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An order property of partition cardinals

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This note studies cardinal numbers κ which have a partition property which amounts to the following. Let ν be a cardinal, η an ordinal limit number and m a positive integer. Let the m-length sequences of finite subsets of κ be partitioned into ν parts. Then there is a sequence $H_1, \ldots, H_m$ of subsets of κ, each having order type η, such that for each choice of non-zero numbers $n_1, \ldots, n_m$ there is some class of the partition inside which fall all sequences having in their i-th place (for $i = 1, \ldots, m$) a subset of H_i which contains exactly n_i elements. The case when $m = 1$ is thus seen to be the well known property $\kappa \to (\eta)_\nu^{<\omega}$. The most interesting results obtained relate to the ordering of the least cardinals with the appropriate properties as m and η vary.

In order to define the partition property to be discussed, the following notation is helpful. Let S be any set, and let m be any positive integer. Then $[S]^m$ denotes the set of those subsets of S which have exactly m elements, ${}^mS$ denotes the set of m-place sequences with values in S and S^{*m} denotes the set $\cup\left\{{}^m\left([S]^n\right);\ n = 1, 2, 3, \ldots\right\}$. The set of all finite subsets of S is denoted $[S]^{<\omega}$. Cardinal numbers are identified with the initial ordinals.

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DEFINITION 1. *Let* κ *and* ν *be cardinals, let* η *be an ordinal limit number and let* m *be a positive integer. Suppose that for any partition* $\Delta = \{\Delta_l \; ; \; l < \nu\}$ *of* κ^{*m} *into* ν *parts the following situation prevails. There is a sequence* $H_1, \ldots, H_m$ (*where each* H_i *is a subset of* κ *having order type* η) *which is homogeneous for* Δ, *in the sense that for each* n *there is* l *less than* ν *such that* $[H_1]^n \times \ldots \times [H_m]^n \subseteq \Delta_l$. *In this case, we say that* κ *has the partition property* $\kappa \to {}^m(\eta)_\nu^{<\omega}$.

The case when $m = 1$ has been extensively studied in the literature (for example, [1] and [4]). In [4], cardinals having such a partition property where η is a cardinal are referred to as Erdös cardinals. Cardinals with the property $\kappa \to {}^1(\kappa)_2^{<\omega}$ are known as Ramsey cardinals.

In this note, some consequences of the partition properties in which $m > 1$ will be listed. These mainly parallel the case when $m = 1$, and for the most part proofs will be merely sketched. The results of greatest interest are Theorems 5 and 7, which deal with the ordering of the various partition cardinals.

Definition 1 is restricted to a consideration of those sequences of finite subsets of κ for which all entries in the sequence have the same number of elements. This is an unnecessary restriction, as the result of Theorem 3 shows. The following lemma is needed.

LEMMA 2. *Let* η *be a limit ordinal, and suppose* $\kappa \to {}^m(\eta)_2^{<\omega}$. *Then* $\kappa \to {}^m(\eta)_{2^{\aleph_0}}^{<\omega}$.

The case $m = 1$ is a theorem of Rowbottom [3].

THEOREM 3. *Let* η *be a limit ordinal; let* $\kappa \to {}^m(\eta)_\nu^{<\omega}$. *Then for any partition* $\Delta = \{\Delta_l \; ; \; l < \nu\}$ *of* ${}^m([\kappa]^{<\omega})$ *there is a sequence* $H_1, \ldots, H_m$ *where each* H_i *is a subset of* κ *having order type* η, *which is homogeneous for* Δ *in the extended sense, that is, for each sequence* $n_1, \ldots, n_m$ *where each* n_i *is non-zero, there is* l *less than*

ν *for which* $[H_1]^{n_1} \times \ldots \times [H_m]^{n_m} \subseteq \Delta_l$.

Proof. Let a partition $\Delta = \{\Delta_l \; ; \; l < \nu\}$ of ${}^m([\kappa]^{<\omega})$ be given. For each positive n , put

$$F_n = \left\{f \; ; \; f \textit{ maps } \{1, \ldots, m\} \textit{ into } \{1, \ldots, n\}\right\} .$$

For each f in F_n define a partition $\Gamma(f) = \{\Gamma_l(f) \; ; \; l < \nu\}$ of ${}^m([\kappa]^n)$ by:

if $\alpha_1(i) < \ldots < \alpha_n(i) < \kappa$ for $i = 1, \ldots, m$ then

$$\left\langle\{\alpha_1(i), \ldots, \alpha_n(i)\}; \; i = 1, \ldots, m\right\rangle \in \Gamma_l(f) \Leftrightarrow$$
$$\left\langle\{\alpha_1(i), \ldots, \alpha_{f(i)}(i)\}; \; i = 1, \ldots, m\right\rangle \in \Delta_l .$$

Then $\Gamma = \left\{\Gamma_l(f) \; ; \; l < \nu \textit{ and } \exists n\left(f \in F_n\right)\right\}$ is a partition of κ^{*m} , and Γ has power at most $\nu \times \aleph_0$. By virtue of the property $\kappa \to {}^m(\eta)^{<\omega}_\nu$, with an appeal to Lemma 2 if ν is finite, it follows that there is a sequence $H_1, \ldots, H_m$ (each H_i a subset of κ having order type η) which is homogeneous for Γ . It is not difficult to see that this same sequence $H_1, \ldots, H_m$ is also homogeneous for Δ in the extended sense. This proves Theorem 3.

The question of the existence of cardinals with the property of Definition 1 is of some interest. The following theorem provides for their existence.

THEOREM 4. *Let* ν *be a cardinal and* η *a limit ordinal. Suppose* κ *is a cardinal such that* $\kappa \to {}^1(\eta.m)^{<\omega}_\nu$. *Then* $\kappa \to {}^m(\eta)^{<\omega}_\nu$.

The proof is obtained by taking any partition $\Delta = \{\Delta_l \; ; \; l < \nu\}$ of κ^{*m} , and defining from this a partition $\Gamma = \{\Gamma_l \; ; \; l < \nu\}$ of $[\kappa]^{<\omega}$ such that if $\langle a_1, \ldots, a_m\rangle$ from κ^{*m} is such that $\max(a_i) < \min(a_{i+1})$

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for each i , then

$$\langle a_1, \ldots, a_m \rangle \in \Delta_l \Longleftrightarrow a_1 \cup \ldots \cup a_m \in \Gamma_l .$$

Any subset H of κ having order type $\eta.m$ which is homogeneous for Γ may be divided $H = H_1 \cup \ldots \cup H_m$ where each H_i has order type η and $\sup(H_i) < \min(H_{i+1})$. But then $H_1, \ldots, H_m$ is a sequence homogeneous for Δ .

Thus in particular, if $\kappa \to {}^1(\aleph_1)_\nu^{<\omega}$ then $\kappa \to {}^m(\aleph_0)_\nu^{<\omega}$ for all m . If κ is Ramsey, then $\kappa \to {}^m(\eta)_2^{<\omega}$ for all m and all η less than κ . In fact, this last is the best that can be hoped for. No cardinal has even the property $\kappa \to {}^2(\kappa)_2^{<\omega}$, as may be seen by considering any partition $\Delta = \{\Delta_0, \Delta_1\}$ of κ^{*2} in which

$$\langle \{\alpha\}, \{\beta\} \rangle \in \Delta_0 \Longleftrightarrow \alpha \leq \beta .$$

We come now to the two main theorems, concerning the ordering of these partition cardinals amongst themselves. The first theorem may be stated immediately.

THEOREM 5. *Let* ζ *and* η *be limit ordinals such that* $\eta.m < \zeta$. *Let* κ *be the least cardinal such that* $\kappa \to {}^m(\eta)_\nu^{<\omega}$, *and let* λ *be any cardinal such that* $\lambda \to {}^1(\zeta)_\nu^{<\omega}$. *Then* $\kappa < \lambda$.

Proof. Let κ and λ be as mentioned in the theorem. By Theorem 4, $\kappa \leq \lambda$. Suppose that in fact $\kappa \to {}^1(\zeta)_\nu^{<\omega}$, and seek a contradiction.

Since any β in κ has power less than κ , there is a partition $\Delta(\beta) = \{\Delta_l(\beta) \ ; \ l < \nu\}$ of β^{*m} which has no homogeneous sequence $H_1, \ldots, H_m$ where each H_i has order type at least η . Take any partition $\Gamma = \{\Gamma_l \ ; \ l < \nu\}$ of $[\kappa]^{<\omega}$ which has the following property: for all n and for all l less than ν , if a_i is in $[\kappa]^n$ (for $i = 1, \ldots, m$) and $\max(a_i) < \min(a_{i+1})$, $\max(a_m) < \alpha < \kappa$, then

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$$a_1 \cup \ldots \cup a_m \cup \{\alpha\} \in \Gamma_l \Leftrightarrow \langle a_1, \ldots, a_m \rangle \in \Delta_l(\alpha) .$$

By virtue of the assumption $\kappa \rightarrow {}^1(\zeta)_\nu^{<\omega}$, there is a subset H of κ having order type ζ, which is homogeneous for Γ. However, for each α in H, let $\{\alpha' \in H ; \alpha' < \alpha\}$ be split into m sets $H_1(\alpha), \ldots, H_m(\alpha)$ all having the same order type, such that $\sup(H_i(\alpha)) < \min(H_{i+1}(\alpha))$. (Any elements of $\{\alpha' \in H ; \alpha' < \alpha\}$ left over may be ignored.) Then $H_1(\alpha), \ldots, H_m(\alpha)$ is a sequence homogeneous for $\Delta(\alpha)$, and so each $H_i(\alpha)$ has order type less than η. From this it follows that the order type of $\{\alpha' \in H ; \alpha' < \alpha\}$ is smaller than $\eta.m$. Hence the order type of H does not exceed $\eta.m$. This contradicts the order type of H being ζ. The proof is complete.

The following lemma is required in order to establish the effect of changing the value of m.

LEMMA 6. *Let* $\kappa \rightarrow {}^m(\eta)_\nu^{<\omega}$ *where* η *is a limit ordinal. Then for any partition* $\Delta = \{\Delta_l ; l < \nu\}$ *of* κ^{*m} *there is a sequence* $H_1, \ldots, H_m$ *homogeneous for* Δ, (*where each* H_i *is a subset of* κ *having order type* η) *for which there is some permutation* σ *of* $\{1, \ldots, m\}$ *such that* $\sup(H_{\sigma(i)}) \leq \min(H_{\sigma(i+1)})$ *for each* i.

The proof depends on Lemma 2 in the case that ν is finite. For details, see [5].

THEOREM 7. *Let* η *be a limit ordinal. Suppose* κ *is the least cardinal such that* $\kappa \rightarrow {}^m(\eta)_\nu^{<\omega}$ *and let* λ *be any cardinal with the property* $\lambda \rightarrow {}^{m+1}(\eta)_\nu^{<\omega}$. *Then* $\kappa < \lambda$.

Proof. Let κ and λ be as mentioned in the theorem. Clearly $\kappa \leq \lambda$. Suppose that κ does have also the property $\kappa \rightarrow {}^{m+1}(\eta)_\nu^{<\omega}$, and seek a contradiction.

As in the proof of Theorem 5, for each β in κ there is a partition $\Delta(\beta) = \{\Delta_l(\beta)\ ;\ l < \nu\}$ of β^{*m} which has no homogeneous sequence $H_1, \ldots, H_m$ where each H_i has order type at least η.

Choose a partition $\Gamma = \{\Gamma_l\ ;\ l < \nu\}$ of κ^{*m+1} which satisfies the following condition: if $\beta_1(i) < \ldots < \beta_n(i)$ for $i = 1, \ldots, m+1$, and there is some j for which $\beta_1(j) > \max\left\{\beta_n(i)\ ;\ i \in \{1, \ldots, m+1\} - \{j\}\right\}$, then for each l less than ν

$$\left\langle\{\beta_1(i), \ldots, \beta_n(i)\}\ ;\ i \in \{1, \ldots, m+1\}\right\rangle \in \Gamma_l \Leftrightarrow$$
$$\left\langle\{\beta_1(i), \ldots, \beta_n(i)\}\ ;\ i \in \{1, \ldots, m+1\} - \{j\}\right\rangle \in \Delta_l(\beta_1(j))\ .$$

Let $H_1, \ldots, H_{m+1}$ be any sequence homogeneous for Γ, where each H_i is a subset of κ having order type η. By Lemma 6, it may be assumed further that in some reordering the H_i are increasing - suppose in fact that always $\sup\left(H_i\right) \leq \min\left(H_{i+1}\right)$. Then if α is the least element of H_{m+1}, it follows that the sequence $H_1, \ldots, H_m$ is homogeneous for $\Delta(\alpha)$. Moreover each H_i (for $i = 1, \ldots, m$) is contained in α, and has order type η. However, this contradicts the choice of the partition $\Delta(\alpha)$. The theorem is thus proved.

Theorems 5 and 7 suffice to determine completely the ordering of the smallest cardinals with the properties $\kappa \to {}^m(\eta)_\nu^{<\omega}$ for η an ordinal power of ω. For any given ν, the least cardinals with the following properties form a strictly increasing sequence:

$$\kappa \to {}^1(\omega)_\nu^{<\omega},\ \kappa \to {}^2(\omega)_\nu^{<\omega},\ \kappa \to {}^3(\omega)_\nu^{<\omega},\ \ldots,\ \kappa \to {}^1(\omega^2)_\nu^{<\omega}\ ,$$
$$\kappa \to {}^2(\omega^2)_\nu^{<\omega},\ \kappa \to {}^3(\omega^2)_\nu^{<\omega},\ \ldots,\ \kappa \to {}^1(\omega_1)_\nu^{<\omega}\ ,$$
$$\kappa \to {}^2(\omega_1)_\nu^{<\omega},\ \kappa \to {}^3(\omega_1)_\nu^{<\omega},\ \ldots,\ \kappa \to {}^1(\kappa)_\nu^{<\omega}\ .$$

In the case that η is not an ordinal power of ω, some questions remain. The first such questions are the following:

PROBLEM 8. *Let* κ *be the least cardinal with the property*

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$\kappa \to {}^1(\omega.2)^{<\omega}_\nu$, *let* λ *be the least cardinal with the property* $\lambda \to {}^2(\omega)^{<\omega}_\nu$ *and let* ι *be the least cardinal with the property* $\iota \to {}^3(\omega)^{<\omega}_\nu$. *Is it true that* $\lambda < \kappa$ *? Or that* $\kappa < \iota$ *?*

The effect of varying ν is also not determined. For the case $m = 1$, Silver [4] has provided a complete solution with the following theorem.

THEOREM 9. *Let* η *be a limit ordinal and let* κ *be the least cardinal such that* $\kappa \to {}^1(\eta)^{<\omega}_2$. *Take any* ν *less than* κ . *Then* $\kappa \to {}^1(\eta)^{<\omega}_\nu$.

I do not know if this theorem holds for the case $m > 1$.

I conclude this note with a few comments concerning the inaccessibility of the various partition cardinals. For the case $m = 1$, the following theorem has been proved by Silver [4, p. 84].

THEOREM 10. *If* η *is a limit ordinal and if* κ *is least such that* $\kappa \to {}^1(\eta)^{<\omega}_2$, *then* κ *is strongly inaccessible.*

For the case $m > 1$, I can prove only a weaker version. Standard arguments (for example [2]) can be extended to yield:

THEOREM 11. *Let* η *be a cardinal, and let* κ *be the least cardinal such that* $\kappa \to {}^m(\eta)^{<\omega}_2$. *Then* κ *is strongly inaccessible.*

Of course, Theorem 11 holds for η any limit ordinal, and likewise Theorem 9 holds for $m > 1$, if Problem 8 leads to the trivial solution that a cardinal has the property $\kappa \to {}^m(\eta)^{<\omega}_2$ only by virtue of having the property $\kappa \to {}^1(\eta.m)^{<\omega}_2$.

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