

MULTIPLICATIVE SPECTRAL FUNCTIONALS ON $C(X)$

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Abstract

If A is a commutative C^* -algebra and if $\phi : A \rightarrow \mathbb{C}$ is a continuous multiplicative functional such that $\phi(x)$ belongs to the spectrum of x for each $x \in A$, then ϕ is linear and hence a character of A . This establishes a multiplicative Gleason–Kahane–Żelazko theorem for $C(X)$.

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1. Introduction

Let A be a complex Banach algebra with identity element $\mathbf{1}$, invertible group $G(A)$ and spectrum $\sigma(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\}$ for each $x \in A$. A famous result in the theory of Banach algebras, colloquially known as the Gleason–Kahane–Żelazko theorem, says that any linear functional $\phi : A \rightarrow \mathbb{C}$ which satisfies $\phi(x) \in \sigma(x)$ for each $x \in A$ is automatically multiplicative and hence a character of A . (For an elementary proof, due to Roitman and Sternfeld, see [1, Theorem 4.1.1].) It is natural to ask whether one can obtain a multiplicative version of the Gleason–Kahane–Żelazko theorem. A first result in this direction was obtained by Maouche in the paper [2]. Specifically, Maouche showed that if A is a complex and unital Banach algebra and if $\phi : A \rightarrow \mathbb{C}$ is a multiplicative functional on A such that $\phi(x) \in \sigma(x)$ for each $x \in A$, then ϕ generates a corresponding character ψ_ϕ on A which agrees with ϕ on $G_1(A)$, the connected component of $G(A)$ containing the identity. He also gave an example in the context of a commutative C^* -algebra (that is, some $C(X)$) which shows that ϕ is not necessarily equal to its induced character ψ_ϕ and, in particular, that ϕ need not be linear. (It follows easily that ϕ is homogeneous, so the difficulty lies with additivity.) With Maouche’s example, the current authors made the important observation that the particular functional ϕ is not continuous and so, in an attempt to obtain a multiplicative Gleason–Kahane–Żelazko theorem, it would not seem unreasonable to include continuity as part of the assumptions on ϕ as well (and even then a proof could be hard). With the classical Gleason–Kahane–Żelazko theorem, continuity of ϕ is not required and is part of the conclusion rather than the assumption.

In [3, Section II], the current authors, together with Schulz, made some progress in the case where A is a C^* -algebra. Amongst a number of results, two of the highlights are stated here as Theorems 1.2 and 1.4. For some results concerning a multiplicative version of the related Kowalski–Słodkowski theorem, one may consult [4].

The aim of this short paper is to establish the multiplicative Gleason–Kahane–Żelazko theorem for $C(X)$ (a proof of which had eluded us for some time). Our arguments, which are elementary but far from trivial, are similar in spirit to Roitman and Sternfeld’s proof of the classical Gleason–Kahane–Żelazko theorem.

The following terminology will be convenient.

DEFINITION 1.1. If A is a complex and unital Banach algebra, then a function $\phi : A \rightarrow \mathbb{C}$ is called a *spectral functional* on A if $\phi(x) \in \sigma(x)$ for each $x \in A$.

THEOREM 1.2 [3, Corollary 3.3 and Theorem 3.12]. *Let A be a C^* -algebra. Then every continuous multiplicative spectral functional $\phi : A \rightarrow \mathbb{C}$ generates a unique corresponding character ψ_ϕ on A which agrees with ψ_ϕ on the closure of $G(A)$. In particular, if A has stable rank one, then ϕ is a character of A .*

We point out the following instance of Theorem 1.2, which will be useful in the next section.

COROLLARY 1.3. *Let A be a C^* -algebra. Then every continuous multiplicative spectral functional $\phi : A \rightarrow \mathbb{C}$ generates a unique corresponding character ψ_ϕ on A which agrees with ψ_ϕ on the collection of self-adjoints.*

THEOREM 1.4 [3, Theorem 3.13]. *Let A be a von Neumann algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional on A . Then ϕ is a character of A .*

2. Multiplicative spectral functionals on $C(X)$

We proceed through a series of lemmas, leading to the proof of Theorem 2.5.

LEMMA 2.1. *Let A be a commutative C^* -algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional on A . If ψ_ϕ is the corresponding character of ϕ , then $\ker \phi = \ker \psi_\phi$.*

PROOF. By Corollary 1.3, $\psi_\phi(x) = \phi(x)$ for all self-adjoint elements x of A . Then, since

$$\begin{aligned}\phi(x)\phi(x^*) &= \phi(xx^*) = \psi_\phi(xx^*) \\ &= \psi_\phi(x)\psi_\phi(x^*) = \psi_\phi(x)\overline{\psi_\phi(x)} \\ &= |\psi_\phi(x)|^2,\end{aligned}$$

it follows that $\phi(x) = 0$ implies $\psi_\phi(x) = 0$. For the reverse implication, suppose that $\psi_\phi(x) = 0$. Consider $\phi(xe^{-nxx^*})$, where $n \in \mathbb{N}$. Using Theorem 1.2, we first calculate

$$\begin{aligned}\phi(xe^{-nxx^*}) &= \phi(x)\phi(e^{-nxx^*}) \\ &= \phi(x)\psi_\phi(e^{-nxx^*}) \\ &= \phi(x)e^{-n\psi_\phi(x)\psi_\phi(x^*)} \\ &= \phi(x).\end{aligned}$$

On the other hand, the continuous functional calculus implies that, for each $n \in \mathbb{N}$,

$$\sigma(xe^{-nxx^*}) = \{\alpha e^{-n|\alpha|^2} : \alpha \in \sigma(x)\}$$

and hence, since ϕ is a spectral functional, there exists a sequence (α_n) in $\sigma(x)$ such that

$$\phi(xe^{-nxx^*}) = \alpha_n e^{-n|\alpha_n|^2}.$$

Then, since the spectrum is compact, we may assume without loss of generality that the sequence (α_n) is convergent; let α be the limit. If $\alpha = 0$, then

$$0 \leq |\phi(x)| = |\alpha_n| e^{-n|\alpha_n|^2} \leq |\alpha_n|$$

implies that $\phi(x) = 0$. If $\alpha \neq 0$, then

$$0 = \lim |\alpha_n| e^{-n|\alpha_n|^2} = |\phi(x)|. \quad \square$$

The following result, which extends Lemma 2.1, will play a crucial role in the proof of the main result.

LEMMA 2.2. *Let A be a commutative C^* -algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional. Then $|\phi(x)| = |\psi_\phi(x)|$ for each $x \in A$.*

PROOF. By Lemma 2.1, we may assume that $x \notin \ker \phi = \ker \psi_\phi$. For each $n \in \mathbb{N}$, define A_n by

$$A_n := \phi(xe^{-n(x-\psi_\phi(x)\mathbf{1})(x^*-\overline{\psi_\phi(x)}\mathbf{1})}).$$

As in the proof of Lemma 2.1, $\phi(x) = A_n = \alpha_n e^{-n|\alpha_n - \psi_\phi(x)|^2}$ for some converging sequence (α_n) in the spectrum of x . Let α be the limit of the sequence (α_n) . If $\alpha \neq \psi_\phi(x)$, then $\lim A_n = 0 = \phi(x)$, which gives a contradiction. Thus, $\alpha = \psi_\phi(x)$ and so $|\phi(x)| \leq |\alpha_n|$ implies that $|\phi(x)| \leq |\psi_\phi(x)|$. Suppose that $0 < |\phi(x)| < |\psi_\phi(x)|$ for some $x \in A$. By Lemma 2.1 and the first part of the proof, $0 < |\phi(x^*)| \leq |\psi_\phi(x^*)|$. Using Corollary 1.3 again,

$$\begin{aligned}|\phi(x)| |\phi(x^*)| &< |\psi_\phi(x)| |\psi_\phi(x^*)| = |\psi_\phi(xx^*)| \\ &= |\phi(xx^*)| = |\phi(x)| |\phi(x^*)|,\end{aligned}$$

which is absurd. So, we may conclude that $|\phi(x)| = |\psi_\phi(x)|$ for each $x \in A$. $\square$

LEMMA 2.3. *Let A be a commutative C^* -algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional. If $\phi(x) = 0$, then $\phi(\mathbf{1} + x) = 1$.*

PROOF. Using Lemma 2.2, we find that $\psi_\phi(x) = 0$. The continuous functional calculus allows us to construct $f(x) = (\mathbf{1} + x)e^{-|x|} \in A$. Evaluating ϕ at $f(x)$ yields

$$\begin{aligned}\phi((\mathbf{1} + x)e^{-|x|}) &= \phi(\mathbf{1} + x)\phi(e^{-|x|}) \\ &= \phi(\mathbf{1} + x)\psi_\phi(e^{-|x|}) = \phi(\mathbf{1} + x)e^{-|\psi_\phi(x)|} \\ &= \phi(\mathbf{1} + x).\end{aligned}$$

From the fact that $\phi((\mathbf{1} + x)e^{-|x|}) \in \sigma((\mathbf{1} + x)e^{-|x|})$, we deduce, as before, that there exists $\alpha \in \sigma(x)$ such that

$$\phi((\mathbf{1} + x)e^{-|x|}) = (1 + \alpha)e^{-|\alpha|} = \phi(\mathbf{1} + x).$$

Therefore, using Lemma 2.2,

$$|1 + \alpha|e^{-|\alpha|} = |\phi(\mathbf{1} + x)| = |\psi_\phi(\mathbf{1} + x)| = 1 \Rightarrow |1 + \alpha| = e^{|\alpha|}.$$

It follows that

$$e^{|\alpha|} = |1 + \alpha| \leq 1 + |\alpha| \leq e^{|\alpha|},$$

from which $1 + |\alpha| = e^{|\alpha|}$ forces $\alpha = 0$ and hence $\phi(\mathbf{1} + x) = (1 + 0)e^{-0} = 1$. $\square$

LEMMA 2.4. *Let A be a commutative C^* -algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional. Then $\phi(\mathbf{1} + x) = 1 + \psi_\phi(x)$.*

PROOF. If $\psi_\phi(x) = -1$, then, by Lemma 2.1, $\psi_\phi(\mathbf{1} + x) = 0 = \phi(\mathbf{1} + x) = 1 + \psi_\phi(x)$. Suppose that $\psi_\phi(x) \neq -1$. Then

$$\psi_\phi(-\psi_\phi(x)\mathbf{1} + x) = 0 = \phi(-\psi_\phi(x)\mathbf{1} + x).$$

Using Lemma 2.3 and the fact that ϕ is homogeneous, we infer that

$$\begin{aligned}\phi(\mathbf{1} + x) &= \phi(\mathbf{1} + \psi_\phi(x)\mathbf{1} - \psi_\phi(x)\mathbf{1} + x) \\ &= \phi(\mathbf{1}(1 + \psi_\phi(x)) + x - \psi_\phi(x)\mathbf{1}) \\ &= (1 + \psi_\phi(x))\phi\left(\mathbf{1} + \frac{x - \psi_\phi(x)\mathbf{1}}{1 + \psi_\phi(x)}\right) \\ &= 1 + \psi_\phi(x).\end{aligned}\quad \square$$

THEOREM 2.5. *Let A be a commutative C^* -algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional. Then ϕ is a character of A .*

PROOF. It suffices to prove that ϕ agrees with ψ_ϕ on all of A . Let $x \in A$ be arbitrary and let n be a positive integer. Then, using Lemma 2.4,

$$\phi(\mathbf{1}/n + x) = \frac{1}{n}\phi(\mathbf{1} + nx) = \frac{1}{n}(1 + n\psi_\phi(x)) = \frac{1}{n} + \psi_\phi(x). \quad (2.1)$$

Letting $n \rightarrow \infty$ on both sides of (2.1) and using the fact that ϕ is continuous at $x \in A$, we deduce that $\phi(x) = \psi_\phi(x)$. $\square$

By the Fuglede–Putnam–Rosenblum theorem [1, Theorem 6.2.5], the bicommutant of a normal element of A is a commutative C^* -subalgebra of A and so, together with the fact that ϕ is multiplicative, we also have the following improvement on [3, Corollaries 3.8 and 3.10].

COROLLARY 2.6. *Let A be a C^* -algebra and let $\phi : A \rightarrow \mathbb{C}$ be a continuous multiplicative spectral functional. Then ϕ agrees with ψ_ϕ on the closure of the set of all finite products of normal elements.*

References

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