed in this paper.

Notation. For any two points X, Y let XY denote either the segment with end points X, Y or the length of this segment. The context will make clear which particular meaning is intended.

For any four distinct points A, B, C, D lying on a convex curve γ let $\gamma(A, B)$ denote the least length of any arc of γ which contains both A and B. Let $\gamma(A, B, C, D)$ be the least length of any arc of γ which contains at least two distinct members of the set A, B, C, D. If the four points A, B, C, D lie in order on γ and divide γ into four arcs of equal length so that

$$\gamma(A, B) = \gamma(B, C) = \gamma(C, D) = \gamma(D, A) = \gamma(A, B, C, D) = \tfrac{1}{4}l,$$

where l is the length of γ, then we say that A, B, C, D is a quadrisection set of γ. The symbols $\gamma(A, B)$, $\gamma(A, B, C, D)$ are defined whether γ is of finite or infinite length; but γ possesses a quadrisection set only if it is of finite length.

Of all the quadrisection sets of a convex curve γ of finite length there is at least one A, B, C, D for which $AC + BD$ attains its least possible value. Such a set is called a minimal quadrisection set.

If A, B, C, D is a quadrisection set of *some* convex curve, then we shall simply say that A, B, C, D is a quadrisection set.

1. Reduction to the case of a quadrilateral. The problem will be solved if we can show that for any convex curve θ and any quadrisection set A, B, C, D

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lying on θ (A, B, C, D is not necessarily a quadrisection set of θ but the points A, B, C, D must necessarily lie in order on θ)

$$AC + BD \geqslant 4(\sqrt{5} - 2)^{\frac{1}{2}}\, \theta(A, B, C, D). \tag{1}$$

For if (1) is true generally, it is true in particular when A, B, C, D is a quadrisection set of θ, in which case $\theta(A, B, C, D)$ is equal to one-quarter of the length of θ and thus (1) would imply the required result.

At each of the four points A, B, C, D select a support line of the convex set which is bounded by θ. The four half-planes which are bounded by these four lines and which contain θ intersect in a convex set whose frontier is a convex curve τ. Then $\tau(A, B, C, D) \geqslant \theta(A, B, C, D)$ and thus (1) would be true if

$$AC + BD \geqslant 4(\sqrt{5} - 2)^{\frac{1}{2}}\, \tau(A, B, C, D). \tag{2}$$

If we perform the construction described in the preceding paragraph but select support lines of the quadrilateral $ABCD$ instead of support lines of the convex set bounded by θ we obtain a class of convex curves which we denote by Γ. τ is one member of Γ. By standard arguments the function $\gamma(A, B, C, D)$ regarded as a function of γ with A, B, C, D fixed and defined for all γ of Γ attains its largest possible value for at least one particular member of Γ. Such a member of Γ is called an extremal quadrilateral of A, B, C, D and we denote one such by σ. Then $\sigma(A, B, C, D) \geqslant \tau(A, B, C, D)$ and (2) would follow if we could prove that

$$AC + BD \geqslant 4(\sqrt{5} - 2)^{\frac{1}{2}}\, \sigma(A, B, C, D). \tag{3}$$

As a step towards the proof of (3) we next prove Theorem 1. We use the following notation. For any member γ of Γ the four lines through A, B, C, D used in defining γ will be denoted by $\gamma_A, \gamma_B, \gamma_C, \gamma_D$, respectively.

THEOREM 1. *Either A, B, C, D is a quadrisection set of σ or σ is a double segment or*

$$AC + BD > 2\sigma(A, B, C, D).$$

If A, B, C are collinear, then $AC = 2\sigma(A, B, C, D)$ and either the above inequality holds or σ is a double segment with D coinciding with B. We assume in what follows that no three of A, B, C, D are collinear.

LEMMA 1. *If $AC + BD \leqslant 2\sigma(A, B, C, D)$ and if r of the numbers $\sigma(A, B)$, $\sigma(B, C)$, $\sigma(C, A)$, $\sigma(A, D)$ are greater than $\sigma(A, B, C, D)$ where $1 \leqslant r \leqslant 3$, then there exists $\sigma^* \in \Gamma$ such that $\sigma^*(A, B, C, D) = \sigma(A, B, C, D)$ and $r + 1$ of the numbers $\sigma^*(A, B)$, $\sigma^*(B, C)$, $\sigma^*(C, A)$, $\sigma^*(A, D)$ are greater than $\sigma^*(A, B, C, D)$.*

By the hypotheses of the lemma we can find amongst the four numbers $\sigma(A, B), \sigma(B, C), \sigma(C, A), \sigma(D, A)$ two of which one is larger than $\sigma(A, B, C, D)$ and the other is equal to $\sigma(A, B, C, D)$ and, moreover, the two arcs concerned have one common end point. Suppose for definiteness that

$$\sigma(A, B) > \sigma(A, B, C, D) = \sigma(D, A)$$

(see Fig. 1).

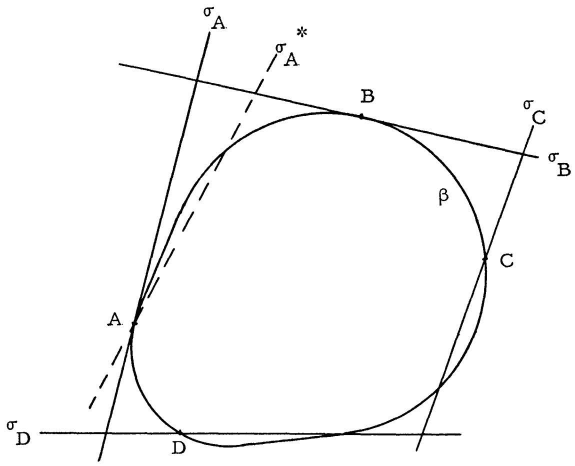

FIGURE 1

Now we know that the points A, B, C, D form a quadrisection set of some convex curve, say of the curve β. The support lines at A, B, C, D to the convex set bounded by β can be used to define a member of Γ, say γ, and

$$(4) \qquad \gamma(A, B, C, D) \geqslant \beta(A, B, C, D) \geqslant AB.$$

In any case $\sigma(A, B, C, D) \geqslant \gamma(A, B, C, D)$. Thus, since $\sigma(A, B) > \sigma(A, B, C, D)$, we conclude that $\sigma(A, B) > AB$. Hence neither σ_A nor σ_B coincides with the line AB.

If we rotate σ_A about A in the appropriate sense and denote this line in its near position by $\sigma_A{}^*$, then if the rotation is small enough, $\sigma_A{}^*$, σ_B, σ_C, σ_D define a member σ^* of Γ such that $\sigma^*(A, B) \leqslant \sigma(A, B)$ and $\sigma^*(D, A) \geqslant \sigma(D, A)$.

Moreover, $\sigma^*(D, A) > \sigma(D, A)$ unless σ_D is the line DA. By choosing the rotation sufficiently small we can ensure that $\sigma^*(A, B, C, D) \geqslant \sigma(A, B, C, D)$ (this means that $\sigma^*(A, B, C, D) = \sigma(A, B, C, D)$ since σ is an extremal quadrilateral of A, B, C, D). Thus either σ_D is the line DA or the lemma is proved.

If σ_D is the line DA (see Fig. 2), then in addition to the rotation of σ_A about A we rotate σ_D about D (in the opposite sense). The combined effect is to increase $\sigma(A, D)$ and of the three numbers $\sigma(A, B)$, $\sigma(B, C)$, $\sigma(C, D)$ we may decrease $\sigma(A, B)$ and $\sigma(C, D)$ (if σ_C is the line CD, we do not reduce $\sigma(C, D)$). Thus, we shall be able again to construct a curve σ^* of the required type unless $\sigma(C, D) = \sigma(A, B, C, D)$.

If σ_D is the line DA and $\sigma(C, D) = \sigma(A, B, C, D)$ (see Fig. 3) we rotate σ_A about A as before and, if possible, σ_D about D, σ_C about C, both in the sense of rotation opposite to that of σ_A about A. We can do this unless σ_C is the line

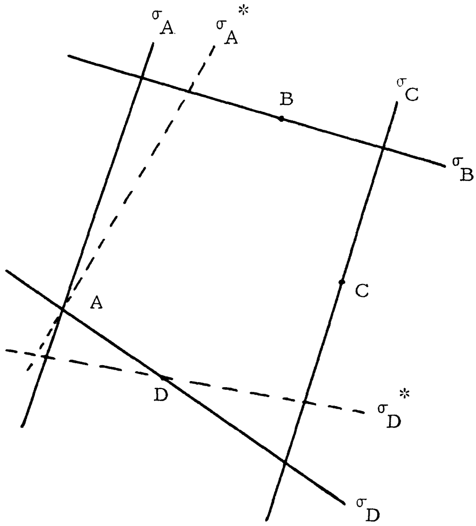

FIGURE 2

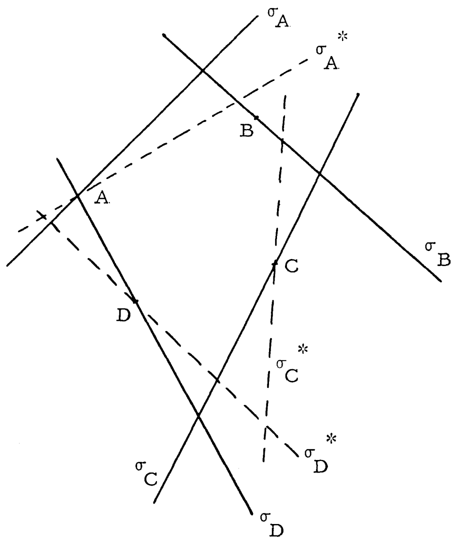

FIGURE 3

BC. Suppose that σ_C is not the line BC; then we can choose the amount of rotation of σ_D about D and σ_C about C so that $\sigma(C, D)$ remains unaltered. The effect of these changes is definitely to increase $\sigma(A, D)$ and possibly to decrease $\sigma(A, B)$ and $\sigma(B, C)$. Thus, by choosing small rotations we can obtain a new curve σ^* of the required type unless $\sigma(B, C) = \sigma(A, B, C, D)$. If $\sigma(B, C)$ is equal to $\sigma(A, B, C, D)$ (see Fig. 4), we can increase $\sigma(B, C)$ by rotating σ_B

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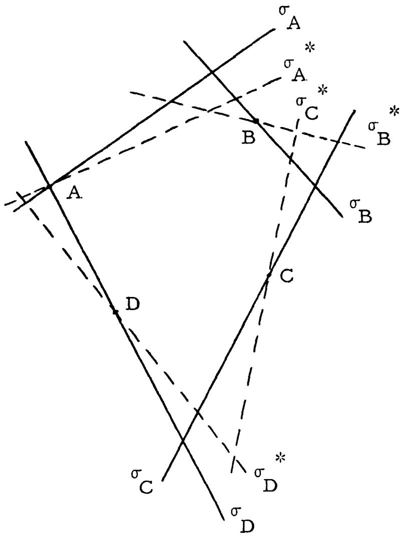

FIGURE 4

about B in addition to all the other rotations in the sense of rotation opposite to that of σ_A about A. This is always possible since σ_B is not the line AB. If we choose the amount of rotation correctly, $\sigma(B, C)$ will remain unaltered. We reduce $\sigma(A, B)$; but if all the rotations are sufficiently small, we obtain a new curve σ^* of the required type.

Finally, if σ_D is the line DA, $\sigma(C, D) = \sigma(A, B, C, D)$ and σ_C is the line BC, let AC meet BD in X and the line AD meet BC in Y (see Fig. 5). The

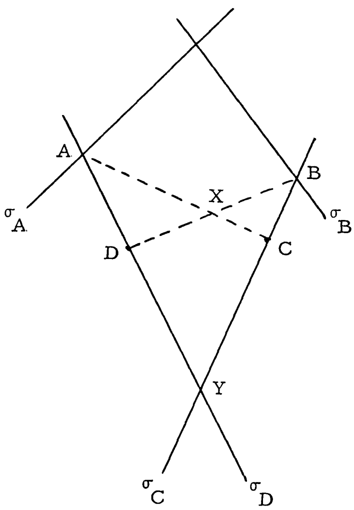

FIGURE 5

intersection of the lines AD, BC must lie on the same side of AB as do the points C, D since otherwise we should have

$$\sigma(C, D) > \sigma(A, B) > \sigma(A, B, C, D).$$

From the triangle inequality applied to the triangles ACY and BDY

$$AC > AY - YC, \qquad BD > BY - YD.$$

Thus,

$$AC + BD > AY + BY - YC - YD = AD + BC \geqslant 2\sigma(A, B, C, D),$$

where $AD \geqslant \sigma(A, B, C, D)$, $BC \geqslant \sigma(A, B, C, D)$ because σ_C is BC and σ_D is DA; thus, segments BC and DA form part of σ. This contradicts the hypothesis of the lemma. Hence, this situation cannot arise and the lemma has been proved.

To complete the proof of Theorem 1 we observe that if $AC + BD \leqslant 2\sigma(A, B, C, D)$ and A, B, C, D is not a quadrisection set of σ, then successive applications of Lemma 1 would lead to a member σ_1 of Γ for which all four of $\sigma_1(A, B)$, $\sigma_1(B, C)$, $\sigma_1(C, D)$, $\sigma_1(D, A)$ would be greater than $\sigma(A, B, C, D)$. This implies $\sigma_1(A, B, C, D) > \sigma(A, B, C, D)$, which is impossible since σ is an extremal quadrilateral of A, B, C, D.

The theorem is proved.

Since $AC + BD > 2\sigma(A, B, C, D)$ implies the required result (since $2 > 4(\sqrt{5} - 2)^{\frac{1}{2}}$), we shall assume in what follows that

$$AC + BD \leqslant 2\sigma(A, B, C, D)$$

and that A, B, C, D is a quadrisection set of σ. Thus, σ is a quadrilateral, either genuine or degenerate, and we need only establish our results in the case of a quadrilateral.

2. Proof of the result for a quadrilateral. Denote by Π the class of all convex quadrilaterals π that have the perimeter length l. Π contains also degenerate quadrilaterals such as double segments and triangles. Denote by $f(\pi)$ the sum $AC + BD$ for a minimal quadrisection set A, B, C, D of π. By standard arguments there is a member of Π, say λ, such that $f(\lambda)$ has the least possible value of all the values $f(\pi)$ for $\pi \in \Pi$. λ is either a double segment, a triangle, or a genuine quadrilateral. We consider these cases separately, and we shall show that

$$f(\lambda) \geqslant (\sqrt{5} - 2)^{\frac{1}{2}}l.$$

This will establish the result.

From the case of a parallelogram we know that

$$f(\lambda) \leqslant \tfrac{1}{2}l = 2\lambda(A, B, C, D).$$

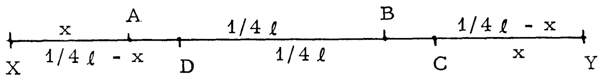

FIGURE 6

Case (i). λ a segment. Let the segment be XY and suppose that A lies distant x from X (see Fig. 6). Then

$$AC + BD = (\tfrac{1}{2}l - 2x) + 2x = \tfrac{1}{2}l,$$

and the result is established in this case.

Case (ii). λ is a triangle. We need the following lemma:

LEMMA 2. *Each side of λ contains at least one of the points A, B, C, D as an interior point.*

Otherwise if a side of λ contains none of the points A, B, C, D as an interior point, we can replace this side of λ by a convex arc joining the two vertices of λ, which with the other two sides of λ forms a convex curve containing A, B, C, D (see Fig. 7). Denote this convex curve by θ.

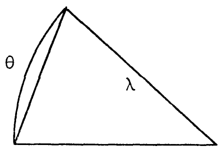

FIGURE 7

λ is an extremal quadrilateral of A, B, C, D; for otherwise there would exist an extremal quadrilateral λ_1 of A, B, C, D and by §1, A, B, C, D would be a quadrisection set of λ_1. If λ is not an extremal quadrilateral of A, B, C, D, then

$$\lambda_1(A, B, C, D) > \lambda(A, B, C, D)$$

and thus the length of λ_1 is greater than l, say it is l_1. A similitude of ratio $l:l_1$ applied to λ_1 transforms it into λ_2, where $\lambda_2 \in \Pi$ and there are four quadrisection points on λ_2 (namely the transforms of A, B, C, D), say A_2, B_2, C_2, D_2, such that

$$A_2 C_2 + B_2 D_2 < AC + BD = f(\lambda).$$

Hence

$$f(\lambda_2) < f(\lambda).$$

But this is impossible from the way in which λ was chosen. Now

$$\theta(A, B, C, D) \geqslant \lambda(A, B, C, D)$$

and if we select support lines to θ at A, B, C, D, then we obtain a convex curve τ that belongs to Γ. Since

$$\tau(A, B, C, D) \geqslant \theta(A, B, C, D) \geqslant \lambda(A, B, C, D),$$

τ must also be an extremal quadrilateral. However, the length of τ is at least that of θ, which is greater than that of λ, and since A, B, C, D must be a quadrisection set both of τ and of λ, we have

$$\tau(A, B, C, D) > \lambda(A, B, C, D).$$

But this contradicts the extremal property of λ.

Thus, the lemma is proved.

Remark. An analogous result holds if λ is a genuine quadrilateral.

There are thus only two essentially distinct possible cases:

(*a*) Exactly one of the four points A, B, C, D is a vertex of λ and the other three points lie one each in the interiors of the three sides of λ.

(*b*) No points of A, B, C, D are vertices of λ. One side of λ contains two of the points A, B, C, D and the other two sides contain one each of these points.

We consider case (*a*) first.

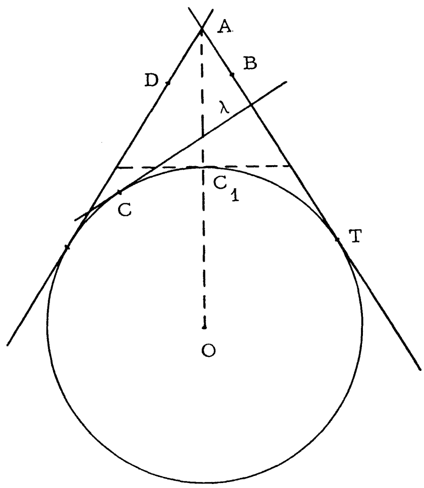

FIGURE 8

Choose the notation so that A is a vertex of λ (see Fig. 8). Let O be the centre of the escribed circle of λ opposite to A. Since λ is divided by the points A and C into two arcs of equal length, it follows that C lies on this escribed circle. Let AO meet the escribed circle in C_1. Then $AC \geqslant AC_1$ and if C is not C_1, then $AC > AC_1$. But in this case A, B, C_1, D is a quadrisection set of the triangle λ_1 formed by the lines AB, AD and the tangent to the escribed circle at C_1. Then

$$f(\lambda_1) \leqslant AC_1 + BD < f(\lambda)$$

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and since λ_1 has perimeter length l we have a contradiction with the extremal property of λ. Thus C lies on AO.

Let T be the point of contact of the line AB with the escribed circle. Then

$$AB = \tfrac{1}{2}AT = AD,$$

$$BD = 2AB \sin \angle BAO = AT \sin \angle BAO,$$

$$AC = AO - OC = AO - OT = AT \sec \angle BAO - AT \tan \angle BAO.$$

Thus, writing α for $\angle BAO$,

$$\begin{aligned} BD + AC &= AT\left[\sin\alpha + \frac{1 - \sin\alpha}{\cos\alpha}\right] \\ &= AT + AT[(\sec\alpha - 1)(1 - \sin\alpha)] \\ &> AT = 2\lambda(A, B, C, D). \end{aligned}$$

Hence, $BD + AC > 2\lambda(A, B, C, D)$. But this is impossible since we know that $BD + AC \leqslant 2\lambda(A, B, C, D)$. Thus this case does not occur.

Next consider case (*b*) illustrated in Fig. 9. Let λ be the triangle LMN

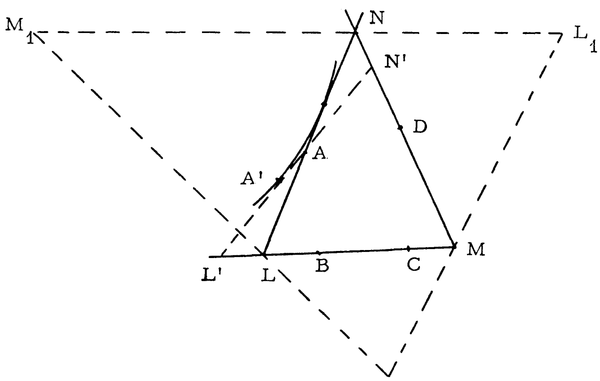

FIGURE 9

and suppose that the points A, B, C, D lie as follows: A on NL; B and C on LM; D on MN. Let M_1, L_1 be the centres of the escribed circles of λ opposite respectively to M and to L. We show first that A lies on $M_1 C$ and D lies on $L_1 B$.

Take N' on the line MN near to N and L' on ML near to L so that $N'L'$ is a tangent to the escribed circle with centre M_1. Now the circle with centre M_1 which passes through A is either tangent to $N'L'$ or meets $N'L'$ in two points. In either case there is a point on this circle and on $N'L'$, say A', such that A', B, C, D is a quadrisection set of the triangle $L'MN'$. By selecting N' either to lie between M and N or on MN beyond N, we can (unless A lies on $M_1 C$) ensure that $A'C < AC$. But then

$$f(\lambda') \leqslant A'C + BD < f(\lambda),$$

which, since $\lambda' \in \Pi$, is a contradiction with the extremal property of λ.

This is impossible. Thus the line AC passes through M_1 and similarly the line BD passes through L_1.

Next let the centre of the escribed circle of λ opposite to N be N_1 (see Fig. 10). If we rotate the line LM about N_1 with points B, C rigidly fixed to LM,

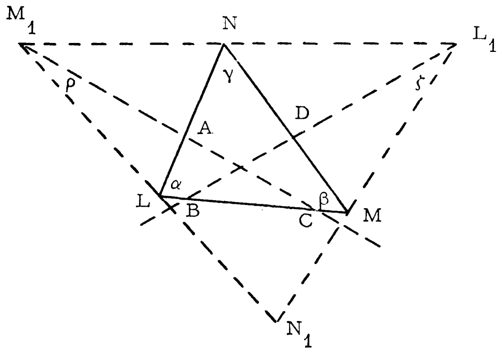

FIGURE 10

the effect is to produce a new triangle for which the points A, D and the new positions of B and C form a quadrisection set. The change in $AC + BD$ to the first order in δ, the angle of rotation, is

$$(BN_1 \sin DBN_1 - CN_1 \sin ACN_1)\delta.$$

This must be zero; otherwise we can again obtain a contradiction with the extremal property of λ. Write $\angle\, LM_1\, C = \rho$ and $\angle\, ML_1\, B = \zeta$. Then

(5) $$\sin \rho \cos \tfrac{1}{2}\alpha = \sin \zeta \cos \tfrac{1}{2}\beta.$$

Denote angle $LM_1\, C$ by ρ and let the angles of LMN at L, M, N be α, β, γ respectively. Now M_1 lies on the bisector of angle NML and the tangent from M to the escribed circle centre M_1 has length $\frac{1}{2}l$. Thus the radius of the escribed circle centre M_1 is $\frac{1}{2}l \tan \frac{1}{2}\beta$. Thus

$$M_1\, L = \tfrac{1}{2}l \tan \tfrac{1}{2}\beta \sec \tfrac{1}{2}\alpha.$$

Similarly

$$L_1\, M = \tfrac{1}{2}l \tan \tfrac{1}{2}\alpha \sec \tfrac{1}{2}\beta.$$

Also

$$LM = \tfrac{1}{2}l(1 - \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta).$$

Writing $a = \tan \frac{1}{2}\alpha$, $b = \tan \frac{1}{2}\beta$, $X = \tan \rho$. $Y = \tan \zeta$, evaluating AL, LC, DM, MB from triangles ALM_1, CLM_1, DML_1, and BML_1 respectively, and using the facts that $AL + LC = BM + MD = \frac{1}{2}l$ and $BC = \frac{1}{4}l$, we obtain (after some manipulations)

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$$(6) \qquad 2bX(1 + a^2) = 1 - a^2X^2,$$

$$(7) \qquad 2aY(1 + b^2) = 1 - b^2Y^2,$$

$$(8) \qquad 2ab + aX + bY = 1.$$

We assume that $a > b$ and show that this assumption leads to a contradiction. From (5), $X > Y$. Subtract (6) from (7) to obtain

$$(9) \qquad (2b + a)X = (2a + b)Y.$$

Add (6) to (7) to obtain

$$(10) \qquad 2bX + 2aY - 2ab = 1 + 2abXY.$$

(8), (10), and $aX + bY > bX + aY$ combine to give

$$(11) \qquad ab < \tfrac{1}{6}.$$

Evaluate Y in terms of a, b from (5) and (9). Substitute this value for Y in (7) to obtain

$$(12) \quad 2a(1 + b^2)(2a + b)[3 - (a^2 + 4ab + b^2)]^{\frac{1}{2}} = 4a^2 + 4ab - 2b^2 + 4ab^3 + a^2b^2 + b^4.$$

Since $ab < \frac{1}{6}$ and $b < a < 1$, we have $a + b < 1\frac{1}{6}$, and applying this in (12) together with $b < a$ we obtain a contradiction.

Thus the assumption $a > b$ is false. Hence $a \leqslant b$. Similarly $b \leqslant a$, and finally we see that $a = b$, i.e. $\alpha = \beta$.

Thus the triangle LMN is isosceles. Let P be the mid-point of LM. Then since M_1AC and L_1DB both bisect the perimeter of LMN, they must meet on NP.

Denote the length LN by a. Then

$$ND = \tfrac{1}{4}a(1 + \sin \tfrac{1}{2}\gamma) = BP.$$

Thus

$$BD^2 = [\tfrac{1}{4}a(1 + \sin \tfrac{1}{2}\gamma)^2]^2 + [[a - \tfrac{1}{4}a(1 + \sin \tfrac{1}{2}\gamma)] \cos \tfrac{1}{2}\gamma]^2.$$

The ratio

$$BD/a(1 + \sin \tfrac{1}{2}\gamma)$$

has its least value when $\sin \frac{1}{2}\gamma = (4 - \sqrt{5})/\sqrt{5}$ and its value then is $(\sqrt{5} - 2)^{\frac{1}{2}}$.

Thus the required result holds in this case.

Case (iii). λ *is a genuine quadrilateral.* By the remark made after Lemma 3, the four points A, B, C, D lie one each on each of the sides of λ and they do not lie at the vertices of λ.

Let the vertices of λ be L, M, N, P and suppose that A lies on LM, B on MN, C on NP, and D on PL (see Fig. 11). Let O be the centre of that circle ω which lies on the side of LM opposite to the side containing C and which touches the lines PL, LM, and MN. Similarly, let O' be the centre of that circle ω' which lies on the side of NP opposite to the side containing A and which touches the lines LP, PN, and NM.

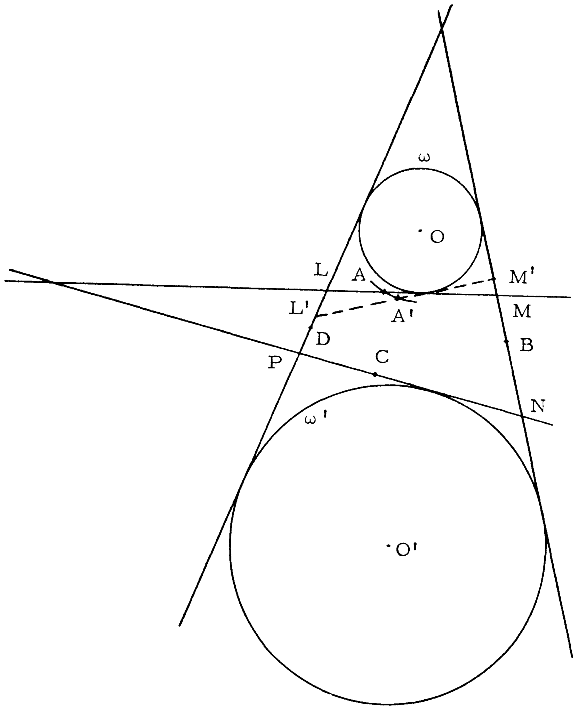

FIGURE 11

If A does not lie on the line OC, we can find a point A' near to A whose distance from O is equal to that of A from O whilst $A'C < AC$. Through A' there pass either one or two tangents to ω. Of these tangents one can be selected which meets line LP in L', line LN in M', and is such that A', B, C, D is a quadrisection set of the quadrilateral λ' with vertices $L'M'NP$ and, moreover, the length of λ' is l. But this means that $f(\lambda') < f(\lambda)$ and since $\lambda' \in \Pi$ we have a contradiction with the extremal property of λ.

Thus, A lies on OC and similarly C lies on AO'. Thus the points $OACO'$ are collinear. AC is the line that either (a) bisects the angle formed by the lines PL, MN or (b) is parallel to both PL, MN and is midway between them. The first alternative holds if PL, MN are not parallel and the second holds when they are parallel.

Similarly, BD either (a) bisects the angle between LM and NP or (b) is parallel to both these lines and midway between them.

We consider the various cases that can occur according as (a) or (b) holds with respect to AC or BD. There are only three essentially different cases.

Case 1. $LM \parallel NP$ and $MN \parallel LP$. $LMNP$ is a parallelogram and AC, BD are parallel to its sides. Thus

$$AC + BD = \tfrac{1}{2}l$$

in this case.

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Case 2. $LM \parallel NP$ *but* $LP \nparallel MN$. Since DB is the line midway between LM and PN, we have $MB = NB$ (see Fig. 12). But $LM \nparallel NP$; thus

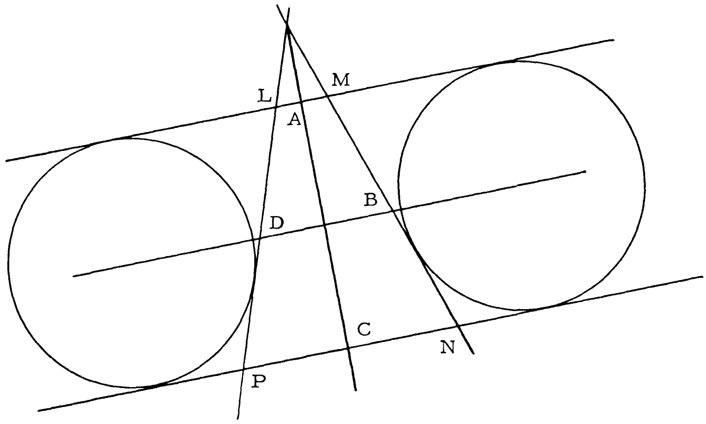

FIGURE 12

$AC \nparallel MN$. Hence $AM \neq NC$ and

$$AM + MB \neq BN + NC.$$

This is impossible as it implies that A, B, C, D is not a quadrisection set of $LMNP$. This case cannot occur.

Case 3. $LM \nparallel NP$ *and* $LP \nparallel MN$. Suppose that LP meets MN in X and that LM meets PN in Y. Let XAC meet YDB in K and suppose for definiteness that of the four vertices L, M, N, P it is L which lies inside the triangle XYK (see Fig. 13).

Denote angles as follows:

$$\angle LXM = 2\theta, \qquad \angle LYP = 2\phi,$$

$$\angle XAM = \alpha, \qquad \angle XBY = \beta,$$

$$\angle XCN = \gamma, \qquad \angle XDY = \delta, \qquad \angle XKY = \chi.$$

Then

$$\delta = \chi + \theta, \qquad \beta = \chi - \theta,$$

$$\alpha = \pi - \chi - \phi, \qquad \gamma = \pi - \chi + \phi.$$

Consider a variation by equal amounts of the points A, B, C, D along the sides of $LMNP$ on which they lie. The fact that A, B, C, D is a minimal quadrisection set of λ implies that

$$\cos\alpha + \cos\beta + \cos\gamma + \cos\delta = 0,$$

i.e. $\chi = \frac{1}{2}\pi$ or $\theta = \phi$.

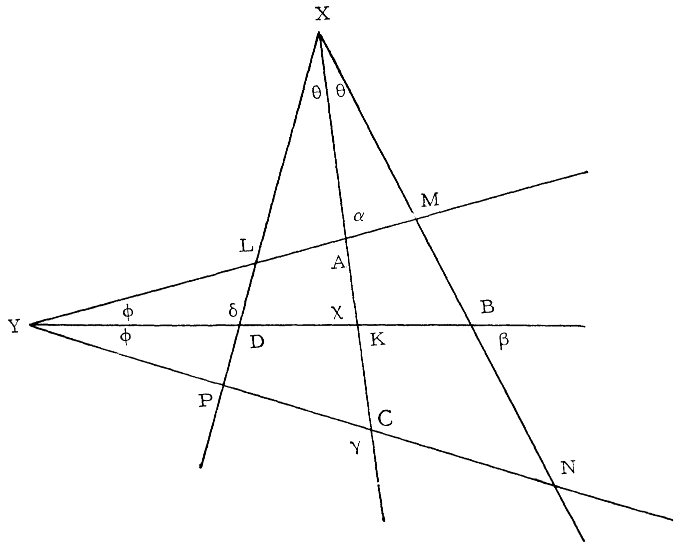

FIGURE 13

If $\chi = \frac{1}{2}\pi$ and we reflect points in the line XC, then B reflects into D and BM reflects into a segment of XD; A remains unaltered. Thus

$$BM + MA = DL + LA$$

only if L is the reflection of M in XC, i.e. only if $LM \parallel DB$. Similarly $PN \parallel DB$, and this is impossible since it means that $LM \parallel PN$, whereas in this particular case $LM \nparallel PN$. Thus $\chi \neq \frac{1}{2}\pi$ and $\theta = \phi$.

We prove next that $XK = YK$. Suppose that $YK > XK$ (see Fig. 14).

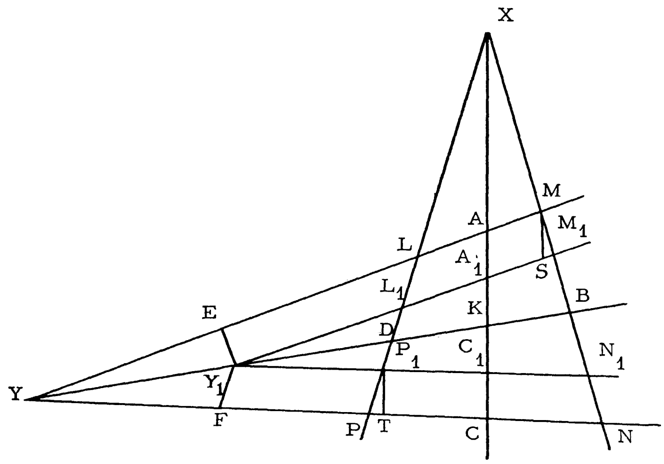

FIGURE 14

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Mark Y_1 on YK so that $Y_1 K = XK$ and draw lines $Y_1 M_1$ and $Y_1 N_1$ parallel to YM and YN respectively. The points of intersection of $Y_1 M_1$ and $Y_1 N_1$ with the lines XP and XC are shown in Fig. 14. If we reflect points in the line which is the bisector of angle $A_1 KD$, we see that $A_1 M_1 BK$ is congruent to $DP_1 C_1 K$ and therefore

$$A_1 M_1 + M_1 B = DP_1 + P_1 C_1.$$

By hypothesis,

$$AM + MB = DP + PC.$$

Now define S so that S lies on $L_1 M_1$ and $MS \parallel AK$. Then

$$AM_1 + M_1 B - AM - MB = SM_1 - MM_1.$$

Similarly, define T on PN so that $P_1 T \parallel AK$. Then

$$DP_1 + P_1 C - DP - PC = -PP_1 - PT.$$

Therefore

$$MM_1 - SM_1 = PT + PP_1.$$

Now $MM_1 = PP_1$; for if we draw $Y_1 E$ and $Y_1 F$ parallel to XM and to XL respectively, then $\angle Y_1 FY = \angle XPY$, $\angle Y_1 EY = \angle NMY$. But

$$\angle XPY + \angle NMY = \pi$$

and thus

$$\angle Y_1 FY + \angle Y_1 EY = \pi,$$

which together with the fact that $Y_1 Y$ bisects $\angle NYM$ proves that $Y_1 F = Y_1 E$. Therefore, $MM_1 = PP_1$.

But then from the equation $MM_1 - SM_1 = PT + PP_1$ it follows that $SM_1 = PT = 0$, which means that Y_1 coincides with Y.

The figure is symmetric about LN and it is possible to prove that for such a quadrilateral the equations

$$DL + LA = AM + MB = BN + NC = CP + PD$$

imply that the quadrilateral is a rhombus. The proof is straightforward and, to save space, is omitted.

Thus, the quadrilateral λ must be a parallelogram. For a parallelogram $AC + BD \geqslant 2\lambda(A, B, C, D)$ and thus the result is established.

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