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COMMUTATIVITY PRESERVING MAPPINGS OF VON NEUMANN ALGEBRAS

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ABSTRACT A map $\theta\colon M \longrightarrow N$ where M and N are rings is said to preserve commutativity in both directions if the elements $a, b \in M$ commute if and only if $\theta(a)$ and $\theta(b)$ commute. In this paper we show that if M and N are von Neumann algebras with no central summands of type I_1 or I_2 and θ is a bijective additive map which preserves commutativity in both directions then $\theta(x) = c\varphi(x) + f(x)$ where c is an invertible element in Z_N, the center of N, $\varphi\colon M \longrightarrow N$ is a Jordan isomorphism of M onto N, and f is an additive map of M into Z_N.

Introduction. By a commutativity preserving mapping of an algebra M into an algebra N we mean a mapping $\theta\colon M \longrightarrow N$ which maps commuting pairs of elements into commuting pairs. We say that θ preserves commutativity in both directions if the elements $a, b, \in M$ commute if and only if $\theta(a)$ and $\theta(b)$ commute. The aim in the study of commutativity preserving mappings is to determine their structure. In this paper, we consider the case when M and N are von Neumann algebras. We shall prove

THEOREM 1. *Let M and N be von Neumann algebras with no central summands of type I_1 or I_2. Let $\theta\colon M \longrightarrow N$ be a bijective additive mapping. If θ preserves commutativity in both directions then it is of the form*

$$\theta(x) = c\varphi(x) + f(x)$$

where c is an invertible element in Z_N, $\varphi\colon M \longrightarrow N$ is a Jordan isomorphism of M onto N, and f is an additive mapping of M into Z_N.

It can be easily shown that Jordan isomorphisms of von Neumann algebras preserve commutativity in both directions (*cf.* [2, Theorem 3.4]). Thus, Theorem 1 characterizes bijective additive mappings preserving commutativity in both directions.

One usually assumes that a commutativity preserving mapping is linear. Our algebraic methods enable us to weaken this assumption and to assume only the additivity of the mapping. Also, mappings such as isomorphisms, anti-isomorphisms and Jordan isomorphisms will be considered in a ring sense—for instance, by a Jordan isomorphism φ of M into N we shall mean an additive bijective mapping satisfying $\varphi(x^2) = \varphi(x)^2$ for all $x \in M$.

A number of authors have characterized commutativity preserving mappings of various algebras. These characterizations are essentially the same as in Theorem 1, although

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in some algebras Jordan isomorphisms can be expressed in a more explicit form It seems that the first result of that kind was given by Watkins in [16] where the form of bijective linear commutativity preserving mappings of $M_n(F)$, the algebra of all $n \times n$ matrices, $n \geq 4$, over a field F, was determined Also, by a simple counterexample it was shown that the situation in case $n = 2$ is quite different (this justifies the assumption in Theorem 1 that von Neumann algebras must not contain central summands of type I_2) The case when $n = 3$ was settled in [1] and [14] In a series of papers [7, 8, 15] mappings preserving commutativity of symmetric matrices were discussed The paper [8] of Choi, Jafarian, and Radjavi also contains some extension of these results to the algebra of all bounded linear operators on an infinite dimensional Hilbert space Subsequently, Omladič [13] described the structure of bijective linear mappings of $\mathcal{B}(X)$, the algebra of all bounded operators on a Banach space X, $\dim X \geq 3$, which preserve commutativity in both directions An analogous result for bijective *-linear mappings of von Neumann factors was obtained by the second named author [12] (note that in Theorem 1 we do not assume that θ preserves adjoints) Finally, in [6] the first named author characterized linear bijective commutativity preserving mappings of prime algebras (satisfying some additional assumptions) Moreover, the assumption that θ preserves commutativity was replaced by a weaker assumption that $\theta(x)$ and $\theta(x^2)$ commute for any element x In this paper we use a similar approach as in [6], and, in fact, the assumption in Theorem 1 that θ preserves commutativity in both directions can be replaced by a quite weaker one (see Theorem 3)

A mapping f of a ring M into itself is said to be *commuting* if $f(x)$ commutes with x for every x in M Additive commuting mappings of prime rings and von Neumann algebras were characterized in [4] and [5], respectively A mapping q $M \longrightarrow M$ is said to be a *trace of a biadditive mapping* if there exists a biadditive mapping B $M \times M \longrightarrow M$ such that $q(x) = B(x, x)$ for all $x \in M$ There is a simple connection between commuting traces of biadditive mappings and commutativity preserving mappings (see the proof of Step 2 of Theorem 3) The fundamental result in [6], upon which all the other results in [6] depend, determines the structure of commuting traces of biadditive mappings of certain prime rings Following the procedure in [6], we will first obtain an analogous result for von Neumann algebras (Theorem 2)

Recall that a bijective additive mapping θ of a ring M onto a ring N is called a *Lie isomorphism* if it preserves commutators, *i e* , $\theta([x, y]) = [\theta(x), \theta(y)]$ for all $x, y \in M$ where $[u, v]$ denotes $uv - vu$ Obviously, these mappings preserve commutativity in both directions Therefore, as a consequence of Theorem 1 we obtain a result concerning Lie isomorphisms of von Neumann algebras (Theorem 4) A similar result was obtained by the second named author in [10] Comparing this result with Theorem 4 we see that in Theorem 4 we do not assume any continuity or *-linearity, but on the other hand, we have to exclude von Neumann algebras containing central summands of type I_1 or I_2 Possibly Theorem 4 holds for arbitrary von Neumann algebras, however, to prove this one should have to use quite different methods

The center of an algebra M will be denoted by Z_M. A ring is called *semi-prime* if $aMa = 0$ implies $a = 0$ for $a \in M$. Any C^*-algebra is semi-prime. We use [9] as a general reference for the theory of operator algebras.

The results. Our first goal is to determine the structure of all commuting traces of biadditive mappings on von Neumann algebras with no central summands of type I_1 or I_2. For this purpose we need some preliminary results.

LEMMA 1. *Let M be a type I von Neumann algebra and let $p \in M$ be a projection. There exist projections e, f_1, f_2 in M such that $p = e+f_1+f_2$, e is abelian, $f_1 \sim f_2$, $f_1 \perp f_2$, and $e \perp f_1+f_2$.*

PROOF. By considering the type I algebra pMp it suffices to assume $p = 1$, the identity of M. Since M is of type I, $M = \oplus_{n\in\mathbf{K}} M_n$ where $\mathbf{K}$ is a set of distinct cardinals and M_n is a homogeneous algebra of type I_n. Now $1 = \sum_{n\in\mathbf{K}} p_n$ where p_n is the identity of M_n and is the sum of n orthogonal equivalent abelian projections. If n is finite and even then $p_n = f_{1n}+f_{2n}$ where $f_{1n} \sim f_{2n}$ and $f_{1n} \perp f_{2n}$. If n is finite and odd then $p_n = e_n+f_{1n}+f_{2n}$ where $e_n \neq 0$ is abelian, $f_{1n} \sim f_{2n}, f_{1n} \perp f_{2n}$. If n is infinite then by breaking up the set of n orthogonal, equivalent abelian projections that sum to p_n into two subsets of the same cardinality we can write $p_n = f_{1n}+f_{2n}, f_{1n} \sim f_{2n}, f_{1n} \perp f_{2n}$. Set $e = \sum e_n, f_1 = \sum f_{1n}$, $f_2 = \sum f_{2n}$. Then e is abelian since it is a sum of abelian projections with mutually disjoint central supports. Moreover $f_1 \sim f_2, f_1 \perp f_2$, $e \perp f_1+f_2$ and $1 = \sum_{n\in\mathbf{K}} p_n = e+f_1+f_2$.

LEMMA 2. *Let M be a von Neumann algebra with no type I_1 or I_2 summands. Then the ideal I of M generated algebraically by $\{[x^2,z]y[x,z]-[x,z]y[x^2,z] : x,y,z \in M\}$ is equal to M.*

PROOF. If $I \neq M$ then $J = \bar{I} \neq M$ where $\bar{I}$ is the uniform closure of I. Let $N = M/J$. Then N is semi-prime since it is a C^*-algebra, and N satisfies $[x^2,z]y[x,z] = [x,z]y[x^2,z]$. Standard polynomial identity theory for semi-prime rings implies that $[x,y]^2 \in Z_N$. If p is in the continuous part of M then $p = f_1+f_2$ where $f_1 \sim f_2, f_1 \perp f_2$. Hence there exists $v \in M$ such that $vv^* = f_1$, $v^*v = f_2$ so that $[v,v^*]^2 = (f_1-f_2)^2 = f_1+f_2$. Hence $\bar{p} = p+J \in Z_N$. Let M_D be the type I part of M where D is a projection in Z_M. If $p \in M_D$ then, by Lemma 1, $p = e+f_1+f_2$ where e is abelian, $f_1 \sim f_2, f_1 \perp f_2$ so that we can apply the previous argument to show that $\overline{f_1+f_2} = f_1+f_2+J \in Z_N$. Let $D = \oplus_{n\in\mathbf{K}} D_n$ where D_n is a homogeneous summand of type n and $\mathbf{K}$ is a set of distinct cardinals. Then $e_n = D_n e$ is an abelian projection in M_{D_n}. Since M has no summand of type I_1 or I_2 we can choose f_n, g_n in M_{D_n} such that $\{e_n, f_n, g_n\}$ is a set of three pairwise orthogonal equivalent projections. Thus $e = \sum e_n, f = \sum f_n, g = \sum g_n$ are pairwise orthogonal and equivalent. By the above argument, $\bar{e}+\bar{f}, \bar{f}+\bar{g}$, and $\bar{e}+\bar{g}$ are in Z_N so that $\bar{e} \in Z_N$. Hence for any projection $p \in M$, $\bar{p} \in Z_N$. We show that for any $m \in M$, $\bar{m} \in Z_N$. It suffices to assume $m = m^*$. By [11, Lemma 2], $Z_N = Z_M+J$ so for each $p \in M$, $p = z+j$ for some $z \in Z_M$, $j \in J$. Given ϵ choose projections $p_\iota \in M$ and scalars λ_ι such that $\|m-\sum\lambda_\iota p_\iota\| < \epsilon$ and then choose $z_\iota \in Z_M$, $j_\iota \in J$ such that $p_\iota = z_\iota+j_\iota$. We have $\|\bar{m}-\sum\lambda_\iota\bar{z}_\iota\| = \inf_{j\in J}\|m-\sum\lambda_\iota z_\iota - j\| \leq \|m-\sum\lambda_\iota z_\iota-\sum\lambda_\iota j_\iota\| = \|m-\sum\lambda_\iota j_\iota\| < \epsilon$.

Hence $\bar{m} \in Z_N$ so that $[M, M] \subseteq J$. By [5, Lemma 2.6] the ideal generated by $[M, M]$ is M so $M = J$ which is a contradiction. ∎

A connection between Lemma 2 and commuting traces of biadditive mappings is indicated in the following lemma, which was proved in [6] (although it is not explicitly stated there, it is clear from the proof of [6, Theorem 1]).

LEMMA 3. *Let M be any ring admitting the operator $\frac{1}{2}$ (i.e., the mapping $x \longrightarrow 2x$ is bijective). If $q: M \longrightarrow M$ is a commuting trace of a biadditive mapping, then there exist mappings $g_1: M \times M \times M \longrightarrow M$ and $g_2, g_3: M \times M \times M \times M \longrightarrow M$ such that*

$$\gamma(x, y, z)uq(w) = g_1(x, y, z)uw^2 + g_2(x, y, z, w)uw + g_3(x, y, z, w)u \tag{1}$$

for all $x, y, z, w, u \in M$, where

$$\gamma(x, y, z) = [x^2, z]y[x, z] - [x, z]y[x^2, z].$$

Moreover, g_2 is additive in the last argument.

We will need the following simple lemma, which is a special case of [2, Lemma 1.2].

LEMMA 4. *Let G be an additive group and M be a semiprime ring. Suppose that additive mappings S and T of G into M satisfy $S(x)MT(x) = \{0\}$ for all $x \in G$. Then $S(x)MT(y) = \{0\}$ for all $x, y \in G$.*

We are now in a position to prove

THEOREM 2. *Let M be a von Neumann algebra with no central summands of type I_1 or I_2. Let $q: M \longrightarrow M$ be a trace of a biadditive mapping. If q is commuting then it is of the form*

$$q(x) = \lambda x^2 + \mu(x)x + \nu(x), \quad x \in M,$$

where $\lambda \in Z_M$, μ and ν are mappings of M into Z_M, and μ is additive.

PROOF. Replacing u by uv in (1), and then comparing the relation so obtained with (1), we obtain

$$\gamma(x, y, z)u[v, q(w)] = g_1(x, y, z)u[v, w^2] + g_2(x, y, z, w)u[v, w]. \tag{2}$$

Let 1 be the identity element of M. By Lemma 2 there exist $t_i, x_i, y_i, z_i, u_i \in M$, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{n} t_i \gamma(x_i, y_i, z_i) u_i = 1.$$

Using (2), we then see that for any $v, w \in M$ we have

$$\begin{aligned}
[v, q(w)] &= 1[v, q(w)] \\
&= \Big\{ \sum_{i=1}^{n} t_i \gamma(x_i, y_i, z_i) u_i \Big\} [v, q(w)] \\
&= \sum_{i=1}^{n} t_i \{ \gamma(x_i, y_i, z_i) u_i [v, q(w)] \} \\
&= \sum_{i=1}^{n} t_i g_1(x_i, y_i, z_i) u_i [v, w^2] + \sum_{i=1}^{n} t_i g_2(x_i, y_i, z_i, w) u_i [v, w].
\end{aligned}$$

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Thus

$$(3) \qquad [v, q(w)] = \lambda[v, w^2] + \mu(w)[v, w], \quad v, w \in M$$

for some $\lambda \in M$ and some map $\mu: M \longrightarrow M$; note that μ is additive since, by Lemma 3, g_2 is additive in the last argument. Our intention is to show that $\lambda \in Z_M$ and that μ maps M into Z_M.

Substituting vy for v in (3) we obtain

$$[v, q(w)]y + v[y, q(w)] = \lambda[v, w^2]y + \lambda v[y, w^2] + \mu(w)[v, w]y + \mu(w)v[y, w].$$

On the other hand, (3) shows that

$$[v, q(w)]y + v[y, q(w)] = \lambda[v, w^2]y + \mu(w)[v, w]y + v\lambda[y, w^2] + v\mu(w)[y, w].$$

Comparing the last two relations we get

$$(4) \qquad [\lambda, v][y, w^2] + [\mu(w), v][y, w] = 0, \quad v, y, w \in M.$$

Replacing v by xv in (4), it follows that

$$x[\lambda, v][y, w^2] + [\lambda, x]v[y, w^2] + x[\mu(w), v][y, w] + [\mu(w), x]v[y, w] = 0.$$

By (4), the sum of the first and the third summands equals zero. Hence

$$(5) \qquad [\lambda, x]v[y, w^2] + [\mu(w), x]v[y, w] = 0, \quad x, v, y, w \in M.$$

In particular,

$$[\lambda, x](v[y, w]z)[y, w^2] + [\mu(w), x](v[y, w]z)[y, w] = 0.$$

But on the other hand, (5) yields

$$\big([\mu(w), x]v[y, w]\big)z[y, w] = -[\lambda, x]v[y, w^2]z[y, w].$$

Comparing the last two relations we arrive at

$$[\lambda, x]v([y, w]z[y, w^2] - [y, w^2]z[y, w]) = 0.$$

Lemma 2 implies that $[\lambda, x]M = 0$ for all $x \in M$, and therefore, $\lambda \in Z_M$. Now, (5) reduces to

$$(6) \qquad [\mu(w), x]M[y, w] = \{0\}, \quad x, y, w \in M.$$

Now fix $x, y \in M$ and introduce additive mappings S and T of M by $S(w) = [\mu(w), x]$, $T(w) = [y, w]$. By (6), we have $S(w)MT(w) = \{0\}$ for all $w \in M$, so it follows from Lemma 4 that $S(w)MT(z) = \{0\}$ for all $w, z \in M$. Thus $[\mu(w), x]v[y, z] = 0$ holds for any $w, x, v, y, z \in M$. In particular, $[\mu(w), x]v[\mu(w), x] = 0$, $w, x \in M$, which shows that $\mu(w) \in Z_M$, $w \in M$. By (3) we now see that $\nu(w) = q(w) - \lambda w^2 - \mu(w)w$ lies in Z_M as well. With this the theorem is proved. ∎

Our next aim is to consider commutativity preserving maps of von Neumann algebras. We need two preliminary results.

LEMMA 5 *Let M be a von Neumann algebra with no central summands of type I_1 If $c \in Z_M$ is such that that $cM \subseteq Z_M$, then $c = 0$*

PROOF We have $[cx, y] = 0$, and therefore, $c[x, y] = 0$ for all $x, y \in M$ Thus $cI = \{0\}$ where I is the ideal of M generated by all commutators in M But $I = M$ [5, Lemma 2 6], and so c must be zero ∎

Recall that a ring M is said to be *torsion-free* if $nx = 0$, where $x \in M$ and n is any positive integer, implies $x = 0$

LEMMA 6 *Let M be a semiprime torsion-free ring and G be an additive group Suppose that mappings ϵ $G \times G \longrightarrow M$ and τ $G \times G \times G \longrightarrow M$ are additive in each argument If $\epsilon(x,x)M\tau(x,x,x) = \{0\}$ for every $x \in G$, then $\epsilon(y,y)M\tau(x,x,x) = \{0\}$ for all $x, y \in G$*

PROOF We have $\epsilon(x,x)r\tau(x,x,x) = 0$ Note that the substitution $x + ny$ for x, where $x, y \in G$ and n is an integer, yields

$$n\Big\{\big(\epsilon(x,y) + \epsilon(y,x)\big)r\tau(x,x,x) + \epsilon(x,x)r\big(\tau(x,x,y) + \tau(x,y,x) + \tau(y,x,x)\big)\Big\} + n^2z_2 + n^3z_3 + n^4z_4 = 0$$

for some elements $z_2, z_3, z_4 \in M$ depending on x, y and r Since n is an arbitrary integer and M is torsion-free, it follows easily that

$$\big(\epsilon(x,y) + \epsilon(y,x)\big)r\tau(x,x,x) + \epsilon(x,x)r\big(\tau(x,x,y) + \tau(x,y,x) + \tau(y,x,x)\big) = 0$$

Multiplying from the right by $s\tau(x,x,x)$, since $\epsilon(x,x)M\tau(x,x,x) = \{0\}$, we arrive at

$$\big(\epsilon(x,y) + \epsilon(y,x)\big)r\tau(x,x,x)s\tau(x,x,x) = 0$$

Since r and s are arbitrary elements in M, the semiprimeness of M implies that

$$\big(\epsilon(x,y) + \epsilon(y,x)\big)M\tau(x,x,x) = \{0\} \tag{7}$$

for all $x, y \in G$ In this relation, replace x by $x+nz$ with $x, z \in G$ and n an integer Arguing similarly as above, one obtains easily that

$$\big(\epsilon(z,y) + \epsilon(y,z)\big)r\tau(x,x,x) + \big(\epsilon(x,y) + \epsilon(y,x)\big)r\big(\tau(z,x,x) + \tau(x,z,x) + \tau(x,x,z)\big) = 0$$

Multiplying from the right by $s\tau(x,x,x)$, and then using (7), we get

$$\big(\epsilon(z,y) + \epsilon(y,z)\big)r\tau(x,x,x)s\tau(x,x,x) = 0$$

Since R is semiprime, it follows that $\big(\epsilon(z,y) + \epsilon(y,z)\big)M\tau(x,x,x) = \{0\}$ A special case of this relation, where $z = y$, gives the assertion of the lemma

We now come to the central theorem of this paper, note that this theorem includes Theorem 1

THEOREM 3. *Let M and N be von Neumann algebras with no central summands of type I_1 or I_2. Let $\theta\colon M \longrightarrow N$ be a bijective additive mapping such that $\theta(Z_M) = Z_N$, $[\theta(x^2), \theta(x)] = 0$ for all $x \in M$, and $[\theta^{-1}(wy), \theta^{-1}(y)] = 0$ for all $y \in N$ and $w \in Z_N$. Then θ is of the form*

$$\theta(x) = c\varphi(x) + f(x)$$

where c is an invertible element in Z_N, f is an additive mapping of M into Z_N, and φ is a Jordan isomorphism of M onto N.

Moreover, there exist central projections $p \in M$ and $q \in N$ such that the restriction of φ to pM is an isomorphism of pM onto qN, and the restriction of φ to $(1-p)M$ is an anti-isomorphism of $(1-p)M$ onto $(1-q)N$.

PROOF. The proof is broken up into a series of steps.

STEP 1. *There is an isomorphism $\alpha\colon Z_M \longrightarrow Z_N$ such that*

$$\begin{cases} \theta(zx) - \alpha(z)\theta(x) \in Z_N & \text{for all } z \in Z_M,\ x \in M, \text{ and} \\ \theta^{-1}(wy) - \alpha^{-1}(w)\theta^{-1}(y) \in Z_M & \text{for all } w \in Z_N,\ y \in N. \end{cases}$$

PROOF OF STEP 1. Take $x \in M$ and $z \in Z_M$. As θ maps Z_M into Z_N, a substitution $x + z$ for x in $[\theta(x^2), \theta(x)] = 0$ gives $[\theta(zx), \theta(x)] = 0$. Denoting $\theta(x)$ by y, we thus have $\left[\theta\left(z\theta^{-1}(y)\right), y\right] = 0$ for arbitrary $z \in Z_M$ and $y \in N$. That is, for any $z \in Z_M$, $y \longrightarrow \theta\left(z\theta^{-1}(y)\right)$ is a commuting additive mapping of a von Neumann algebra N. By [5, Theorem 2.1] it follows that there exists an element w in Z_N (depending on z) such that $\theta\left(z\theta^{-1}(y)\right) - wy \in Z_N$ for all $y \in N$; or equivalently, $\theta(zx) - w\theta(x) \in Z_N$ for all $x \in M$. We set $w = \alpha(z)$, and claim that the mapping $z \longrightarrow \alpha(z)$ is an isomorphism of Z_M onto Z_N. Our key relation is

$$\theta(zx) - \alpha(z)\theta(x) \in Z_N \text{ for all } x \in M,\ z \in Z_M. \tag{8}$$

Let us first prove that α is additive. Take $z_1, z_2 \in Z_M$. According to (8), for any $x \in M$ we have

$$\theta\left((z_1 + z_2)x\right) \in \alpha(z_1 + z_2)\theta(x) + Z_N.$$

On the other hand,

$$\theta\left((z_1 + z_2)x\right) = \theta(z_1x) + \theta(z_2x) \in \alpha(z_1)\theta(x) + \alpha(z_2)\theta(x) + Z_N.$$

Comparing, we get $\left(\alpha(z_1 + z_2) - \alpha(z_1) - \alpha(z_2)\right)\theta(x) \in Z_N$. Since θ is onto, Lemma 5 implies that $\alpha(z_1 + z_2) = \alpha(z_1) + \alpha(z_2)$.

Next, let us show that α is multiplicative. On the one hand, for $z_1, z_2 \in Z_M$, $x \in M$, we have

$$\theta(z_1z_2x) \in \alpha(z_1z_2)\theta(x) + Z_N,$$

while on the other hand,

$$\begin{aligned} \theta\left(z_1(z_2x)\right) \in \alpha(z_1)\theta(z_2x) + Z_N &\subseteq \alpha(z_1)\left(\alpha(z_2)\theta(x) + Z_N\right) + Z_N \\ &= \alpha(z_1)\alpha(z_2)\theta(x) + Z_N. \end{aligned}$$

Hence $\big(\alpha(z_1z_2) - \alpha(z_1)\alpha(z_2)\big)\theta(x) \in Z_N$, and so $\alpha(z_1z_2) = \alpha(z_1)\alpha(z_2)$ by Lemma 5

Suppose that $\alpha(z) = 0$ for some $z \in Z_M$ By (8), we then have $\theta(zx) \in Z_N$ for every $x \in M$ Since we assumed that $\theta(Z_M) = Z_N$ this implies $zx \in Z_M, x \in M$, and so Lemma 5 yields $z = 0$ Thus α is one-to-one

Let us show that α is onto Take $w \in Z_N$ By assumption, we have $[\theta^{-1}(wy), \theta^{-1}(y)] = 0$ for all $y \in N$ Writing y as $\theta(x)$, we get $\big[\theta^{-1}\big(w\theta(x)\big), x\big] = 0$ That is, $x \longrightarrow \theta^{-1}\big(w\theta(x)\big)$ is a commuting additive mapping of M By [5, Theorem 2 1] there is an element $\beta(w) \in Z_M$ such that $\theta^{-1}\big(w\theta(x)\big) - \beta(w)x \in Z_M$ for al $x \in M$, or equivalently,

$$\theta^{-1}(wy) - \beta(w)\theta^{-1}(y) \in Z_M \text{ for all } y \in N \tag{9}$$

As $\theta(Z_M) = Z_N$ it follows that $wy - \theta\big(\beta(w)\theta^{-1}(y)\big) \in Z_N$, $y \in N$ In view of (8), $\theta\big(\beta(w)\theta^{-1}(y)\big) \in \alpha\big(\beta(w)\big)y + Z_N$, and therefore $\big(w - \alpha(\beta(w)\big)y \in Z_N$, $y \in N$ But then $w = \alpha\big(\beta(w)\big)$ by Lemma 5 This implies that α is onto Of course, $\beta = \alpha^{-1}$, and so, according to (9), the assertion of Step 1 is proved

STEP 2 *There exist an element $\lambda \in Z_N$, an additive mapping μ $M \longrightarrow Z_N$ and a mapping ν $M \longrightarrow Z_N$ such that*

$$\theta(x^2) = \lambda\theta(x)^2 + \mu(x)x + \nu(x) \textit{ for all } x \in M \tag{10}$$

PROOF OF STEP 2 The relation $[\theta(x), \theta(x^2)] = 0, x \in M$, can be written in the form $\big[y, \theta\big(\theta^{-1}(y)^2\big)\big] = 0$, $y \in N$ Thus, $q(y) = \theta\big(\theta^{-1}(y)^2\big)$ is a commuting mapping of N Since q is a trace of a biadditive mapping $B(y, z) = \theta\big(\theta^{-1}(y)\theta^{-1}(z)\big)$, Theorem 2 can be applied Hence there are $\lambda \in Z_N$, an additive mapping μ_1 $N \longrightarrow Z_N$ and a mapping ν_1 $N \longrightarrow Z_N$ such that

$$\theta\big(\theta^{-1}(y)^2\big) = \lambda y^2 + \mu_1(y)y + \nu_1(y)$$

for all $y \in N$ Note that this implies (10) where $\mu = \mu_1\theta$ and $\nu = \nu_1\theta$

STEP 3 *λ is invertible*

PROOF OF STEP 3 We have

$$x^2 = \theta^{-1}\big(\lambda\theta(x)^2 + \mu(x)\theta(x) + \nu(x)\big)$$

Applying Step 1 and the assumption that θ^{-1} maps Z_N into Z_M it follows that

$$x^2 - \alpha^{-1}(\lambda)\theta^{-1}\big(\theta(x)^2\big) - \alpha^{-1}\big(\mu(x)\big)x \in Z_M$$

Consequently

$$[x^2, u] = \alpha^{-1}(\lambda)\big[\theta^{-1}\big(\theta(x)^2\big), u\big] - \alpha^{-1}\big(\mu(x)\big)[x, u]$$

holds for all $x, u \in M$ From this relation we see that

$$[x^2, u]v[x, u] - [x, u]v[x^2, u] = \alpha^{-1}(\lambda)\Big\{\big[\theta^{-1}\big(\theta(x)^2\big), u\big]v[x, u] - [x, u]v\big[\theta^{-1}\big(\theta(x)^2\big), u\big]\Big\}$$

for all $x, u, v \in M$ Since the ideal generated by elements of the form $[x^2, u]v[v, u] - [x, u]v[x^2, u]$ is equal to M (Lemma 2), it follows that $1 \in \alpha^{-1}(\lambda)M$, which means that $\alpha^{-1}(\lambda)$ is invertible But then λ is invertible

STEP 4. *A mapping* $\varphi\colon M \longrightarrow N$, *defined by*

$$\varphi(x) = \lambda\theta(x) + \frac{1}{2}\mu(x)$$

is a Jordan homomorphism.

PROOF OF STEP 4. We will argue similarly as in the proof of [6, Theorem 2]. We have

$$\begin{aligned}\varphi(x^2) &= \lambda\theta(x^2) + \frac{1}{2}\mu(x^2)\\ &= \lambda^2\theta(x)^2 + \lambda\mu(x)\theta(x) + \lambda\nu(x) + \frac{1}{2}\mu(x^2),\end{aligned}$$

and

$$\begin{aligned}\varphi(x)^2 &= \Big(\lambda\theta(x) + \frac{1}{2}\mu(x)\Big)^2\\ &= \lambda^2\theta(x)^2 + \lambda\mu(x)\theta(x) + \frac{1}{4}\mu(x)^2.\end{aligned}$$

Comparing these two relations we get

$$\varphi(x^2) - \varphi(x)^2 \in Z_N \text{ for all } x \in M. \tag{11}$$

Define the mapping $\epsilon\colon M \times M \longrightarrow N$ by

$$\epsilon(x,y) = \varphi(xy + yx) - \varphi(x)\varphi(y) - \varphi(y)\varphi(x).$$

Obviously, ϵ is biadditive and it satisfies $\epsilon(x,y) = \epsilon(y,x)$ for all $x, y \in M$. Replacing x by $x+y$ in (11) we that ϵ in fact maps into Z_N. In order to show that φ is a Jordan homomorphism we must prove that $\epsilon = 0$.

Note that $\varphi(x^2) = \varphi(x)^2 + \frac{1}{2}\epsilon(x,x)$. Next, we have

$$\begin{aligned}\varphi(x^3) &= \frac{1}{2}\varphi(x^2x + xx^2)\\ &= \frac{1}{2}\{\varphi(x^2)\varphi(x) + \varphi(x)\varphi(x^2) + \epsilon(x^2,x)\}\\ &= \frac{1}{2}\Big\{\Big(\varphi(x)^2 + \frac{1}{2}\epsilon(x,x)\Big)\varphi(x) + \varphi(x)\Big(\varphi(x)^2 + \frac{1}{2}\epsilon(x,x)\Big) + \epsilon(x^2,x)\Big\}\\ &= \varphi(x)^3 + \frac{1}{2}\epsilon(x,x)\varphi(x) + \frac{1}{2}\epsilon(x^2,x).\end{aligned}$$

Hence

$$\begin{aligned}\varphi(x^4) &= \frac{1}{2}\varphi(xx^3 + x^3x)\\ &= \frac{1}{2}\{\varphi(x)\varphi(x^3) + \varphi(x^3)\varphi(x) + \epsilon(x^3,x)\}\\ &= \frac{1}{2}\Big\{\varphi(x)\Big(\varphi(x)^3 + \frac{1}{2}\epsilon(x,x)\varphi(x) + \frac{1}{2}\epsilon(x^2,x)\Big) + \Big(\varphi(x)^3 + \frac{1}{2}\epsilon(x,x)\varphi(x)\\ &\qquad + \frac{1}{2}\epsilon(x^2,x)\Big)\varphi(x) + \epsilon(x^3,x)\Big\}\\ &= \varphi(x)^4 + \frac{1}{2}\epsilon(x,x)\varphi(x)^2 + \frac{1}{2}\epsilon(x^2,x)\varphi(x) + \frac{1}{2}\epsilon(x^3,x).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\varphi(x^4) &= \varphi\big((x^2)^2\big) = \varphi(x^2)^2 + \frac{1}{2}\epsilon(x^2, x^2) \\ &= \Big(\varphi(x)^2 + \frac{1}{2}\epsilon(x, x)\Big)^2 + \frac{1}{2}\epsilon(x^2, x^2) \\ &= \varphi(x)^4 + \epsilon(x, x)\varphi(x)^2 + \frac{1}{4}\epsilon(x, x)^2 + \frac{1}{2}\epsilon(x^2, x^2)\end{aligned}$$

Comparing the two relations, so obtained for $\varphi(x^4)$, we get

$$\epsilon(x, x)\varphi(x)^2 - \epsilon(x^2, x)\varphi(x) \in Z_N$$

where x is an arbitrary element in M This implies that

$$\epsilon(x, x)[\varphi(x)^2, u] = \epsilon(x^2, x)[\varphi(x), u]$$

for all $x \in M$, $u \in N$, and therefore,

$$\epsilon(x, x)([\varphi(x)^2, u]y[\varphi(x), u] - [\varphi(x), u]y[\varphi(x)^2, u]) = 0$$

for all $x \in M$, $y, u \in N$ Now pick $y, u \in N$ and define τ $M \times M \times M \longrightarrow N$ by

$$\tau(x_1, x_2, x_3) = [\varphi(x_1)\varphi(x_2), u]y[\varphi(x_3), u] - [\varphi(x_1), u]y[\varphi(x_2)\varphi(x_3), u]$$

and note that $\epsilon(x, x)\tau(x, x, x) = 0$ for all $x \in M$ Since ϵ maps in the center of N, we also have $\epsilon(x, x)M\tau(x, x, x) = \{0\}$ Thus, the mappings ϵ and τ satisfy the requirements of Lemma 6, and so

$$\epsilon(v, v)N\tau(x, x, x) = \{0\} \text{ for all } x, v \in M$$

Using the definition of φ, we see that

$$\tau(x, x, x) = \lambda^3\big([\theta(x)^2, u]y[\theta(x), u] - [\theta(x), u]y[\theta(x)^2, u]\big)$$

Thus, since λ is invertible and θ is onto, we have

$$\epsilon(v, v)N([s^2, u]y[s, u] - [s, u]y[s^2, u]) = \{0\}$$

for all $v \in M$ and $s, y, u \in N$ Applying Lemma 2 we see that $\epsilon(v, v)$ must be zero This proves that φ is a Jordan homomorphism

We set $c = \lambda^{-1}$ and $f(x) = -\frac{1}{2}\lambda^{-1}\mu(x)$, so we have $\theta(x) = c\varphi(x) + f(x)$

STEP 5 *φ is one-to-one and onto*

PROOF OF STEP 5 Suppose $\varphi(a) = 0$ for some $a \in M$ Then $\theta(a) = f(a) \in Z_N$ By assumption, this yields $a \in Z_M$ Therefore, $\varphi(ax) = \frac{1}{2}\varphi(ax + xa) = \frac{1}{2}\big(\varphi(a)\varphi(x) + \varphi(x)\varphi(a)\big) = 0$ for every $x \in M$ As above, this implies that $ax \in Z_M$ But then $a = 0$ by Lemma 5

Let us show that the restriction of φ to Z_M is equal to α. Take $z \in M$. Then $\theta(z) \in Z_N$, and therefore, $\varphi(z) \in Z_N$. Hence

$$\varphi(zx) = \frac{1}{2}\varphi(zx + xz) = \frac{1}{2}\big((\varphi(z)\varphi(x) + \varphi(x)\varphi(z)\big) = \varphi(z)\varphi(x)$$

holds for any x in M. Consequently

$$\theta(zx) = c\varphi(zx) + f(zx) = c\varphi(z)\varphi(x) + f(zx).$$

Thus $\theta(zx) = c\varphi(z)\varphi(x) \in Z_N$ for all $x \in M$. On the other hand, by Step 1 we have $\theta(zx) - \alpha(z)\theta(x) \in Z_N$, that is, $\theta(zx) - c\alpha(z)\varphi(x) \in Z_N$. Comparing, we get $\big(\varphi(z) - \alpha(z)\big)c\varphi(x) \in Z_N$, and therefore, as $c\varphi(x) = \theta(x) - f(x)$, we have $\big(\varphi(z) - \alpha(z)\big)\theta(x) \in Z_N$. By Lemma 5 it follows that $\varphi(z) = \alpha(z)$.

Since α is onto, there is $c_1 \in Z_M$ such that $c = \alpha(c_1) = \varphi(c_1)$. Similary, for every $x \in M$ there is $f_1(x) \in Z_M$ such that $\varphi\big(f_1(x)\big) = f(x)$. Thus $\theta(x) = \varphi(c_1)\varphi(x) + \varphi\big(f_1(x)\big)$. As shown above, we have $\varphi(c_1)\varphi(x) = \varphi(c_1 x)$, which gives $\theta(x) = \varphi\big(c_1 x + f_1(x)\big)$. Thus, since θ is onto, φ is onto as well.

It remains to prove

STEP 6. *There exist central projections $p \in M$ and $q \in N$ such that the restriction of φ to pM is an isomorphism of pM onto qN, and the restriction of φ to $(1-p)M$ is an anti-isomorphism of $(1-p)M$ onto $(1-q)N$.*

PROOF OF STEP 6. This assertion follows immediately from [3, Theorem 1]. Namely, this theorem tells us that if φ is a Jordan isomorphism of a ring M onto a 2-torsion-free semiprime ring N in which the annihilator of any ideal is a direct summand (*i.e.*, for any ideal I in N, we have $N = \mathrm{Ann}(I) \oplus J$ for some ideal J of N—von Neumann algebras certainly satisfy this condition), then there exist ideals U and V of M and ideals U' and V' of N such that $U \oplus V = M$, $U' \oplus V' = N$, the restriction of φ to U is an isomorphism of U onto U', and the restriction of φ to V is an anti-isomorphism of V to V'. By standard arguments one shows that in case M and N are von Neumann algebras, these ideals must be of the form $U = pM$, $V = (1-p)M$ for some central projection p in M, and $U' = qN$, $V' = (1-q)N$ for some central projection q in N.

The proof of the theorem is thereby completed.

Our last goal is to determine the structure of Lie isomorphisms of von Neumann algebras. For this purpose we need a refinement of Lemma 5.

LEMMA 7. *Let M be a von Neumann algebra with no central summands of type I_1. If $c \in Z_M$ is such that $c[x,y] \in Z_M$ for all $x, y \in M$, then $c = 0$.*

PROOF. We have $c\big[[x,y],u\big] = 0$ for all $x, y, u \in M$. Replacing y by yx it follows that

$$0 = c\big[[x,y]x, u\big] = c[x,y][x,u] + c\big[[x,y],u\big]x = c[x,y][x,u].$$

Thus $c[x,y][x,u] = 0$ for all $x, y, u \in M$. Substituting uv for u and using the relation $[x,uv] = [x,u]v + u[x,v]$, we then get $c[x,y]M[x,v] = \{0\}$ for all $x, y, v \in M$. Note that Lemma 4 implies that $c[x,y]M[u,v] = \{0\}$ for all $x, y, u, v \in M$. Using the fact that the ideal generated by all commutators in M is equal to M [5, Lemma 2.6], it follows easily that $c = 0$.

THEOREM 4 *Let M and N be von Neumann algebras with no central summands of type I_1 or I_2 If θ $M \longrightarrow N$ is a Lie isomorphism then it is of the form $\theta = \psi + f$ where f is an additive mapping M into Z_N sending commutators to zero, and, for some central projections $p \in M$ and $q \in N$, the restriction of ψ to pM is an isomorphism of pM onto qN and the restriction of ψ to $(1-p)M$ is a negative of an anti-isomorphism of $(1-p)M$ onto $(1-q)N$*

PROOF Clearly, θ satisfies the requirements of Theorem 3, and it is, therefore, of the form described in the statement of Theorem 3

Take $x, y \in pM$ Since the restriction of φ to pM is a homomorphism, we have

$$\theta([x,y]) = c\varphi([x,y]) + f([x,y]) = c[\varphi(x), \varphi(y)] + f([x,y])$$

On the other hand,

$$\theta([x,y]) = [\theta(x), \theta(y)] = c^2[\varphi(x), \varphi(y)]$$

Comparing, we get $(c^2 - c)[\varphi(x), \varphi(y)] = f([x,y]) \in Z_N$ Since φ maps pM onto qN it follows that $(c^2 - c)q = 0$, and therefore, since c is invertible, $cq = q$ Note that this implies that $f([x,y]) = 0$ for all $x, y \in pM$

Similarly, by computing $\theta([x,y])$, $x, y \in (1-p)M$, in two ways, one shows that $(c^2 + c)[\varphi(x), \varphi(y)] = f([x,y]) \in Z_N$ This yields $c(1-q) = -(1-q)$ and $f([x,y]) = 0$, $x, y \in (1-p)M$

Now it can be easily shown that the mapping $\psi(x) = c\varphi(x)$ satisfies the desired conclusions

Addendum. The question arises as to whether the assumption of *-linearity for θ will imply the *-linearity of φ in Theorem 3 since in general we can only conclude that φ is a ring isomorphism if θ is only assumed additive

COROLLARY *If θ is *-linear then so is φ*

PROOF We first prove φ is linear if θ is linear Let $\lambda \in \mathbb{C}$, $x \in M$ Since $c \in Z_N$ and is invertible the linearity of θ implies

$$(*) \qquad \varphi(\lambda x) - \lambda\varphi(x) \in Z_N$$

If $x \in pM$, $y \in qN$ and $x_0 \in pM$ is such that $\varphi(x_0) = y$ then $\big(\varphi(\lambda x) - \lambda\varphi(x)\big)y = \big(\varphi(\lambda x) - \lambda\varphi(x)\big)\varphi(x_0) = \varphi(\lambda x x_0) - \lambda\varphi(x x_0) \in Z_{qN}$ by $(*)$ and the fact that φ is a ring isomorphism from pM onto qN By Lemma 5 applied to qN we have $\varphi(\lambda x) = \lambda\varphi(x)$ for $x \in pM$ Similarly, if $x \in (1-p)M$, $y \in (1-q)N$, and $x_0 \in (1-p)M$ is such that $\varphi(x_0) = y$ then $\big(\varphi(\lambda x) - \lambda\varphi(x)\big)y = \big(\varphi(\lambda x) - \lambda\varphi(x)\big)\varphi(x_0) = \varphi(\lambda x_0 x) - \lambda\varphi(x_0 x) \in Z_{(1-q)N}$ by $(*)$ and the fact that φ is a ring anti-isomorphism from $(1-p)M$ onto $(1-q)N$ As before $\varphi(\lambda x) = \lambda\varphi(x)$ for $x \in (1-p)M$

To prove adjoint preservation we first notice that $\theta(x^*) = \theta(x)^*$ implies $c\varphi(x^*) - c^*\varphi(x)^* \in Z_N$ for all $x \in M$ Assume for a moment that $p = 1$ Since φ is a linear ring isomorphism of M and N, there exist a *-isomorphism ρ $M \longrightarrow N$ and a positive invertible

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element $s \in M$ such that $\varphi(x) = \rho(sxs^{-1})$ by [17, Theorem I]. Hence $c^*\varphi(x)^* - c\varphi(x^*) = c^*\big(\rho(sxs^{-1})\big)^* - c\rho(sx^*s^{-1}) = c^*\rho(s^{-1}x^*s) - c\rho(sx^*s^{-1}) \in Z_N$ for all $x \in M$. Let $c_0 \in Z_M$ be such that $\rho(c_0) = c$. Then c_0 is invertible and $c_0^*s^{-1}x^*s - c_0sx^*s^{-1} \in Z_M$ for all $x \in M$ since ρ is a *-isomorphism. Replacing x by sx^*s we see that $c_0^*xs^2 - c_0s^2x \in Z_M$ for all $x \in M$. Let $w = c_0s^2$ so $w^* = c_0^*s^2$. Since $c_0 \in Z_M$ we have $c_0^*xs^2 - c_0s^2x = xc_0^*s^2 - c_0s^2x = xw^* - wx \in Z_M$ for all $x \in M$. Taking $x = 1$ we see that $w^* - w \in Z_M$. Now $[x,w] = xw - wx = xw - xw^* + z$ for some $z \in Z_M$ depending on x. Hence $\big[x,[x,w]\big] = 0$ since $w - w^* \in Z_M$. By the Kleinecke-Sirokov Theorem [18], $[x,w]$ is quasi-nilpotent for all x. Since $w = c_0s^2$ and c_0 is central and invertible we have $[x,s^2]$ is quasi-nilpotent for all x. Taking $x = x^*$ we see that $[ix,s^2]$ is self-adjoint and quasi-nilpotent so that $[x,s^2] = 0$ for all $x = x^*$ in M. This implies $s^2 \in Z_M$. Since $s \geq 0$ we have $s \in Z_M$. Hence $\varphi(x) = \rho(sxs^{-1}) = \rho(x)$ and φ is a *-isomorphism.

If $q = 1$ and $\varphi: M \longrightarrow N$ was a linear anti-isomorphism, we define M^{op} to be the von Neumann algebra obtained from M by defining a new multiplication $a \bullet b := ba$ and keeping the same adjoint and linear structure as that of M. Then $\varphi: M^{\mathrm{op}} \longrightarrow N$ is a linear isomorphism and $c\varphi(x^*) - c^*\varphi(x)^* \in Z_N\ \forall x \in M$. By the first part of the argument φ preserves the adjoint on M^{op} and hence on M.

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