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SPATIO-TEMPORAL VARIOGRAMS AND COVARIANCE MODELS

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Abstract

Variograms and covariance functions are the fundamental tools for modeling dependent data observed over time, space, or space–time. This paper aims at constructing nonseparable spatio-temporal variograms and covariance models. Special attention is paid to an intrinsically stationary spatio-temporal random field whose covariance function is of Schoenberg–Lévy type. The correlation structure is studied for its increment process and for its partial derivative with respect to the time lag, as well as for the superposition over time of a stationary spatio-temporal random field. As another approach, we investigate the permissibility of the linear combination of certain separable spatio-temporal covariance functions to be a valid covariance, and obtain a subclass of stationary spatio-temporal models isotropic in space.

Keywords: Covariance; intrinsically stationary; isotropic; positive definite; power-law decay; Schoenberg–Lévy kernel; stationary; variogram

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1. Introduction

The world is dynamic on many scales in space and time. There is now considerable interest in spatio-temporal data mining (e.g. Ladner *et al.* (2002)), and spatio-temporal modeling in the environmental, information, and physical sciences. Whenever possible and available, a rational approach to modeling spatio-temporal data should start from a theory or mechanism that explains the underlying physical facts. In reality, however, no obvious mechanism may exist, and frequently such a theory must be developed from observational or experimental study. For this purpose, statistical techniques are often very important tools. Static and dynamic models to describe the spatio-temporal mechanism are prominent among these.

Dynamic or stochastic spatio-temporal models in the literature are often expressed as stochastic difference equations (e.g. Bartlett (1975) and Stoffer (1986)), stochastic partial differential or integral equations (e.g. Whittle (1954), (1962), Heine (1955), Jones and Zhang (1997), Brown *et al.* (2000), Storvik *et al.* (2002), and Anh *et al.* (2003)), the Kalman filter (e.g. Huang and Cressie (1996), Meiring *et al.* (1998), Wikle and Cressie (1999), and Stroud *et al.* (2001)), or wavelets (e.g. Ruiz-Medina and Angulo (2002)). The major uses of stochastic models include description, interpolation, and prediction.

The spatio-temporal covariance function is one of the prerequisites for optimal prediction or kriging. It may be calculated from a fitted stochastic model or deduced from an empirical covariance of the data. In case the closed form of a stochastic model is not available, a common strategy in practice is to select a covariance function from a parametric or semiparametric

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family whose members are known to be positive-definite functions; see, for example, Christakos (1984), Weber and Talkner (1993), and Gaspari and Cohn (1999). The selection is theoretically legitimate in the sense that, according to a celebrated existence theorem of Kolmogorov, a positive-definite function on a space–time domain can always be thought of as the covariance function of a zero-mean Gaussian random field.

The variogram and the covariance function are two of the most commonly used measures of spatio-temporal dependence. Owing to a lack of nonseparable models, separable covariance models had been used for modeling space–time interaction, with undesirable properties. It is thus important to have nonseparable models when describing wide-range space–time interaction. Cressie and Huang (1999) provided a detailed treatment for deriving nonseparable, stationary spatio-temporal covariance functions, using Bochner's theorem and the (inverse) Fourier transform approach. A closely related approach, the cosine transform, as well as other simple approaches, were proposed by Ma (2003a), and certain covariance families developed. For instance, the model (3.3) in Ma (2003a) contains the main model (11) of Gneiting (2002) as a special case (see Ma (2005b)).

The kernel method has a long pedigree going back to the first half of the twentieth century; the fundamental results were obtained by Schoenberg (1938a), (1938b). For references, see Gangolli (1967a), (1967b), Berg *et al.* (1984), and Ma (2003b). In Section 2, we study kernels associated with an intrinsically stationary spatio-temporal random field, and, in Section 3, we explore the correlation structure of its partial derivative with respect to the time lag. The partial integral of a stationary spatio-temporal random field with respect to the time lag is considered in Section 4.

A nonnegative linear combination of two covariance functions is also a covariance function; so is the product of two covariance functions. These basic properties allow us to derive many nonseparable spatio-temporal covariance functions via mixture methods (e.g. de Iaco *et al.* (2002) and Ma (2002), (2003a), (2003c)), where the mixing weights are constrained to be nonnegative. However, the validity of the mixture could be questionable, were the nonnegativity constraints to be relaxed. For instance, a linear combination of two covariance functions may not be a covariance function if one of the mixing weights is negative. In Section 5, the permissible conditions on mixing weights are studied for certain separable spatio-temporal covariance functions whose spatial component is isotropic. This results in a subclass of stationary spatio-temporal covariance models isotropic in space. See Ma (2005a) for other subclasses, as well as our motivations for looking at the permissibility of the difference or the linear combination of two covariance functions. Our theorems are proved in Section 6.

2. Kernels related to an intrinsically stationary random field

Suppose that $\{Z(\boldsymbol{s};t),\ (\boldsymbol{s};t)\in\mathcal{S}\times\mathcal{T}\}$ is a real-valued stochastic process or random field on the space–time domain $\mathcal{S}\times\mathcal{T}$, where $\mathcal{S}$ equals $\mathbb{R}^d$ or $\mathbb{Z}^d$ and $\mathcal{T}$ equals $\mathbb{R}$ or $\mathbb{Z}$. When the second-order moments of the random field exist, its covariance function is defined by

$$\begin{aligned}C(&\boldsymbol{s}_1,\boldsymbol{s}_2;t_1,t_2)\\&=\mathrm{E}[\{Z(\boldsymbol{s}_1;t_1)-\mathrm{E}\,Z(\boldsymbol{s}_1;t_1)\}\{Z(\boldsymbol{s}_2;t_2)-\mathrm{E}\,Z(\boldsymbol{s}_2;t_2)\}],\qquad(\boldsymbol{s}_1;t_1),(\boldsymbol{s}_2;t_2)\in\mathcal{S}\times\mathcal{T}.\end{aligned}$$

Weak (or second-order) stationarity and intrinsic stationarity are two particularly important assumptions made in spatio-temporal modeling. The random field $\{Z(\boldsymbol{s};t),\ (\boldsymbol{s};t)\in\mathcal{S}\times\mathcal{T}\}$ is said to be stationary in space and time if its mean function $\mathrm{E}\,Z(\boldsymbol{s};t)$ is a constant for all $(\boldsymbol{s};t)$ and its covariance function $C(\boldsymbol{s}_0,\boldsymbol{s}_0+\boldsymbol{s};t_0,t_0+t)$ depends only on the space lag $\boldsymbol{s}$ and time lag t for all $(\boldsymbol{s}_0;t_0)\in\mathcal{S}\times\mathcal{T}$, in which case we simply write $C(\boldsymbol{s};t)$ instead of $C(\boldsymbol{s}_0,\boldsymbol{s}_0+\boldsymbol{s};t_0,t_0+t)$.

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The concept of intrinsic stationarity stems from a traditional approach to achieving stationarity that involves taking the difference of a process. The random field $\{Z(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ is said to be intrinsically stationary in space and time (or to have stationary increments in space and time) if, for every fixed $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$, the increment

$$Z(\mathbf{s} + \mathbf{s}_0; t + t_0) - Z(\mathbf{s}; t), \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T},$$

is a spatio-temporal random field stationary in space and time. For an intrinsically stationary random field, its covariance is not necessarily well defined, but its variogram is. The latter is defined as

$$\gamma(\mathbf{s}; t) = \tfrac{1}{2} \operatorname{var}(Z(\mathbf{s}_0 + \mathbf{s}; t_0 + t) - Z(\mathbf{s}_0; t_0)), \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T},$$

and does not depend on $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$. A characteristic property of an intrinsically stationary variogram $\gamma(\mathbf{s}; t)$ is that it is nonnegative and (conditionally) negative definite with $\gamma(\mathbf{0}; 0) = 0$.

We are particularly interested in a spatio-temporal random field $\{Z(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ with the covariance function

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \gamma(\mathbf{s}_1; t_1) + \gamma(\mathbf{s}_2; t_2) - \gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2), \qquad (\mathbf{s}_k; t_k) \in \mathcal{S} \times \mathcal{T},\ k = 1, 2. \tag{1}$$

Following Gangolli (1967a), (1967b), we call (1) the Schoenberg–Lévy kernel. The random field possesses an orthogonal decomposition (Ma (2003b))

$$\begin{aligned} Z(\mathbf{s}; t) &= Z_+(\mathbf{s}; t) + Z_-(\mathbf{s}; t), && (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}, \\ Z_+(\mathbf{s}; t) &= \tfrac{1}{2}\{Z(\mathbf{s}; t) + Z(-\mathbf{s}; -t)\}, && (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}, \\ Z_-(\mathbf{s}; t) &= \tfrac{1}{2}\{Z(\mathbf{s}; t) - Z(-\mathbf{s}; -t)\}, && (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}, \end{aligned}$$

where the random fields $\{Z_+(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ and $\{Z_-(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ are uncorrelated, $\{Z_+(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ has nonnegative covariance

$$\gamma(\mathbf{s}_1; t_1) + \gamma(\mathbf{s}_2; t_2) - \tfrac{1}{2}\gamma(\mathbf{s}_1 + \mathbf{s}_2; t_1 + t_2) - \tfrac{1}{2}\gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2),$$

and $\{Z_-(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ has covariance

$$\tfrac{1}{2}\{\gamma(\mathbf{s}_1 + \mathbf{s}_2; t_1 + t_2) - \gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2)\}, \qquad (\mathbf{s}_i; t_i) \in \mathcal{S} \times \mathcal{T}\ (i = 1, 2).$$

A necessary and sufficient condition for the Schoenberg–Lévy kernel (1) to be a covariance function is that $\gamma(\mathbf{s}; t)$ be an intrinsically stationary variogram on $\mathcal{S} \times \mathcal{T}$. More precisely, $\gamma(\mathbf{s}; t)$ is the variogram corresponding to the random field $\{Z(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$, since

$$\begin{aligned} &\tfrac{1}{2} \operatorname{var}(Z(\mathbf{s}_1; t_1) - Z(\mathbf{s}_2; t_2)) \\ &\quad = \tfrac{1}{2}\{\operatorname{var}(Z(\mathbf{s}_1; t_1)) + \operatorname{var}(Z(\mathbf{s}_2; t_2)) - 2\operatorname{cov}(Z(\mathbf{s}_1; t_1), Z(\mathbf{s}_2; t_2))\} \\ &\quad = \tfrac{1}{2}[2\gamma(\mathbf{s}_1; t_1) + 2\gamma(\mathbf{s}_2; t_2) - 2\{\gamma(\mathbf{s}_1; t_1) + \gamma(\mathbf{s}_2; t_2) - \gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2)\}] \\ &\quad = \gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2). \end{aligned}$$

In other words, a spatio-temporal random field with covariance function of Schoenberg–Lévy type is indeed an intrinsically stationary random field. Thus, for every $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$, the increment process $\{Z(\mathbf{s}+\mathbf{s}_0; t+t_0) - Z(\mathbf{s}; t),\ (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}\}$ is stationary, with covariance

$$\begin{aligned}
&\operatorname{cov}(Z(\mathbf{s}+\mathbf{s}_0; t+t_0) - Z(\mathbf{s}; t), Z(\mathbf{s}_0; t_0) - Z(\mathbf{0}; 0)) \\
&\quad = \operatorname{cov}(Z(\mathbf{s}+\mathbf{s}_0; t+t_0), Z(\mathbf{s}_0; t_0)) + \operatorname{cov}(Z(\mathbf{s}; t), Z(\mathbf{0}; 0)) \\
&\qquad - \operatorname{cov}(Z(\mathbf{s}+\mathbf{s}_0; t+t_0), Z(\mathbf{0}; 0)) - \operatorname{cov}(Z(\mathbf{s}; t), Z(\mathbf{s}_0; t_0)) \\
&\quad = \gamma(\mathbf{s}+\mathbf{s}_0; t_0+t) + \gamma(\mathbf{s}-\mathbf{s}_0; t-t_0) - 2\gamma(\mathbf{s}; t), \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}.
\end{aligned}$$

We summarize this observation in the following lemma, of which a purely mathematical proof can be found in Berg *et al.* (1984, p. 103).

Lemma 1. *If $\gamma(\mathbf{s}; t)$ is an intrinsically stationary variogram on $\mathcal{S} \times \mathcal{T}$ then, for every $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$,*

$$C(\mathbf{s}; t) = \gamma(\mathbf{s}+\mathbf{s}_0; t_0+t) + \gamma(\mathbf{s}-\mathbf{s}_0; t-t_0) - 2\gamma(\mathbf{s}; t), \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}, \tag{2}$$

is a stationary covariance function on $\mathcal{S} \times \mathcal{T}$.

In particular, if $C(\mathbf{s}; t)$ is a stationary covariance function, then $C(\mathbf{0}; 0) - C(\mathbf{s}; t)$ is a stationary variogram and $2C(\mathbf{s}; t) - \{C(\mathbf{s}+\mathbf{s}_0; t+t_0) + C(\mathbf{s}-\mathbf{s}_0; t-t_0)\}$ is a stationary covariance function on $\mathcal{S} \times \mathcal{T}$.

A natural question is whether the converse of Lemma 1 holds. Before presenting our finding and conjecture, let us look at an example in which $\gamma(\mathbf{s}; t)$ is not a spatio-temporal variogram even if $\gamma(\mathbf{s}+\mathbf{s}_0; t) + \gamma(\mathbf{s}-\mathbf{s}_0; t) - 2\gamma(\mathbf{s}; t)$ is a stationary covariance function on $\mathcal{S} \times \mathcal{T}$ for every $\mathbf{s}_0 \in \mathcal{S}$.

Example 1. Let $\gamma_{\mathrm{S}}(\mathbf{s})$ be a purely spatial, intrinsically stationary variogram on $\mathcal{S}$. Consider the function

$$f(\mathbf{s}; t) = \gamma_{\mathrm{S}}(\mathbf{s}) + t^4, \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T}.$$

This is not a valid spatio-temporal variogram since, otherwise, its temporal margin $f(\mathbf{0}; t) = t^4$ would be a variogram on $\mathcal{T}$. However, by Lemma 1,

$$f(\mathbf{s}+\mathbf{s}_0; t) + f(\mathbf{s}-\mathbf{s}_0; t) - 2f(\mathbf{s}; t) = \gamma_{\mathrm{S}}(\mathbf{s}+\mathbf{s}_0) + \gamma_{\mathrm{S}}(\mathbf{s}-\mathbf{s}_0) - 2\gamma_{\mathrm{S}}(\mathbf{s}), \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathcal{T},$$

is a spatio-temporal stationary covariance function for every fixed $\mathbf{s}_0 \in \mathcal{S}$.

In view of Example 1, we conjecture that an even function $\gamma(\mathbf{s}; t)$ on $\mathcal{S} \times \mathcal{T}$, with $\gamma(\mathbf{0}; 0) = 0$, is an intrinsically stationary variogram if and only if (2) is a stationary covariance function for every $(\mathbf{s}_0; t_0)$ in a certain subset of $\mathcal{S} \times \mathcal{T}$. An interesting special result along this line is as follows.

Theorem 1. *Suppose that $\gamma(\mathbf{s}; t)$ is an even function on $\mathcal{S} \times \mathcal{T}$, with $\gamma(\mathbf{0}; 0) = 0$, and reaches a limiting value as $\mathbf{s}$ and t tend to (positive or negative) infinity simultaneously (componentwise, in the case of $\mathbf{s}$). If, for a fixed $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$ with nonzero coordinates, $\gamma(\mathbf{s}+\mathbf{s}_0; t+t_0) + \gamma(\mathbf{s}-\mathbf{s}_0; t-t_0) - 2\gamma(\mathbf{s}; t)$ is a stationary covariance function, then $\gamma(\mathbf{s}; t)$ is a stationary variogram on $\mathcal{S} \times \mathcal{T}$.*

Theorem 1 might not hold if one of the coordinates of $(\mathbf{s}_0; t_0)$ is zero, as Example 1 shows.

Corollary 1. *For an even function $C(\mathbf{s}; t)$ on $\mathcal{S} \times \mathcal{T}$ with a limiting value as $\mathbf{s}$ and t tend to (positive or negative) infinity simultaneously, if $2C(\mathbf{s}; t) - C(\mathbf{s}+\mathbf{s}_0; t+t_0) - C(\mathbf{s}-\mathbf{s}_0; t-t_0)$ is a stationary covariance function on $\mathcal{S} \times \mathcal{T}$ for a fixed $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$ with nonzero coordinates, then so is $C(\mathbf{s}; t)$.*

3. Stationary covariance functions based on partial differentiation

For a spatio-temporal random field $\{Z(\mathbf{s}; t),\ \mathcal{S} \times \mathbb{R}\}$, its partial derivative with respect to t is defined by

$$\frac{\partial}{\partial t} Z(\mathbf{s}; t) = \lim_{h \to 0} \frac{Z(\mathbf{s}; t+h) - Z(\mathbf{s}; t)}{h}, \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R},$$

where the limit is to be taken in the sense of convergence in the mean square.

Consider an intrinsically stationary spatio-temporal random field $\{Z(\mathbf{s}; t),\ \mathcal{S} \times \mathbb{R}\}$ with the variogram $\gamma(\mathbf{s}; t)$. Assume that all its increments have zero mean. If $\gamma(\mathbf{s}; t)$ has a second partial derivative with respect to t, then the partial derivative $(\partial/\partial t) Z(\mathbf{s}; t)$ exists and its covariance function is formally derived to be

$$\begin{aligned}
&\operatorname{cov}\bigg(\frac{\partial}{\partial u} Z(\mathbf{s}_0; u)\bigg|_{u=t_0}, \frac{\partial}{\partial u} Z(\mathbf{s}_0 + \mathbf{s}; u)\bigg|_{u=t_0+t}\bigg) \\
&\quad = \operatorname{cov}\bigg(\lim_{h \to 0} \frac{Z(\mathbf{s}_0; t_0 + h) - Z(\mathbf{s}_0; t_0)}{h}, \\
&\qquad\qquad \lim_{h' \to 0} \frac{Z(\mathbf{s}_0 + \mathbf{s}; t_0 + t + h') - Z(\mathbf{s}_0 + \mathbf{s}; t_0 + t)}{h'}\bigg) \\
&\quad = \lim_{h \to 0,\ h' \to 0} (hh')^{-1} \operatorname{cov}(Z(\mathbf{s}_0; t_0 + h) - Z(\mathbf{s}_0; t_0), \\
&\qquad\qquad\qquad Z(\mathbf{s}_0 + \mathbf{s}; t_0 + t + h') - Z(\mathbf{s}_0 + \mathbf{s}; t_0 + t)) \\
&\quad = \lim_{h \to 0,\ h' \to 0} (hh')^{-1} \{\gamma(\mathbf{s}; t - h) + \gamma(\mathbf{s}; t + h') - \gamma(\mathbf{s}; t + h' - h) - \gamma(\mathbf{s}; t)\} \\
&\quad = \frac{\partial^2}{\partial t^2} \gamma(\mathbf{s}; t),
\end{aligned}$$

where the interchange of the limits and the integration requires further justification. Nevertheless, a rigorous proof that $(\partial^2/\partial t^2)\gamma(\mathbf{s}; t)$ is a valid covariance function can be easily established using the kernel (2).

Theorem 2. *For an intrinsically stationary variogram $\gamma(\mathbf{s}; t)$ on $\mathcal{S} \times \mathbb{R}$, if it is twice differentiable with respect to t for every fixed $\mathbf{s} \in \mathcal{S}$, then $(\partial^2/\partial t^2)\gamma(\mathbf{s}; t)$ is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$. If, in addition, $\gamma(\mathbf{s}; t)$ is symmetric with respect to either $\mathbf{s}$ or t, in the sense that $\gamma(\pm\mathbf{s}; t) = \gamma(\mathbf{s}; \pm t)$ for all $(\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R}$, then $(\partial/\partial t)\gamma(\mathbf{s}; t)$ vanishes at $t = 0$.*

A purely temporal version of Theorem 2 can be found in Gneiting *et al.* (2001, Theorem 7) and Ma (2004, Lemma 2). Unlike in the purely temporal case, the converse of Theorem 2 is not necessarily true. To see this, consider the function

$$f(\mathbf{s}; t) = \rho_S(\mathbf{s})(1 - \cos t), \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R},$$

where $\rho_S(\mathbf{s})$ is a purely spatial, stationary correlation function on $\mathcal{S}$ that is not identically equal to 1. Clearly $f(\mathbf{0}; 0) = 0$, $f(\mathbf{s}; t)$ is even, and $(\partial^2/\partial t^2) f(\mathbf{s}; t) = \rho_S(\mathbf{s}) \cos t$ is a separable correlation function on $\mathcal{S} \times \mathbb{R}$. By Lemma 2 of Ma (2003c), however, $\rho_S(\mathbf{s})(1 - \cos t)$ is not a spatio-temporal variogram.

In Theorem 2, $(\partial/\partial t)\gamma(\mathbf{s}; t)$ may not vanish at $t = 0$ without an additional symmetry assumption. For example,

$$\gamma(\mathbf{s}; t) = (\boldsymbol{\theta}^\top \mathbf{s} + t)^2, \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R},$$

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where $\boldsymbol{\theta}$ is a fixed nonzero vector, is an intrinsically stationary variogram, but $(\partial/\partial t)\gamma(\boldsymbol{s}; t) = 2(\boldsymbol{\theta}^\top \boldsymbol{s} + t)$ does not vanish at $t = 0$.

Corollary 2. *If a stationary covariance function $C(\boldsymbol{s}; t)$ on $\mathcal{S} \times \mathbb{R}$ is twice differentiable with respect to t for every $\boldsymbol{s} \in \mathcal{S}$, then $-(\partial^2/\partial t^2)C(\boldsymbol{s}; t)$ is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$. If, in addition, $C(\boldsymbol{s}; t)$ is symmetric with respect to either $\boldsymbol{s}$ or t, in the sense that $C(\pm\boldsymbol{s}; t) = C(\boldsymbol{s}; \pm t)$ for all $(\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}$, then $(\partial/\partial t)C(\boldsymbol{s}; t)$ vanishes at $t = 0$.*

A purely temporal version of Corollary 2 is Slutsky's classical result (see Theorem 1.4 of Doob (1944)) derived from frequency domain analysis. The conditions in Corollary 2 can be relaxed. For instance, for a covariance function $C(\boldsymbol{s}_1, \boldsymbol{s}_2; t)$ on $\mathcal{S} \times \mathbb{R}$ that is stationary in time, if it is twice differentiable with respect to t for all fixed $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}$, then $-(\partial^2/\partial t^2)C(\boldsymbol{s}_1, \boldsymbol{s}_2; t)$ is also a covariance function on $\mathcal{S} \times \mathbb{R}$ that is stationary in time. See Ma (2005b) for some semiparametric examples.

A nonnegative, continuous function $f(x)$ on $[0, \infty)$ is called a completely monotone function if it has derivatives of all orders and if

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x) \geq 0 \quad \text{for all } x > 0 \text{ and } k \in \mathbb{N},$$

and is called a Bernstein function (Berg and Forst (1975)), or a completely monotone mapping (Bochner (1955)), if it has a completely monotone derivative, i.e. if

$$(-1)^{k-1} \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x) \geq 0 \quad \text{for all } x > 0 \text{ and } k \in \mathbb{N}.$$

We refer the reader to Miller and Samko (2001) for a recent expository survey of properties of completely monotone functions, as well as various examples. In the construction of spatio-temporal variograms, the importance of the Bernstein function lies in the fact (see Bochner (1955)) that if $\gamma(\boldsymbol{s}; t)$ is an intrinsically stationary variogram on $\mathcal{S} \times \mathcal{T}$, then so is $f(\gamma(\boldsymbol{s}; t))$, provided that $f(x)$ is a Bernstein function on $[0, \infty)$ with $f(0) = 0$. If $\ell(x)$ is a completely monotone function on $[0, \infty)$ and $\gamma(\boldsymbol{s}; t)$ is an intrinsically stationary variogram on $\mathcal{S} \times \mathcal{T}$, then $\ell(\gamma(\boldsymbol{s}; t))$ is known to be a stationary covariance function on $\mathcal{S} \times \mathcal{T}$.

Corollary 3. *Let $\ell(x)$ be a completely monotone function on $[0, \infty)$. If an intrinsically stationary variogram $\gamma(\boldsymbol{s}; t)$ on $\mathcal{S} \times \mathbb{R}$ is twice differentiable with respect to t, for every $\boldsymbol{s} \in \mathcal{S}$, then*

$$C(\boldsymbol{s}; t) = \ell(\gamma(\boldsymbol{s}; t)) \frac{\partial^2}{\partial t^2} \gamma(\boldsymbol{s}; t) + \ell'(\gamma(\boldsymbol{s}; t)) \frac{\partial}{\partial t} \gamma(\boldsymbol{s}; t), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}, \tag{3}$$

where a prime denotes differentiation, is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$.

The function (3) is obtained by taking the second partial derivative of $\int_0^{\gamma(\boldsymbol{s};t)} \ell(u)\,\mathrm{d}u$ with respect to t. This integral is an intrinsically stationary variogram on $\mathcal{S} \times \mathbb{R}$ since $\int_0^x \ell(u)\,\mathrm{d}u$ is a Bernstein function on $[0, \infty)$.

Corollary 4. *Let $\ell(x)$ be a completely monotone function on $[0, \infty)$, and let $\gamma_{\mathrm{S}}(\boldsymbol{s})$ and $\gamma_{\mathrm{T}}(t)$ be intrinsically stationary variograms on $\mathcal{S}$ and $\mathbb{R}$, respectively. If $\gamma_{\mathrm{T}}(t)$ is twice differentiable then*

$$C(\boldsymbol{s}; t) = \ell(\gamma_{\mathrm{S}}(\boldsymbol{s}) + \gamma_{\mathrm{T}}(t))\gamma_{\mathrm{T}}''(t) + \ell'(\gamma_{\mathrm{S}}(\boldsymbol{s}) + \gamma_{\mathrm{T}}(t))\gamma_{\mathrm{T}}'(t), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R},$$

is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$.

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Corollary 4 follows as a special case of Corollary 3 with $\gamma(\boldsymbol{s}; t) = \gamma_{\mathrm{S}}(\boldsymbol{s}) + \gamma_{\mathrm{T}}(t)$. Sources of purely spatial variograms $\gamma_{\mathrm{S}}(\boldsymbol{s})$ are Cressie (1993) and Chilès and Delfiner (1999), among others. There are many twice-differentiable variograms on the real line. For instance, a nonparametric class (see Ma (2004)) is

$$\gamma_{\mathrm{T}}(t) = \int_0^{|t|} (|t| - u) C_{\mathrm{T}}(u)\, \mathrm{d}u, \qquad t \in \mathbb{R},$$

where $C_{\mathrm{T}}(t)$ is an integrable stationary covariance function on the real line.

We now exemplify the use of Theorem 2 in deriving new spatio-temporal covariance functions.

Example 2. Assume that $\rho_{\mathrm{S}}(\boldsymbol{s})$ is a stationary correlation function on $\mathcal{S}$, that $\rho_{\mathrm{T}}(t)$ is a twice-differentiable stationary correlation function on $\mathbb{R}$, and that α_1, α_2, and α_{12} are nonnegative constants such that $0 < \alpha_1 + \alpha_2 + \alpha_{12} < 1$. From Example 4 of Ma (2002),

$$-\log(1 - \alpha_1 \rho_{\mathrm{S}}(\boldsymbol{s}) - \alpha_2 \rho_{\mathrm{T}}(t) - \alpha_{12} \rho_{\mathrm{S}}(\boldsymbol{s}) \rho_{\mathrm{T}}(t)), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R},$$

is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$. It is capable of modeling positive and/or negative spatio-temporal correlations based on an appropriate choice of $\rho_{\mathrm{S}}(\boldsymbol{s})$ and $\rho_{\mathrm{T}}(t)$ with the same or different signs. By Corollary 2,

$$\begin{aligned} C(\boldsymbol{s}; t) = &- \frac{(\alpha_2 + \alpha_{12} \rho_{\mathrm{S}}(\boldsymbol{s})) \rho_{\mathrm{T}}''(t)}{1 - \alpha_1 \rho_{\mathrm{S}}(\boldsymbol{s}) - \alpha_2 \rho_{\mathrm{T}}(t) - \alpha_{12} \rho_{\mathrm{S}}(\boldsymbol{s}) \rho_{\mathrm{T}}(t)} \\ &- \frac{(\alpha_2 + \alpha_{12} \rho_{\mathrm{S}}(\boldsymbol{s}))^2 \{\rho_{\mathrm{T}}'(t)\}^2}{\{1 - \alpha_1 \rho_{\mathrm{S}}(\boldsymbol{s}) - \alpha_2 \rho_{\mathrm{T}}(t) - \alpha_{12} \rho_{\mathrm{S}}(\boldsymbol{s}) \rho_{\mathrm{T}}(t)\}^2} \end{aligned}$$

is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$.

Examples of twice-differentiable correlation functions $\rho_{\mathrm{T}}(t)$ on the real line are

(i) $(1 + |t|) \exp(-|t|)$, $t \in \mathbb{R}$;

(ii) $(1 - 2\alpha t^2) \exp(-\alpha t^2)$, $t \in \mathbb{R}$, where α is a positive constant;

(iii) $(\beta - \alpha)^{-1} \{\beta \exp(-\alpha |t|) - \alpha \exp(-\beta |t|)\}$, $t \in \mathbb{R}$, where α and β are distinct positive constants;

(iv) $\exp(-\alpha |t|) \{\cos(\beta t) + (\alpha / \beta) \sin(\beta t)\}$, $t \in \mathbb{R}$, where α and β are positive constants;

(v) $(1 + \alpha t^2)^{-\beta}$, $t \in \mathbb{R}$, where α and β are positive constants; and

(vi) $(\log \alpha - \log \beta)^{-1} \{\log(\alpha + t^2) - \log(\beta + t^2)\}$, $t \in \mathbb{R}$, where α and β are distinct positive constants.

Example 3. For an intrinsically stationary variogram $\gamma_{\mathrm{S}}(\boldsymbol{s})$ on $\mathcal{S}$ and a completely monotone function $\ell(x)$ on $[0, \infty)$, it was shown by Ma (2003a) that

$$\frac{1}{2} \{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})\}^{-1/2} \ell\left(\frac{t^2}{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})} \right), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R},$$

is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$. A direct application of Corollary 2 gives a spatio-temporal covariance

$$C(\boldsymbol{s}; t) = -\{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})\}^{-3/2} \ell'\left(\frac{t^2}{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})}\right) - 2t^2\{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})\}^{-5/2} \ell''\left(\frac{t^2}{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})}\right), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}. \tag{4}$$

Various choices for $\ell(\cdot)$ may now be made. For example, taking $\ell(x) = \exp(-x)$, $x \geq 0$, in (4), we obtain

$$C(\boldsymbol{s}; t) = \{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})\}^{-5/2}\{1 + \gamma_{\mathrm{S}}(\boldsymbol{s}) - 2t^2\} \exp\left(-\frac{t^2}{1 + \gamma_{\mathrm{S}}(\boldsymbol{s})}\right), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}.$$

Many other nonseparable spatio-temporal stationary covariance models can be obtained from (4) by the appropriate selection of $\ell(\cdot)$.

One type of separable spatio-temporal covariance function factorizes as

$$C(\boldsymbol{s}; t) = C_{\mathrm{S}}(\boldsymbol{s}) C_{\mathrm{T}}(t), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathcal{T},$$

where $C_{\mathrm{S}}(\boldsymbol{s})$ and $C_{\mathrm{T}}(t)$ are purely spatial and purely temporal covariances, respectively. This represents a rather limited description of space–time interaction, as pointed out by Cressie and Huang (1999), Kyriakidis and Journel (1999), and Stein (2005). It is thus of considerable interest to derive nonseparable covariance models. Occasionally, however, it transpires that a seemingly nonseparable candidate is actually just a separable model. An example of this kind is presented in Example 4.

Example 4. Let $\gamma_{\mathrm{S}}(\boldsymbol{s})$ be an unbounded, intrinsically stationary variogram on $\mathcal{S}$, and let α_k and β_k $(k = 0, 1, 2)$ be constants. We are going to show that

$$C(\boldsymbol{s}; t) = \exp(-\gamma_{\mathrm{S}}(\boldsymbol{s})(\alpha_0 + \alpha_1 t + \alpha_2 t^2) - (\beta_0 + \beta_1 t + \beta_2 t^2)), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}, \tag{5}$$

is a valid covariance on $\mathcal{S} \times \mathbb{R}$ if and only if $\alpha_0 \geq 0$, $\beta_2 \geq 0$, and $\alpha_1 = \alpha_2 = \beta_1 = 0$; that is, if and only if it is a separable model.

In fact, a necessary condition for (5) to be a covariance is $C(\boldsymbol{s}; t) \leq C(\boldsymbol{0}; 0) = \exp(-\beta_0)$, or

$$\gamma_{\mathrm{S}}(\boldsymbol{s})(\alpha_0 + \alpha_1 t + \alpha_2 t^2) + \beta_1 t + \beta_2 t^2 \geq 0 \qquad \text{for all } (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R},$$

for which we must have

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 \geq 0 \quad \text{and} \quad \beta_1 t + \beta_2 t^2 \geq 0 \qquad \text{for all } t \in \mathbb{R}.$$

This implies that $\alpha_0 \geq 0$, $\alpha_2 \geq 0$, $\alpha_1^2 \leq 4\alpha_0\alpha_2$, and $\beta_2 \geq 0$. In addition, (5) is symmetric, in the sense that $C(\pm\boldsymbol{s}; t) = C(\boldsymbol{s}; \pm t)$, and twice differentiable with respect to t for every $\boldsymbol{s} \in \mathcal{S}$. If (5) is a covariance then, from Corollary 2 and the fact that

$$0 = \frac{\partial}{\partial t} C(\boldsymbol{s}; t) \bigg|_{t=0} = -\{\gamma_{\mathrm{S}}(\boldsymbol{s})\alpha_1 + \beta_1\} C(\boldsymbol{s}; 0) \quad \text{for all } \boldsymbol{s} \in \mathcal{S},$$

we obtain $\alpha_1 = \beta_1 = 0$. Moreover, by Corollary 2,

$$-\frac{\partial^2}{\partial t^2} C(\boldsymbol{s}; t) = \{2\alpha_2 \gamma_S(\boldsymbol{s}) + 2\beta_2 - (2\alpha_2 t \gamma_S(\boldsymbol{s}) + 2\beta_2 t)^2\} C(\boldsymbol{s}; t)$$

has to be a stationary covariance function on $\mathcal{S} \times \mathbb{R}$, meaning that, for any $\boldsymbol{s} \in \mathcal{S}$,

$$-\frac{\partial^2}{\partial t^2} C(\boldsymbol{s}; t)\bigg|_{t=0} \le -\frac{\partial^2}{\partial t^2} C(\boldsymbol{s}; t)\bigg|_{\boldsymbol{s}=\boldsymbol{0},\ t=0} = 2\beta_2 \exp(-\beta_0)$$

or, equivalently,

$$\{\alpha_2 \gamma_S(\boldsymbol{s}) + \beta_2\} \exp(-\gamma(\boldsymbol{s})\alpha_0) \le \beta_2.$$

Thus, for a vector $\boldsymbol{s}$ with $\gamma(\boldsymbol{s}) > 0$,

$$\alpha_2 \le \frac{1 - \exp(-\gamma(\boldsymbol{s})\alpha_0)}{\gamma_S(\boldsymbol{s})} \beta_2.$$

Letting $\boldsymbol{s}$ approach infinity (componentwise) yields $\alpha_2 \le 0$. Consequently, α_2 must vanish.

A model closely related to (5) is

$$C(\boldsymbol{s}; t) = \exp(-\gamma_S(\boldsymbol{s})(\alpha_0 + \alpha_1|t| + \alpha_2 t^2) - (\beta_0 + \beta_1|t| + \beta_2 t^2)), \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}. \quad (6)$$

A particular case of (6) was proposed for modeling space–time data in the literature. There, $\gamma_S(\boldsymbol{s})$ is the usual Euclidean norm: $\|\boldsymbol{s}\| = (\sum_{k=1}^{d} s_k^2)^{1/2}$, $\boldsymbol{s} \in \mathbb{R}^d$. Its validity was critically examined by Gneiting (2002) in a nonseparable case in which either α_1 or α_2 is nonzero. Simple necessary conditions for the permissibility of (6) are $\alpha_0 \ge 0$, $\alpha_2 \ge 0$, $\beta_1 \ge 0$, and $\beta_2 \ge 0$. It would be of interest to derive a necessary and sufficient condition on the parameters α_k and β_k ($k = 0, 1, 2$) such that (6) is a valid spatio-temporal covariance function.

4. Partial integration with respect to time

In this section, we investigate the partial integral of a stationary spatio-temporal random field with respect to the time lag.

Suppose that $\{Z_0(\boldsymbol{s}; t),\ (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}\}$ is a stationary zero-mean random field whose covariance function $C_0(\boldsymbol{s}; t)$ is continuous in t. Define its partial integral with respect to t, i.e.

$$\int_0^t Z_0(\boldsymbol{s}; u)\, \mathrm{d}u, \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R}, \quad (7)$$

as the limit (in the mean square) of the corresponding approximating sum with respect to u. This type of superposition over time is important in the study of space–time rainfall; see, for example, Rodriguez-Iturbe *et al.* (1998).

The integral (7) is a spatio-temporal random field intrinsically stationary in space. In fact, its mean function is

$$\mathrm{E}\bigg(\int_0^t Z_0(\boldsymbol{s}; u)\, \mathrm{d}u\bigg) = \int_0^t \mathrm{E}\, Z_0(\boldsymbol{s}; u)\, \mathrm{d}u = 0, \qquad (\boldsymbol{s}; t) \in \mathcal{S} \times \mathbb{R},$$

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and its variogram,

$$\begin{aligned}
&\frac{1}{2}\operatorname{var}\biggl(\int_0^{t_1} Z_0(\boldsymbol{s}+\boldsymbol{s}';u)\,\mathrm{d}u - \int_0^{t_2} Z_0(\boldsymbol{s}';v)\,\mathrm{d}v\biggr)\\
&\quad= \frac{1}{2}\,\mathrm{E}\biggl(\int_0^{t_1} Z_0(\boldsymbol{s}+\boldsymbol{s}';u)\,\mathrm{d}u\biggr)^2 + \frac{1}{2}\,\mathrm{E}\biggl(\int_0^{t_2} Z_0(\boldsymbol{s}';u)\,\mathrm{d}u\biggr)^2\\
&\qquad- \mathrm{E}\biggl(\int_0^{t_1} Z_0(\boldsymbol{s}+\boldsymbol{s}';u)\,\mathrm{d}u \int_0^{t_2} Z_0(\boldsymbol{s}';v)\,\mathrm{d}v\biggr)^2\\
&\quad= \frac{1}{2}\int_0^{t_1}\int_0^{t_1} C_0(\boldsymbol{0};u-v)\,\mathrm{d}u\,\mathrm{d}v + \frac{1}{2}\int_0^{t_2}\int_0^{t_2} C_0(\boldsymbol{0};u-v)\,\mathrm{d}u\,\mathrm{d}v\\
&\qquad- \int_0^{t_1}\int_0^{t_2} C_0(\boldsymbol{s};u-v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad= \int_0^{t_1}(t_1-|u|)C_0(\boldsymbol{0};u)\,\mathrm{d}u + \int_0^{t_2}(t_2-|u|)C_0(\boldsymbol{0};u)\,\mathrm{d}u - \int_0^{t_1}\int_0^{t_2} C_0(\boldsymbol{s};u-v)\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned}$$

depends on t_1, t_2, and $\boldsymbol{s}$ only. Thus, the aggregation in time of a stationary spatio-temporal random field produces a spatio-temporal random field intrinsically stationary in space.

We next show that taking the difference of (7) with respect to time gives rise to a spatio-temporal random field stationary in both space and time. More specifically, for a fixed $\tau_0 > 0$, consider the increment of (7), i.e.

$$Z(\boldsymbol{s};t) = \int_0^{t+\tau_0} Z_0(\boldsymbol{s};u)\,\mathrm{d}u - \int_0^{t} Z_0(\boldsymbol{s};u)\,\mathrm{d}u, \qquad (\boldsymbol{s};t) \in \mathcal{S}\times\mathbb{R}. \tag{8}$$

Alternatively, (8) can be written as

$$Z(\boldsymbol{s};t) = \int_t^{t+\tau_0} Z_0(\boldsymbol{s};u)\,\mathrm{d}u = \int_0^{\tau_0} Z_0(\boldsymbol{s};t+u)\,\mathrm{d}u, \qquad (\boldsymbol{s};t) \in \mathcal{S}\times\mathbb{R},$$

which may be regarded as a stochastic convolution of $\{Z_0(\boldsymbol{s};t),\ (\boldsymbol{s};t)\in\mathcal{S}\times\mathbb{R}\}$ and the indicator function of the interval $[0,\tau_0]$, in the sense of Chilès and Delfiner (1999, Section 2.4.1). This is a stationary spatio-temporal random field with mean

$$\mathrm{E}\,Z(\boldsymbol{s};t) = \mathrm{E}\biggl(\int_0^{\tau_0} Z(\boldsymbol{s};t+u)\,\mathrm{d}u\biggr) = \int_0^{\tau_0}\mathrm{E}\,Z(\boldsymbol{s};t+u)\,\mathrm{d}u = 0, \qquad (\boldsymbol{s};t)\in\mathcal{S}\times\mathbb{R},$$

and covariance

$$\begin{aligned}
C(\boldsymbol{s};t) &= \operatorname{cov}\biggl(\int_0^{\tau_0} Z(\boldsymbol{s}+\boldsymbol{s}';t+t'+u)\,\mathrm{d}u, \int_0^{\tau_0} Z(\boldsymbol{s}';t'+v)\,\mathrm{d}v\biggr)\\
&= \int_0^{\tau_0}\int_0^{\tau_0}\operatorname{cov}\Bigl(Z(\boldsymbol{s}+\boldsymbol{s}';t+t'+u), Z(\boldsymbol{s}';t'+v)\Bigr)\,\mathrm{d}u\,\mathrm{d}v\\
&= \int_0^{\tau_0}\int_0^{\tau_0} C_0(\boldsymbol{s};t+u-v)\,\mathrm{d}u\,\mathrm{d}v\\
&= \int_{-\tau_0}^{\tau_0}(\tau_0-|u|)C_0(\boldsymbol{s};t-u)\,\mathrm{d}u, \qquad (\boldsymbol{s};t)\in\mathcal{S}\times\mathbb{R}.
\end{aligned}$$

The above argument results in the following lemma.

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Lemma 2. *If a stationary covariance function $C_0(\mathbf{s}; t)$ on $\mathcal{S} \times \mathbb{R}$ is continuous with respect to t for each $\mathbf{s} \in \mathcal{S}$, then, for a fixed $\tau_0 > 0$,*

$$C(\mathbf{s}; t) = \int_{-\tau_0}^{\tau_0} (\tau_0 - |u|) C_0(\mathbf{s}; t - u)\, \mathrm{d}u, \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R}, \tag{9}$$

is a stationary covariance function on $\mathcal{S} \times \mathbb{R}$.

It can be shown that (9) is equivalent to

$$C(\mathbf{s}; t) = \int_0^{t+\tau_0} \int_0^u C_0(\mathbf{s}; v)\, \mathrm{d}v\, \mathrm{d}u + \int_0^{t-\tau_0} \int_0^u C_0(\mathbf{s}; v)\, \mathrm{d}v\, \mathrm{d}u - 2 \int_0^t \int_0^u C_0(\mathbf{s}; v)\, \mathrm{d}v\, \mathrm{d}u, \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R}.$$

Its temporal margin, $C(\mathbf{0}; t)$, was studied by Barndorff-Nielsen and Shephard (2001) and Ma (2004). However, unlike in the purely temporal case, $\int_0^t \int_0^u C_0(\mathbf{s}; v)\, \mathrm{d}v\, \mathrm{d}u$ is not necessarily a variogram even if the covariance function $C_0(\mathbf{s}; t)$ is continuous with respect to t for each $\mathbf{s} \in \mathcal{S}$. For instance, let

$$C_0(\mathbf{s}; t) = \cos(\boldsymbol{\theta}^\top \mathbf{s}) C_{\mathrm{T}}(t), \qquad (\mathbf{s}; t) \in \mathbb{R}^d \times \mathbb{R},$$

where $\boldsymbol{\theta} \in \mathbb{R}^d$ is a nonzero vector and $C_{\mathrm{T}}(t)$ is a nonnegative, continuous, and stationary covariance function on $\mathbb{R}$. However,

$$\int_0^t \int_0^u C_0(\mathbf{s}; v)\, \mathrm{d}v\, \mathrm{d}u = \cos(\boldsymbol{\theta}^\top \mathbf{s}) \int_0^t \int_0^u C_{\mathrm{T}}(v)\, \mathrm{d}v\, \mathrm{d}u, \qquad (\mathbf{s}; t) \in \mathcal{S} \times \mathbb{R},$$

is not always nonnegative and, thus, not a variogram on $\mathcal{S} \times \mathbb{R}$.

5. Linear combination of separable spatio-temporal covariance models

A stationary covariance function $C(\mathbf{s}; t)$ on $\mathbb{R}^d \times \mathcal{T}$ is said to be isotropic in space if it depends on the spatial lag $\mathbf{s}$ only through the Euclidean norm $\|\mathbf{s}\|$. We refer the reader to Ma (2003c) for the characterization of stationary spatio-temporal covariance models isotropic in space. A subclass of these is constructed in this section, via linear combinations of separable spatio-temporal covariance models.

Assume that α_k and β_k ($k = 1, 2$) are positive constants with $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. Clearly, for $k = 1, 2$,

$$\exp(-\alpha_k \|\mathbf{s}\| - \beta_k |t|), \qquad (\mathbf{s}; t) \in \mathbb{R}^d \times \mathbb{R},$$

is a separable covariance function on $\mathbb{R}^d \times \mathbb{R}$. We start by asking a simple question: when does their linear combination

$$C(\mathbf{s}; t) = a_1 \exp(-\alpha_1 \|\mathbf{s}\| - \beta_1 |t|) + a_2 \exp(-\alpha_2 \|\mathbf{s}\| - \beta_2 |t|), \qquad (\mathbf{s}; t) \in \mathbb{R}^d \times \mathbb{R},$$

where a_1 and a_2 are constants subject to the obvious assumption $\operatorname{var}(Z(\mathbf{s}; t)) = a_1 + a_2 \ge 0$, define a valid spatio-temporal covariance function? Neglecting the trivial degenerate case in which $a_1 + a_2 = 0$, let $a_1 + a_2 > 0$ and consider the function

$$\rho(\mathbf{s}; t) = \frac{C(\mathbf{s}; t)}{C(\mathbf{0}; 0)} = \frac{a_1}{a_1 + a_2} \exp(-\alpha_1 \|\mathbf{s}\| - \beta_1 |t|) + \frac{a_2}{a_1 + a_2} \exp(-\alpha_2 \|\mathbf{s}\| - \beta_2 |t|), \qquad (\mathbf{s}; t) \in \mathbb{R}^d \times \mathbb{R}.$$

After the reparameterization $\theta = a_1/(a_1 + a_2)$, this becomes

$$\rho(\boldsymbol{s}; t) = \theta \exp(-\alpha_1 \|\boldsymbol{s}\| - \beta_1 |t|) + (1-\theta) \exp(-\alpha_2 \|\boldsymbol{s}\| - \beta_2 |t|), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}. \quad (10)$$

By inspection, (10) is a spatio-temporal correlation function if $0 \le \theta \le 1$, as it is a convex combination of two separable correlation functions. Otherwise, i.e. for other values of θ, it is the difference of two separable correlation functions. The full domain of θ over which (10) is a valid correlation function is described in Corollary 5(i).

A general form, which includes (10) as a special case, is

$$\begin{aligned} C(\boldsymbol{s}; t) = {} & \theta (\alpha_1 \|\boldsymbol{s}\|)^{\nu_1} K_{\nu_1}(\alpha_1 \|\boldsymbol{s}\|)(\beta_1 |t|)^{\nu_2} K_{\nu_2}(\beta_1 |t|) \\ & + (1-\theta)(\alpha_2 \|\boldsymbol{s}\|)^{\nu_1} K_{\nu_1}(\alpha_2 \|\boldsymbol{s}\|)(\beta_2 |t|)^{\nu_2} K_{\nu_2}(\beta_2 |t|), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}, \quad (11) \end{aligned}$$

where ν_1 and ν_2 are positive constants and $K_\nu(x)$ stands for the modified Bessel function of the second kind of order ν (see Gradshteyn and Ryzhik (2000)).

The spatial covariance model $(\alpha \|\boldsymbol{s}\|)^\nu K_\nu(\alpha \|\boldsymbol{s}\|)$, $\boldsymbol{s} \in \mathbb{R}^d$, was proposed by von Kármán (1948) for $\nu = \frac{1}{3}$ and $\boldsymbol{s} \in \mathbb{R}^3$, and constructed in the plane by Whittle (1954) via the stochastic partial differential equation. It reduces to $\sqrt{\pi/2} \exp(-\alpha \|\boldsymbol{s}\|)$ and $\sqrt{\pi/2}(1+\alpha \|\boldsymbol{s}\|) \exp(-\alpha \|\boldsymbol{s}\|)$, $\boldsymbol{s} \in \mathbb{R}^d$, when $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$, respectively. Some properties of the von Kármán–Whittle model, which is often named the Matérn model in the statistical literature, were demonstrated in Matérn (1986), Kent (1989), and Stein (1999).

Theorem 3. *Let α_k, β_k, and ν_k ($k = 1, 2$) be positive constants with $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. The function (11) is a stationary covariance function on $\mathbb{R}^d \times \mathbb{R}$ if and only if the constant θ satisfies*

$$\left(1 - \frac{\alpha_2^d \beta_2}{\alpha_1^d \beta_1}\right)^{-1} \le \theta \le \left\{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{2\nu_1} \left(\frac{\beta_1}{\beta_2}\right)^{2\nu_2}\right\}^{-1}. \quad (12)$$

In the case that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, (12) is interpreted as for all real numbers θ. If $\alpha_1 = \alpha_2$ or if $\beta_1 = \beta_2$, (11) reduces to a separable spatio-temporal model.

The upper bound on θ in (12) does not depend on the dimension d. In contrast, however, the lower bound in (12) decreases as d increases, and tends to 0 as d approaches infinity.

The upper bound on θ in (12) also depends on the parameters ν_1 and ν_2, while the lower bound does not. In particular, the two parts of Corollary 5, below, are obtained by taking $(\nu_1, \nu_2) = (\frac{1}{2}, \frac{1}{2})$ and $(\nu_1, \nu_2) = (p + \frac{1}{2}, \frac{1}{2})$, respectively, where p is a nonnegative integer.

Corollary 5. *Assume that $0 < \alpha_1 \le \alpha_2$ and $0 < \beta_1 \le \beta_2$.*

(i) *The function $\rho(\boldsymbol{s}; t)$ in (10) is a stationary correlation function on $\mathbb{R}^d \times \mathbb{R}$ if and only if the constant θ satisfies*

$$\left(1 - \frac{\alpha_2^d \beta_2}{\alpha_1^d \beta_1}\right)^{-1} \le \theta \le \left(1 - \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2}\right)^{-1}. \quad (13)$$

(ii) *For a nonnegative integer p,*

$$\begin{aligned} C(\boldsymbol{s}; t) = \sum_{k=0}^{p} \frac{2^k (2p-k)!}{k!\,(p-k)!} \{ & \theta (\alpha_1 \|\boldsymbol{s}\|)^k \exp(-\alpha_1 \|\boldsymbol{s}\| - \beta_1 |t|) \\ & + (1-\theta)(\alpha_2 \|\boldsymbol{s}\|)^k \exp(-\alpha_2 \|\boldsymbol{s}\| - \beta_2 |t|)\}, \\ & \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}, \end{aligned}$$

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is a stationary covariance function on $\mathbb{R}^d \times \mathbb{R}$ *if and only if the constant* θ *satisfies*

$$\left(1 - \frac{\alpha_2^d \beta_2}{\alpha_1^d \beta_1}\right)^{-1} \le \theta \le \left(1 - \frac{\alpha_1^{2p+1} \beta_1}{\alpha_2^{2p+1} \beta_2}\right)^{-1}.$$

It follows from Corollary 2 that the partial derivative of (10) with respect to t exists only if $\alpha_1 = \alpha_2$, in which case (10) is a separable model.

An important feature of the bounds in (13), as well as those in (12), is that they depend on the quotients α_1/α_2 and β_1/β_2, instead of the αs and βs individually. This observation leads us to Corollary 6, which is a special case of Theorem 4 of Ma (2003c) when $\theta = 0$ or $\theta = 1$.

Corollary 6. *Assume that* $0 < \alpha_1 \le \alpha_2$*,* $0 < \beta_1 \le \beta_2$*, and that* θ *satisfies (13). If* $\ell(u, v)$ *is the Laplace transform of a nonnegative bivariate random vector* (U, V)*, i.e.* $\ell(u, v) = \mathrm{E}\exp(-Uu - Vv)$*, then*

$$\rho(\boldsymbol{s}; t) = \theta \ell(\alpha_1 \|\boldsymbol{s}\|, \beta_1 |t|) + (1 - \theta) \ell(\alpha_2 \|\boldsymbol{s}\|, \beta_2 |t|), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}, \tag{14}$$

is a stationary correlation function on $\mathbb{R}^d \times \mathbb{R}$*.*

Suppose that $\{Z_0(\boldsymbol{s}; t),\ (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}\}$ is a stationary zero-mean random field with correlation function (10), and is independent of the random vector (U, V), which possesses joint distribution function $F(u, v)$. One proof of Corollary 6 proceeds by constructing a new spatio-temporal random field whose correlation function coincides with (14), namely

$$Z(\boldsymbol{s}; t) = Z_0(\boldsymbol{s}U; tV), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}.$$

In fact, under such a construction, we have

$$\begin{aligned}
&\operatorname{cov}(Z(\boldsymbol{s} + \boldsymbol{s}_0; t + t_0), Z(\boldsymbol{s}_0; t_0)) \\
&\quad = \int_0^\infty \int_0^\infty \operatorname{cov}(Z_0((\boldsymbol{s} + \boldsymbol{s}_0)u; (t + t_0)v), Z(\boldsymbol{s}_0 u; t_0 v))\, \mathrm{d}F(u, v) \\
&\quad = \int_0^\infty \int_0^\infty \{\theta \exp(-\alpha_1 \|\boldsymbol{s}\| u - \beta_1 |t| v) + (1 - \theta) \exp(-\alpha_2 \|\boldsymbol{s}\| v - \beta_2 |t| v)\}\, \mathrm{d}F(u, v) \\
&\quad = \theta \ell(\alpha_1 \|\boldsymbol{s}\|, \beta_1 |t|) + (1 - \theta) \ell(\alpha_2 \|\boldsymbol{s}\|, \beta_2 |t|),
\end{aligned}$$

where $(\boldsymbol{s}_0; t_0) \in \mathbb{R}^d \times \mathbb{R}$ is arbitrary.

Two particular cases of (14) are worth mentioning.

1. If the nonnegative random variables U and V are independent, so that $\ell(u, v)$ factorizes as $\ell(u, v) = \ell_1(u)\ell_2(v)$, $u \ge 0$, $v \ge 0$, where $\ell_1(\cdot)$ and $\ell_2(\cdot)$ are completely monotone functions on $[0, \infty)$, then (14) becomes

$$\rho(\boldsymbol{s}; t) = \theta \ell_1(\alpha_1 \|\boldsymbol{s}\|) \ell_2(\beta_1 |t|) + (1 - \theta) \ell_1(\alpha_2 \|\boldsymbol{s}\|) \ell_2(\beta_2 |t|), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}.$$

   Its spatial margin is presented in Theorem 2 of Ma (2005a).

2. When (U, V) has mass confined to the line $u = v$, $\ell(u, v)$ reduces to a completely monotone function $\ell(\cdot)$ on $[0, \infty)$, and (14) becomes

$$\rho(\boldsymbol{s}; t) = \theta \ell(\alpha_1 \|\boldsymbol{s}\| + \beta_1 |t|) + (1 - \theta) \ell(\alpha_2 \|\boldsymbol{s}\| + \beta_2 |t|), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}.$$

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Corollary 7 follows from Theorem 3 by letting $\beta_1 = \beta_2$ and then considering the spatial margin.

Corollary 7. *For $\nu > 0$ and α_1 and α_2 such that $0 < \alpha_1 < \alpha_2$, the function*

$$C(\boldsymbol{s}) = \theta(\alpha_1\|\boldsymbol{s}\|)^{\nu} K_{\nu}(\alpha_1\|\boldsymbol{s}\|) + (1-\theta)(\alpha_2\|\boldsymbol{s}\|)^{\nu} K_{\nu}(\alpha_2\|\boldsymbol{s}\|), \qquad \boldsymbol{s} \in \mathbb{R}^d, \tag{15}$$

is a stationary covariance function on $\mathbb{R}^d$ if and only if the constant θ satisfies

$$\left(1 - \frac{\alpha_2^d}{\alpha_1^d}\right)^{-1} \le \theta \le \left\{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{2\nu}\right\}^{-1}.$$

When $\nu = \frac{1}{2}$ and $\theta = \alpha_2/(\alpha_2 - \alpha_1)$, (15) simplifies to

$$C(\boldsymbol{s}) = \frac{\alpha_2}{\alpha_2 - \alpha_1}\exp(-\alpha_1\|\boldsymbol{s}\|) - \frac{\alpha_1}{\alpha_2 - \alpha_1}\exp(-\alpha_2\|\boldsymbol{s}\|), \qquad \boldsymbol{s} \in \mathbb{R}^d,$$

where we have omitted a positive constant. This is a spatial second-order autoregression proposed by Shkarofsky (1968) as a three-dimensional turbulence model and by Buell (1972) for modeling wind and geopotential on isobaric surfaces.

Notice that, for $\nu > 0$, the function $x^{\nu/2}K_{\nu}(x^{1/2})$, $x \ge 0$, is completely monotone on $[0, \infty)$. This implies that $x^{\nu}K_{\nu}(x)$, $x \ge 0$, is decreasing on $[0, \infty)$. As a consequence, (11) is nonnegative if $0 \le \theta \le \{1 - (\alpha_1/\alpha_2)^{2\nu_1}(\beta_1/\beta_2)^{2\nu_2}\}^{-1}$. However, (11) may take negative values when θ is negative. This observation motivates Theorem 4. Actually, there is a 'dual' relationship between (11) and (16), which may simply be interpreted as one of the covariance functions being a positive multiple of the (inverse) Fourier transform of the other.

Theorem 4. *Let α_k, β_k, and ν_k $(k = 1, 2)$ be positive constants with $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$.*

(i) *If θ is a nonnegative constant then*

$$\begin{aligned} C(\boldsymbol{s}; t) = {} & \theta\alpha_1^{2\nu_1}\beta_1^{2\nu_2}(\|\boldsymbol{s}\|^2 + \alpha_1^2)^{-(\nu_1 + d/2)}(t^2 + \beta_1^2)^{-(\nu_2+1/2)} \\ & + (1-\theta)\alpha_2^{2\nu_1}\beta_2^{2\nu_2}(\|\boldsymbol{s}\|^2 + \alpha_2^2)^{-(\nu_1 + d/2)}(t^2 + \beta_2^2)^{-(\nu_2+1/2)}, \\ & (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}, \end{aligned} \tag{16}$$

is a stationary covariance function on $\mathbb{R}^d \times \mathbb{R}$.

(ii) *In the case that $\alpha_1 \ne \alpha_2$ or $\beta_1 \ne \beta_2$, $\theta \ge 0$ is also a necessary condition for (16) to be a covariance function on $\mathbb{R}^d \times \mathbb{R}$.*

In general, there appear to be no simple necessary and sufficient conditions for the validity of a linear combination of more than two pairs of separable spatio-temporal covariances. A sufficient condition is given in Theorem 5.

Theorem 5. *Assume that $C_{\mathrm{S}}(\boldsymbol{s})$ is an isotropic covariance function on $\mathbb{R}^d$ whose spectral density function $f_{\mathrm{S}}(\boldsymbol{\omega})$ is decreasing in $\|\boldsymbol{\omega}\|$, $\boldsymbol{\omega} \in \mathbb{R}^d$, and that $C_{\mathrm{T}}(t)$ is a purely temporal covariance function on $\mathbb{R}$ whose spectral density function $f_{\mathrm{T}}(\omega_0)$ is decreasing in $|\omega_0|$, where $\omega_0 \in \mathbb{R}$. If α_k, β_k, and θ_k $(k = 1, \dots, p)$ are constants such that $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_p$, $0 < \beta_1 \le \beta_2 \le \cdots \le \beta_p$, and $\sum_{i=1}^k \theta_i \ge 0$, $k = 1, \dots, p$, then*

$$C(\boldsymbol{s}; t) = \sum_{k=1}^{p} \theta_k \alpha_k^{-d}\beta_k^{-1} C_{\mathrm{S}}\left(\frac{\boldsymbol{s}}{\alpha_k}\right) C_{\mathrm{T}}\left(\frac{t}{\beta_k}\right), \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}, \tag{17}$$

is a stationary covariance function on $\mathbb{R}^d \times \mathbb{R}$.

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A necessary and sufficient condition to have an isotropic covariance function $C_{\mathrm{S}}(\boldsymbol{s})$ on $\mathbb{R}^d$ whose spectral density function $f_{\mathrm{S}}(\boldsymbol{\omega})$ is decreasing in $\|\boldsymbol{\omega}\|$ is that $C_{\mathrm{S}}(\boldsymbol{s})$ be an isotropic covariance function on $\mathbb{R}^{d+2}$; see Zolotarev (1981, p. 288) and Gneiting (1998, p. 145). The univariate result is due to Khinchin (see Lukacs (1970, p. 92)). In particular, an isotropic covariance function in arbitrary dimension possesses an integral representation (Schoenberg (1938b))

$$C_{\mathrm{S}}(\boldsymbol{s}) = \int_0^\infty \exp(-\|\boldsymbol{s}\|^2 u)\,\mathrm{d}F(u) = \ell(\|\boldsymbol{s}\|^2), \qquad \boldsymbol{s} \in \mathbb{R}^d, \tag{18}$$

where $F(u)$ is nondecreasing and bounded for $u \geq 0$, and $\ell(x) = \int_0^\infty \exp(-xu)\,\mathrm{d}F(u)$, $x \geq 0$, is the Laplace transform of $F(u)$. The spectral density of (18),

$$f_{\mathrm{S}}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} C_{\mathrm{S}}(\boldsymbol{s}) \exp(\mathrm{i}\boldsymbol{s}^\top \boldsymbol{\omega})\,\mathrm{d}\boldsymbol{s} = \pi^{d/2} \int_0^\infty u^{-d/2} \exp\biggl(-\frac{1}{4u}\|\boldsymbol{\omega}\|^2\biggr)\,\mathrm{d}F(u), \qquad \boldsymbol{\omega} \in \mathbb{R}^d,$$

is a decreasing function of $\|\boldsymbol{\omega}\|$, whenever the integral in the right-most expression exists.

As an example, let $F(u)$ in (18) be a gamma distribution function with density

$$\frac{1}{\Gamma(\nu_1)} u^{\nu_1 - 1} \exp(-u), \qquad u > 0,$$

so that

$$C_{\mathrm{S}}(\boldsymbol{s}) = (1 + \|\boldsymbol{s}\|^2)^{-(\nu_1 + d/2)}, \qquad \boldsymbol{s} \in \mathbb{R}^d.$$

Similarly, choose

$$C_{\mathrm{T}}(t) = (1 + t^2)^{-(\nu_2 + 1/2)}, \qquad t \in \mathbb{R}.$$

Then (17) is a spatio-temporal covariance function with power-law decay:

$$C(\boldsymbol{s}; t) = \sum_{k=1}^p \theta_k \alpha_k^{2\nu_1} \beta_k^{2\nu_2} (\|\boldsymbol{s}\|^2 + \alpha_k^2)^{-(\nu_1 + d/2)} (t^2 + \beta_k^2)^{-(\nu_2 + 1/2)}, \qquad (\boldsymbol{s}; t) \in \mathbb{R}^d \times \mathbb{R}. \tag{19}$$

This reduces to (16) when $p = 2$. Letting $\beta_1 = \cdots = \beta_p$ and $\nu_1 = \nu$ in the spatial margin of (19), i.e. $C(\boldsymbol{s}; 0)$, yields a power-law decaying covariance function over $\mathbb{R}^d$:

$$C(\boldsymbol{s}) = \sum_{k=1}^p \theta_k \alpha_k^{2\nu} (\|\boldsymbol{s}\|^2 + \alpha_k^2)^{-(\nu + d/2)}, \qquad \boldsymbol{s} \in \mathbb{R}^d.$$

Whittle (1962) derived isotropic covariances in the plane with a power law at large distances in order to explain the so-called Fairfield Smith law of environmental variation, conjectured on the basis of agricultural uniformity trials made by Fairfield Smith. As suggested by Ma (2005b), one can derive long-range-dependent spatio-temporal random fields by randomizing the time scale of a given spatio-temporal random field.

So far, we have considered stationary spatio-temporal covariance functions isotropic in space. Simple anisotropic functional forms can be readily introduced by replacing $\|\boldsymbol{s}\|^2$ with $\boldsymbol{s}^\top \boldsymbol{A}\boldsymbol{s}$, where $\boldsymbol{A}$ is a $d \times d$ nonnegative-definite matrix. Alternatively, nonstationary models may be obtained by replacing $\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|$ with $\|\boldsymbol{g}(\boldsymbol{s}_1) - \boldsymbol{g}(\boldsymbol{s}_2)\|$, where $\boldsymbol{g}(\boldsymbol{s})$ is a nonlinear transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$.

6. Proofs

6.1. Proof of Theorem 1

For a nonnegative integer k, write

$$C_k(\boldsymbol{s};t) = \gamma(\boldsymbol{s}+2^k\boldsymbol{s}_0; t+2^k t_0) + \gamma(\boldsymbol{s}-2^k\boldsymbol{s}_0; t-2^k t_0) - 2\gamma(\boldsymbol{s};t), \qquad (\boldsymbol{s};t) \in \mathcal{S}\times\mathcal{T}.$$

We shall show, by induction, that $C_k(\boldsymbol{s};t)$ is a stationary covariance function on $\mathcal{S}\times\mathcal{T}$ for every positive integer $k \in \mathbb{N}$.

We begin by proving that $C_1(\boldsymbol{s};t)$ is a stationary covariance function on $\mathcal{S}\times\mathcal{T}$. By assumption, $C_0(\boldsymbol{s};t)$ is a stationary covariance function. This implies that

$$\begin{aligned} C_0(\boldsymbol{s}+\boldsymbol{s}_0; t+t_0) &+ C_0(\boldsymbol{s}-\boldsymbol{s}_0; t-t_0) + 2C_0(\boldsymbol{s};t) \\ &= \{\gamma(\boldsymbol{s}+2\boldsymbol{s}_0; t+2t_0) + \gamma(\boldsymbol{s};t) - 2\gamma(\boldsymbol{s}-\boldsymbol{s}_0; t-t_0)\} \\ &\quad + \{\gamma(\boldsymbol{s};t) + \gamma(\boldsymbol{s}-2\boldsymbol{s}_0; t-2t_0) - 2\gamma(\boldsymbol{s}-\boldsymbol{s}_0; t-t_0)\} \\ &\quad + 2\{\gamma(\boldsymbol{s}+\boldsymbol{s}_0; t+t_0) + \gamma(\boldsymbol{s}-\boldsymbol{s}_0; t-t_0) - 2\gamma(\boldsymbol{s};t)\} \\ &= C_1(\boldsymbol{s};t) \end{aligned}$$

is a stationary spatio-temporal covariance function.

Suppose now that $C_k(\boldsymbol{s};t)$ is a stationary covariance function on $\mathcal{S}\times\mathcal{T}$. Then, so is

$$C_{k+1}(\boldsymbol{s};t) = C_k(\boldsymbol{s}+2^k\boldsymbol{s}_0; t+2^k t_0) + C_k(\boldsymbol{s}-2^k\boldsymbol{s}_0; t-2^k t_0) + 2C_k(\boldsymbol{s};t), \qquad (\boldsymbol{s};t) \in \mathcal{S}\times\mathcal{T}.$$

By induction, $C_k(\boldsymbol{s};t)$ is a stationary spatio-temporal covariance function for every $k \in \mathbb{N}$.

As k tends to infinity, the limit of $\frac{1}{2}C_k(\boldsymbol{s};t)$ exists and equals

$$\gamma((\infty,\infty,\dots,\infty);\infty) - \gamma(\boldsymbol{s};t),$$

which is also a stationary covariance function and possesses the variogram $\gamma(\boldsymbol{s};t)$.

6.2. Proof of Theorem 2

By assumption and Lemma 1, $\gamma(\boldsymbol{s};t+h)+\gamma(\boldsymbol{s};t-h)-2\gamma(\boldsymbol{s};t)$ is a stationary covariance function on $\mathcal{S}\times\mathbb{R}$ for any fixed $h>0$; so also is

$$\frac{1}{h^2}\{\gamma(\boldsymbol{s};t+h)+\gamma(\boldsymbol{s};t-h)-2\gamma(\boldsymbol{s};t)\}, \qquad (\boldsymbol{s};t)\in\mathcal{S}\times\mathbb{R}.$$

Letting $h \to 0_+$ reveals that $(\partial^2/\partial t^2)\gamma(\boldsymbol{s};t)$ is a stationary covariance function on $\mathcal{S}\times\mathbb{R}$. When $\gamma(\boldsymbol{s};t)$ is symmetric, we have

$$\begin{aligned} \lim_{h\to 0}\frac{\gamma(\boldsymbol{s};h)-\gamma(\boldsymbol{s};0)}{h} &= \lim_{h\to 0}\frac{\gamma(\boldsymbol{s};h)+\gamma(\boldsymbol{s};-h)-2\gamma(\boldsymbol{s};0)}{h^2}\cdot\frac{h}{2} \\ &= \frac{1}{2}\frac{\partial^2}{\partial t^2}\gamma(\boldsymbol{s};t)\bigg|_{t=0}\cdot 0 \\ &= 0. \end{aligned}$$

Hence, $(\partial/\partial t)\gamma(\boldsymbol{s};t)$ vanishes at $t=0$.

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6.3. Proof of Theorem 3

The Fourier transform of the von Kármán–Whittle model $(\alpha\|\boldsymbol{s}\|)^{\nu}K_{\nu}(\alpha\|\boldsymbol{s}\|),\ \boldsymbol{s}\in\mathbb{R}^d$, is

$$\int_{\mathbb{R}^d}(\alpha\|\boldsymbol{s}\|)^{\nu}K_{\nu}(\alpha\|\boldsymbol{s}\|)\cos(\boldsymbol{s}^{\top}\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{s}=c_0\alpha^{2\nu}(\|\boldsymbol{\omega}\|^2+\alpha^2)^{-(\nu+d/2)},\qquad \boldsymbol{\omega}\in\mathbb{R}^d,$$

where c_0 is a positive constant not related to α (see Yaglom (1987, Equation (4.130)) or Stein (1999, p. 49)). Clearly, (11) is bounded and continuous on $\mathbb{R}^d\times\mathbb{R}$, with Fourier transform

$$\begin{aligned}f(\boldsymbol{\omega};\omega_0)&=\int_{\mathbb{R}^d\times\mathbb{R}}C(\boldsymbol{s};t)\exp\{\mathrm{i}(\boldsymbol{s}^{\top}\boldsymbol{\omega}+t\omega_0)\}\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}t\\&=c_0\{\theta\alpha_1^{2\nu_1}\beta_1^{2\nu_2}(\|\boldsymbol{\omega}\|^2+\alpha_1^2)^{-(\nu_1+d/2)}(\omega_0^2+\beta_1^2)^{-(\nu_2+1/2)}\\&\quad+(1-\theta)\alpha_2^{2\nu_1}\beta_2^{2\nu_2}(\|\boldsymbol{\omega}\|^2+\alpha_2^2)^{-(\nu_1+d/2)}(\omega_0^2+\beta_2^2)^{-(\nu_2+1/2)}\},\\&\qquad\qquad(\boldsymbol{\omega};\omega_0)\in\mathbb{R}^d\times\mathbb{R},\end{aligned}$$

where, now, c_0 is a positive constant not related to α_i or β_i ($i=1,2$). By Bochner's theorem and Fourier inversion, the condition for (11) to be a correlation function is the same as that for $f(\boldsymbol{\omega};\omega_0)$ to be nonnegative on $\mathbb{R}^d\times\mathbb{R}$. Thus, it suffices to show that (12) is a necessary and sufficient condition to have $f(\boldsymbol{\omega};\omega_0)\ge 0,\ (\boldsymbol{\omega};\omega_0)\in\mathbb{R}^d\times\mathbb{R}$.

Two simple necessary conditions are

$$f(\boldsymbol{0};0)\ge 0\quad\text{and}\quad\lim_{\|\boldsymbol{\omega}\|\to\infty,\ \omega_0\to\infty}f(\boldsymbol{\omega};\omega_0)\|\boldsymbol{\omega}\|^{2\nu_1+d}\omega_0^{2\nu_2+1}\ge 0,$$

which are equivalent to

$$\theta\alpha_1^{-d}\beta_1^{-1}+(1-\theta)\alpha_2^{-d}\beta_2^{-1}\ge 0 \tag{20}$$

and

$$\theta\alpha_1^{2\nu_1}\beta_1^{2\nu_2}+(1-\theta)\alpha_2^{2\nu_1}\beta_2^{2\nu_2}\ge 0. \tag{21}$$

Solving (20) and (21) simultaneously yields (12)

On the other hand, suppose that (12) holds or, equivalently, that (20) and (21) hold. If $(1-\alpha_2^d\beta_2/\alpha_1^d\beta_1)^{-1}\le\theta\le 0$, then $1-\theta\ge 0$ and

$$\begin{aligned}c_0^{-1}f(\boldsymbol{\omega};\omega_0)&=\theta\alpha_1^{-d}\beta_1^{-1}\left(\frac{\alpha_1^2}{\|\boldsymbol{\omega}\|^2+\alpha_1^2}\right)^{\nu_1+d/2}\left(\frac{\beta_1^2}{\omega_0^2+\beta_1^2}\right)^{\nu_2+1/2}\\&\quad+(1-\theta)\alpha_2^{-d}\beta_2^{-1}\left(\frac{\alpha_2^2}{\|\boldsymbol{\omega}\|^2+\alpha_2^2}\right)^{\nu_1+d/2}\left(\frac{\beta_2^2}{\omega_0^2+\beta_2^2}\right)^{\nu_2+1/2}\\&\ge\{\theta\alpha_1^{-d}\beta_1^{-1}+(1-\theta)\alpha_2^{-d}\beta_2^{-1}\}\left(\frac{\alpha_1^2}{\|\boldsymbol{\omega}\|^2+\alpha_1^2}\right)^{\nu_1+d/2}\left(\frac{\beta_1^2}{\omega_0^2+\beta_1^2}\right)^{\nu_2+1/2}\\&\ge 0,\end{aligned}$$

where the last inequality follows from (20) and the first inequality follows from

$$\frac{\alpha_1^2}{\|\boldsymbol{\omega}\|^2+\alpha_1^2}\le\frac{\alpha_2^2}{\|\boldsymbol{\omega}\|^2+\alpha_2^2}\quad\text{and}\quad\frac{\beta_1^2}{\omega_0^2+\beta_1^2}\le\frac{\beta_2^2}{\omega_0^2+\beta_2^2},\qquad(\boldsymbol{\omega};\omega_0)\in\mathbb{R}^d\times\mathbb{R}.$$

If $0 \le \theta \le \{1-(\alpha_1/\alpha_2)^{2\nu_1}(\beta_1/\beta_2)^{2\nu_2}\}^{-1}$, we obtain $f(\boldsymbol{\omega};\omega_0) \ge 0$, $(\boldsymbol{\omega};\omega_0) \in \mathbb{R}^d \times \mathbb{R}$, from

$$\begin{aligned} c_0^{-1} f(\boldsymbol{\omega};\omega_0)(\|\boldsymbol{\omega}\|^2+\alpha_2^2)^{\nu_1+d/2}(\omega_0^2+\beta_2^2)^{\nu_2+1/2} \\ &= \theta\alpha_1^{2\nu_1}\beta_1^{2\nu_2}\left(\frac{\|\boldsymbol{\omega}\|^2+\alpha_2^2}{\|\boldsymbol{\omega}\|^2+\alpha_1^2}\right)^{\nu_1+d/2}\left(\frac{\omega_0^2+\beta_2^2}{\omega_0^2+\beta_1^2}\right)^{\nu_2+1/2} + (1-\theta)\alpha_2^{2\nu_1}\beta_2^{2\nu_2} \\ &\ge \theta\alpha_1^{2\nu_1}\beta_1^{2\nu_2}\cdot 1 + (1-\theta)\alpha_2^{2\nu_1}\beta_2^{2\nu_2} \\ &\ge 0, \end{aligned}$$

where the last inequality follows from (21).

6.4. Proof of Theorem 4

Part (i) is a special case of Theorem 5.

To prove part (ii) notice that, from the proof of Theorem 3, the Fourier transform of (11) is a positive multiple of (16) with $(\boldsymbol{s};t)$ replaced by $(\boldsymbol{\omega};\omega_0)$. Thus, the inverse Fourier transform of (16), $f(\boldsymbol{\omega};\omega_0)$, is a positive multiple of (11) with $(\boldsymbol{s};t)$ replaced by $(\boldsymbol{\omega};\omega_0)$, i.e.

$$\begin{aligned} c_1 f(\boldsymbol{\omega};\omega_0) &= \theta(\alpha_1\|\boldsymbol{\omega}\|)^{\nu_1}K_{\nu_1}(\alpha_1\|\boldsymbol{\omega}\|)(\beta_1|\omega_0|)^{\nu_2}K_{\nu_2}(\beta_1|\omega_0|) \\ &\quad + (1-\theta)(\alpha_2\|\boldsymbol{\omega}\|)^{\nu_1}K_{\nu_1}(\alpha_2\|\boldsymbol{\omega}\|)(\beta_2|\omega_0|)^{\nu_2}K_{\nu_2}(\beta_2|\omega_0|), \\ &\qquad (\boldsymbol{\omega};\omega_0) \in \mathbb{R}^d \times \mathbb{R}, \end{aligned}$$

where c_1 is a positive constant. Suppose that (16) is a covariance. Then $f(\boldsymbol{\omega};\omega_0)$ is nonnegative for all $(\boldsymbol{\omega};\omega_0) \in \mathbb{R}^d \times \mathbb{R}$ and, thus,

$$\theta + (1-\theta)\frac{(\alpha_2\|\boldsymbol{\omega}\|)^{\nu_1}K_{\nu_1}(\alpha_2\|\boldsymbol{\omega}\|)(\beta_2|\omega_0|)^{\nu_2}K_{\nu_2}(\beta_2|\omega_0|)}{(\alpha_1\|\boldsymbol{\omega}\|)^{\nu_1}K_{\nu_1}(\alpha_1\|\boldsymbol{\omega}\|)(\beta_1|\omega_0|)^{\nu_2}K_{\nu_2}(\beta_1|\omega_0|)} \ge 0$$

or, equivalently,

$$\theta + (1-\theta)\left(\frac{\alpha_2}{\alpha_1}\right)^{\nu_1}\frac{K_{\nu_1}(\alpha_2\|\boldsymbol{\omega}\|)}{K_{\nu_1}(\alpha_1\|\boldsymbol{\omega}\|)}\left(\frac{\beta_2}{\beta_1}\right)^{\nu_2}\frac{K_{\nu_2}(\beta_2|\omega_0|)}{K_{\nu_2}(\beta_1|\omega_0|)} \ge 0, \qquad (\boldsymbol{\omega};\omega_0) \in \mathbb{R}^d \times \mathbb{R}. \quad (22)$$

According to Kent (1978) and Ismail and Kelker (1979), the functions

$$\left(\frac{\alpha_2}{\alpha_1}\right)^{\nu_1}\frac{K_{\nu_1}(\alpha_2 x^{1/2})}{K_{\nu_1}(\alpha_1 x^{1/2})}, \quad x \ge 0, \qquad \text{and} \qquad \left(\frac{\beta_2}{\beta_1}\right)^{\nu_2}\frac{K_{\nu_2}(\beta_2 x^{1/2})}{K_{\nu_2}(\beta_1 x^{1/2})}, \quad x \ge 0,$$

are the Laplace transforms of infinitely divisible probability distributions, and consequently tend to 0 as x approaches infinity. Finally, we obtain $\theta \ge 0$ from (22) by letting $\|\boldsymbol{\omega}\| \to \infty$ and $|\omega_0| \to \infty$.

6.5. Proof of Theorem 5

By assumption, we may write $f_{\mathrm{S}}(\|\boldsymbol{\omega}\|)$ and $f_{\mathrm{T}}(|\omega_0|)$ for the Fourier transforms of $C_{\mathrm{S}}(\boldsymbol{s})$ and $C_{\mathrm{T}}(t)$, respectively, where $f_{\mathrm{S}}(x)$ and $f_{\mathrm{T}}(x)$ are decreasing on $[0,\infty)$. Then, for $k = 1,\dots,p$,

$$\int_{\mathbb{R}^d} C_{\mathrm{S}}\left(\frac{\boldsymbol{s}}{\alpha_k}\right)\exp(\mathrm{i}\boldsymbol{s}^\top\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{s} = \alpha_k^d f_{\mathrm{S}}(\alpha_k\|\boldsymbol{\omega}\|), \qquad \int_{\mathbb{R}} C_{\mathrm{T}}\left(\frac{t}{\beta_k}\right)\exp(\mathrm{i}t\omega)\,\mathrm{d}t = \beta_k f_{\mathrm{T}}(\beta_k|\omega_0|).$$

The Fourier transform of (17) is

$$f(\boldsymbol{\omega};\omega_0) = \sum_{k=1}^{p}\theta_k f_{\mathrm{S}}(\alpha_k\|\boldsymbol{\omega}\|)f_{\mathrm{T}}(\beta_k|\omega_0|), \qquad (\boldsymbol{\omega};\omega_0) \in \mathbb{R}^d \times \mathbb{R}.$$

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An alternative expression is obtained by applying Abel's lemma, which gives

$$f(\boldsymbol{\omega}; \omega_0) = f_{\mathrm{S}}(\alpha_p \|\boldsymbol{\omega}\|) f_{\mathrm{T}}(\beta_p |\omega_0|) \sum_{i=1}^{p} \theta_i + \sum_{k=1}^{p-1} \left(\sum_{i=1}^{k} \theta_i \right) \{ f_{\mathrm{S}}(\alpha_k \|\boldsymbol{\omega}\|) f_{\mathrm{T}}(\beta_k |\omega_0|) - f_{\mathrm{S}}(\alpha_{k+1} \|\boldsymbol{\omega}\|) f_{\mathrm{T}}(\beta_{k+1} |\omega_0|) \}.$$

This is nonnegative on $\mathbb{R}^d \times \mathbb{R}$, under the assumptions of Theorem 5.

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