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ON SOME FRACTIONAL INTEGRALS AND THEIR APPLICATIONS

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1. The symmetric operators

In previous papers [3, 4] the author has discussed the symmetric generalised Erdélyi–Kober operators of fractional integration defined by

$$\mathfrak{I}_{\lambda}(\eta,\alpha)F(x)=2^{\alpha}\lambda^{1-\alpha}x^{-2(\alpha+\eta)}\int_{0}^{x}u^{2\eta+1}(x^2-u^2)^{(\alpha-1)/2}J_{\alpha-1}\{\lambda\sqrt{(x^2-u^2)}\}F(u)\,du, \tag{1}$$

$$\mathfrak{R}_{\lambda}(\eta,\alpha)F(x)=2^{\alpha}\lambda^{1-\alpha}x^{2\eta}\int_{x}^{\infty}u^{1-2(\alpha+\eta)}(u^2-x^2)^{(\alpha-1)/2}J_{\alpha-1}\{\lambda\sqrt{(u^2-x^2)}\}F(u)\,du, \tag{2}$$

where $\alpha>0$, $\lambda\geqq 0$ and the operators $\mathfrak{I}_{i\lambda}(\eta,\alpha)$ and $\mathfrak{R}_{i\lambda}(\eta,\alpha)$ defined as in equations (1) and (2) respectively but with $J_{\alpha-1}$, the Bessel function of the first kind replaced by $I_{\alpha-1}$, the modified Bessel function of the first kind.

In this paper we introduce two new operators of fractional integration and discuss some of their properties together with a number of their applications.

2. The unsymmetric operators

In the definitions (1) and (2) $\lambda\geqq 0$ is a constant. If we now set $\lambda=kx$, $k\geqq 0$ we find, after a simple change of variables, that they become the unsymmetric operators defined by

$$I_k(\eta,\alpha)f(x)=2^{\alpha-1}k^{1-\alpha}x^{-(\alpha+\eta)}\int_{0}^{x}u^{\eta}\left[\frac{x-u}{x}\right]^{(\alpha-1)/2}J_{\alpha-1}\{k\sqrt{(x^2-xu)}\}f(u)\,du, \tag{3}$$

$$K_k(\eta,\alpha)f(x)=2^{\alpha-1}k^{1-\alpha}x^{\eta}\int_{x}^{\infty}u^{-(\alpha+\eta)}\left[\frac{u-x}{x}\right]^{(\alpha-1)/2}J_{\alpha-1}\{k\sqrt{(xu-x^2)}\}f(u)\,du, \tag{4}$$

where $\alpha>0$ and the operators $I_{ik}(\eta,\alpha)$ and $K_{ik}(\eta,\alpha)$ defined by equations (3) and (4) respectively with $J_{\alpha-1}$ replaced by $I_{\alpha-1}$.

When $k=0$ the above operators reduce to the familiar Erdélyi–Kober operators of

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fractional integration given by

$$I_0(\eta,\alpha)f(x)=I_x^{\eta,\alpha}f(x)=x^{-(\alpha+\eta)}I_x^{\alpha}x^{\eta}f(x), \tag{5}$$

$$K_0(\eta,\alpha)f(x)=K_x^{\eta,\alpha}f(x)=x^{\eta}K_x^{\alpha}x^{-(\alpha+\eta)}f(x), \tag{6}$$

where

$$I_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-u)^{\alpha-1}f(u)\,du, \quad \alpha>0, \tag{7}$$

$$K_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(u-x)^{\alpha-1}f(u)\,du, \quad \alpha>0. \tag{8}$$

From the definitions (3) to (8) it can easily be shown that the unsymmetric operators have the following properties,

$$I_p(\eta,\alpha)x^{\sigma}f(x)=x^{\sigma}I_p(\eta+\sigma,\alpha)f(x), \tag{9}$$

$$K_p(\eta,\alpha)x^{\sigma}f(x)=x^{\sigma}K_p(\eta-\sigma,\alpha)f(x), \tag{10}$$

$$I_p(\eta,\alpha)I_x^{\eta-\beta,\beta}f(x)=I_p(\eta-\beta,\alpha+\beta)f(x), \tag{11}$$

$$K_p(\eta,\alpha)K_x^{\eta+\alpha,\beta}f(x)=K_p(\eta,\alpha+\beta)f(x), \tag{12}$$

where $\alpha,\beta>0$ and $p=k$ or $p=ik$, $k\geqq 0$.

In this paper we shall confine our attention to the operators $I_p(\eta,\alpha)$ and postpone a consideration of the operators $K_p(\eta,\alpha)$ until a later date.

3. The operators $I_p(\eta,\alpha)$, $\alpha>0$

We shall now show that there is a useful connection between the operators $I_p(\eta,\alpha)$ and the differential operator $M_{\gamma}^{(x)}$ defined by

$$M_{\gamma}^{(x)}=x^{-(\gamma-1)}Dx^{\gamma+1}D=x^2D^2+x(\gamma+1)D, \tag{13}$$

where

$$D=\frac{d}{dx}.$$

Theorem 1. *If* $\alpha>0$, $f\in C^2(0,b)$, $b>0$, $x^{\eta+m}D^m f(x)$, $m=0,1,2$ *are integrable at the origin and* $x^{\eta+1}f(x)\to 0$ *as* $x\to 0+$; *then*

$$I_p(\eta,\alpha)M_{2(\alpha+\eta)}^{(x)}f(x)=[M_{2(\alpha+\eta)}^{(x)}+(px)^2]I_p(\eta,\alpha)f(x), \quad x>0, \tag{14}$$

where $p=k$ *or* $p=ik$, $k\geqq 0$.

Proof. We set

$$H(x)=I_k(\eta,\alpha)f(x)=2^{\alpha-1}(kx)^{1-\alpha}\int_0^1 t^{\eta}(1-t)^{(\alpha-1)/2}J_{\alpha-1}(\xi)f(xt)\,dt, \tag{15}$$

where $\alpha>0$ and $\xi=kx\sqrt{(1-t)}$.

Since $H(x)$ is differentiable we have

$$H'(x)=-2^{\alpha-1}k(kx)^{1-\alpha}\int_0^1 t^{\eta}(1-t)^{\alpha/2}J_{\alpha}(\xi)f(xt)\,dt$$
$$+2^{\alpha-1}(kx)^{1-\alpha}\int_0^1 t^{1+\eta}(1-t)^{(\alpha-1)/2}J_{\alpha-1}(\xi)f'(xt)\,dt. \tag{16}$$

An application of the operator $x^{-(\eta-1)}Dx^{\eta+1}$ to both sides of the above equation yields the expression

$$M_{\eta}^{(x)}H(x)=I_k(\eta,\alpha)M_{\eta}^{(x)}f(x)$$
$$+2^{\alpha-1}(kx)^{2-\alpha}\int_0^1 t^{\eta}(1-t)^{\alpha/2}[\xi J_{\alpha+1}(\xi)-(\eta+2)J_{\alpha}(\xi)]f(xt)\,dt$$
$$-2^{\alpha}x(kx)^{2-\alpha}\int_0^1 t^{1+\eta}(1-t)^{\alpha/2}J_{\alpha}(\xi)f'(xt)\,dt. \tag{17}$$

Integrating the last integral by parts and noting that by assumption the integrated part vanishes, we find, after some manipulation, that equation (17) can be brought to the form

$$M_{\eta}^{(x)}H(x)+(kx)^2H(x)=I_k(\eta,\alpha)M_{\eta}^{(x)}f(x)$$
$$+2^{\alpha-1}(kx)^{2-\alpha}(2\alpha+\eta)\int_0^1 t^{\eta}(1-t)^{\alpha/2}J_{\alpha}(\xi)f(xt)\,dt. \tag{18}$$

Finally, on using equation (16), we have

$$M_{\eta}^{(x)}H(x)+x(2\alpha+\eta)H'(x)+(kx)^2H(x)=I_k(\eta,\alpha)[M_{\eta}^{(x)}f(x)+x(2\alpha+\eta)f'(x)], \tag{19}$$

which is the required result.

Similarly we can prove the theorem when $p=ik$.

4. Applications

(*a*) As a first example we consider the generalised biaxially symmetric potential equation (GBSPE)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\alpha}{x}\frac{\partial u}{\partial x}+\frac{2\beta}{y}\frac{\partial u}{\partial y}=0, \quad \alpha,\beta>0 \tag{20}$$

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and confine our attention to solutions $u(x,y)\in C^2$ in some neighbourhood of the origin that are even in x and y. In this case we must have $u_x(0,y)=u_y(x,0)=0$.

Expressed in polar coordinates $x=r\cos\theta, y=r\sin\theta$, with $u=u(r,\theta)$, the above equation is

$$M^{(r)}_{2(\alpha+\beta)}u+\frac{\partial^2 u}{\partial\theta^2}+2(\beta\cot\theta-\alpha\tan\theta)\frac{\partial u}{\partial\theta}=0, \tag{21}$$

where

$$M^{(r)}_{\gamma}=r^2\frac{\partial^2}{\partial r^2}+r(\gamma+1)\frac{\partial}{\partial r}.$$

On separating the variables we find that a complete set of solutions of equation (21) that are analytic in a neighbourhood of the origin is given by

$$u_n(r,\theta)=a_n r^{2n}P_n^{(\beta-1/2,\alpha-1/2)}(\cos 2\theta),\quad n=0,1,2,\ldots, \tag{22}$$

where the $P_n^{(a,b)}(\xi)$ are the Jacobi polynomials [5] and the a_n are constants.

In order to obtain a complete set of solutions of the corresponding generalised biaxially symmetric Helmholtz equation (GBSHE)

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{2\alpha}{x}\frac{\partial v}{\partial x}+\frac{2\beta}{y}\frac{\partial v}{\partial y}+k^2 v=0,\quad k\geqq 0, \tag{23}$$

we can use the result of Theorem 1 in the following way.

Applying the operator $I_k(\beta,\alpha)$ to equations (21) and (22) we find that a complete set of solutions of the GBSHE

$$M^{(r)}_{2(\alpha+\beta)}v+\frac{\partial^2 v}{\partial\theta^2}+2(\beta\cot\theta-\alpha\tan\theta)\frac{\partial v}{\partial\theta}+(kr)^2 v=0, \tag{24}$$

that are analytic about the origin, is given by

$$v_n(r,\theta)=I_k(\beta,\alpha)u_n(r,\theta)=a_n P_n^{(\beta-1/2,\alpha-1/2)}(\cos 2\theta)I_k(\beta,\alpha)r^{2n} \tag{25}$$

On using the definition (3) we have that

$$\begin{aligned}I_k(\beta,\alpha)r^{2n}&=2^{\alpha-1}r^{2n}(kr)^{1-\alpha}\int_0^1 t^{\beta+2n}(1-t)^{(\alpha-1)/2}J_{\alpha-1}\{kr\sqrt{(1-t)}\}\,dt\\&=2^{\alpha}r^{2n}(kr)^{1-\alpha}\int_0^{\pi/2} J_{\alpha-1}(kr\sin\phi)\sin^{\alpha}\phi(\cos\phi)^{4n+2\beta+1}\,d\phi\\&=\left(\frac{2}{k}\right)^{2n+\alpha+\beta}\Gamma(2n+\beta+1)r^{-\alpha-\beta}J_{2n+\alpha+\beta}(kr),\end{aligned} \tag{26}$$

where the integral has been evaluated by using a result in [5].

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In this way we find that the required set of solutions of the GBSHE is

$$v_n(r,\theta)=A_n r^{-\alpha-\beta}J_{2n+\alpha+\beta}(kr)P_n^{(\beta-1/2,\alpha-1/2)}(\cos 2\theta),\quad n=0,1,2,\dots, \tag{27}$$

where the A_n are constants and this agrees with the result found in [**1**].

(*b*) We next turn our attention to the generalised axially symmetric potential equation in $(n+1)$-variables (GASPEN)

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+\frac{\partial^2 u}{\partial \rho^2}+\frac{s}{\rho}\frac{\partial u}{\partial \rho}=0,\quad s>-1. \tag{28}$$

Introducing the zonal coordinates

$$x_i=r\theta_i,\ i=1,2,\dots,n;\quad \rho=r\left[1-\sum_{i=1}^{n}\theta_i^2\right]^{1/2};\quad r^2=\rho^2+\sum_{i=1}^{n}x_i^2, \tag{29}$$

we see that the GASPEN becomes [**1**]

$$M_{n+s-1}^{(r)}u+n(s-1)u+\sum_{i=1}^{n}\frac{\partial}{\partial\theta_i}\left\{\frac{\partial u}{\partial\theta_i}-\theta_i\left[\sum_{k=1}^{n}\theta_k\frac{\partial u}{\partial\theta_k}+(s-1)u\right]\right\}=0, \tag{30}$$

where $u=u(r;\theta)$ and $\theta=(\theta_1,\theta_2,\dots,\theta_n)$.

By separating the variables it can be shown that the family of functions

$$u_M(r;\theta)=b_M r^{\mu}V_M^{(s)}(\theta), \tag{31}$$

where the b_M are constants, form a complete system of solutions of the GASPEN which is analytic about $r=0$.

The $V_M^{(s)}(\theta)$ are polynomial functions uniquely determined by their generating function

$$[1-2(a,\theta)+\|a\|^2]^{1/2(1-n-s)}=\sum_{M=0}^{\infty}a^M V_M^{(s)}(\theta), \tag{32}$$

where

$$(a,\theta)=\sum_{i=1}^{n}a_i\theta_i,\ \|a\|^2=(a,a),\ a^M=\prod_{i=1}^{n}a_i^{m_i}$$

$$M=(m_1,m_2,\dots,m_n),\ \sum_{M=0}^{\infty}=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_n=0}^{\infty},$$

and

$$\mu=|M|=m_1+m_2+\cdots+m_n. \tag{33}$$

Applying the operator $I_k(-\tfrac{1}{2},\tfrac{1}{2}n+\tfrac{1}{2}s)$ to equations (30) and (31) and using Theorem 1

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we find that the solutions of the generalised axially symmetric Helmholtz equation in $(n+1)$-variables (GASHEN)

$$\sum_{i=1}^{n}\frac{\partial^2 w}{\partial x_i^2}+\frac{\partial^2 w}{\partial \rho^2}+\frac{s}{\rho}\frac{\partial w}{\partial \rho}+k^2 w=0, \quad k\geqq 0, \quad s>-1, \tag{34}$$

which are analytic in a neighbourhood of $r=0$, are of the form

$$\begin{aligned} w_M(r;\theta) &= b_M V_M^{(s)}(\theta) I_k(-\tfrac{1}{2}, \tfrac{1}{2}n+\tfrac{1}{2}s) r^{\mu} \\ &= B_M V_M^{(s)}(\theta) r^{-(n+s-1)/2} J_{\mu+1/2(n+s-1)}(kr), \end{aligned} \tag{35}$$

where the B_M are constants.

5. The operators $I_p(\eta,\alpha)$, $\alpha\leqq 0$

To obtain expressions for the operators $I_p(\eta,\alpha)$ when α is zero or negative, we write

$$I_x^{\eta-\beta,\beta} f(x)=g(x), \; f(x)=I_x^{\eta,-\beta} g(x), \tag{36}$$

in equation (11) to find that it becomes

$$\begin{aligned} I_p(\eta,\alpha)g(x) &= I_p(\eta-\beta,\alpha+\beta) I_x^{\eta,-\beta} g(x) \\ &= I_p(\eta-\beta,\alpha+\beta) x^{\beta-\eta} I_x^{-\beta} x^{\eta} g(x) \\ &= x^{\beta-\eta} I_p(0,\alpha+\beta) I_x^{-\beta} x^{\eta} g(x), \end{aligned} \tag{37}$$

where we have used the results (5) and (9).

The right hand side of equation (37) is defined when $\alpha+\beta>0$. Therefore taking $\beta=m$, the positive integer for which $0<\alpha+m\leqq 1$, when $\alpha\leqq 0$ and noting that $I_x^{-m}=D^m$, we deduce that when $\alpha\leqq 0$ the operators are defined by

$$I_p(\eta,\alpha)g(x)=x^{m-\eta} I_p(0,\alpha+m) D^m x^{\eta} g(x). \tag{38}$$

In particular, when $\alpha=0$, $m=1$, $Dx^{\eta}f(x)$ is integrable at the origin and $x^{\eta}f(x)\to 0$ as $x\to 0$, we have the zero-order operators

$$\begin{aligned} I_k(\eta,0)f(x) &= x^{1-\eta} I_k(0,1) D x^{\eta} f(x) \\ &= x^{-\eta}\int_0^x J_0\{k\sqrt{(x^2-xu)}\} D u^{\eta} f(u)\,du \\ &= f(x)-\frac{kx^{1/2-\eta}}{2}\int_0^x \frac{u^{\eta}}{\sqrt{(x-u)}} J_1\{k\sqrt{(x^2-xu)}\} f(u)\,du \end{aligned} \tag{39}$$

and

$$I_{ik}(\eta,0)f(x)=x^{-\eta}\int_0^x I_0\{k\surd(x^2-xu)\}Du^{\eta}f(u)\,du. \tag{40}$$

Using the Laplace transform we can establish, for suitable functions f, the following expressions for the inverse operators of zero-order.

$$I_k^{-1}(\eta,0)f(x)=x^{1-\eta}\frac{\partial}{\partial x}\int_0^x u^{\eta-1}I_0\{k\surd(ux-u^2)\}f(u)\,du, \tag{41}$$

$$I_{ik}^{-1}(\eta,0)f(x)=x^{1-\eta}\frac{\partial}{\partial x}\int_0^x u^{\eta-1}J_0\{k\surd(ux-u^2)\}f(u)\,du. \tag{42}$$

When $\eta=0$ the operators defined by equations (39) and (41) are identical with those introduced by Vekua [**6**, p. 59].

The following theorem can be proved in a fairly straightforward way.

Theorem 2. *If $f\in C^2(0,b)$, $b>0$, $x^{\eta+m-1}D^m f(x)$, $m=0,1,2$, are integrable at the origin and $x^{\eta+m}D^m f(x)\to 0$ as $x\to 0+$; then*

$$I_p(\eta,0)M_{2\eta}^{(x)}f(x)=[M_{2\eta}^{(x)}+(px)^2]I_p(\eta,0)f(x),\quad x>0, \tag{43}$$

where $p=k$ or $p=ik$, $k\geqq 0$.

6. Applications

(*a*) The generalised axially symmetric potential equation (GASPE)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2\alpha}{y}\frac{\partial u}{\partial y}=0,\quad \alpha>0, \tag{44}$$

when expressed in the polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ becomes

$$M_{2\alpha}^{(r)}u+\frac{\partial^2 u}{\partial\theta^2}+2\alpha\cot\theta\frac{\partial u}{\partial\theta}=0. \tag{45}$$

It is well known that a complete set of solutions of this equation that are analytic in a neighbourhood of the origin is

$$u_n(r,\theta)=a_n r^n C_n^{\alpha}(\cos\theta),\quad n=0,1,2,\ldots, \tag{46}$$

where the a_n are constants and $C_n^{\alpha}(\cos\theta)$ the Gegenbauer polynomials [**5**].

Applying the operator $I_k(\alpha,0)$ to equations (45) and (46) and using Theorem 2, we find that the corresponding set of solutions of the generalised axially symmetric Helmholtz equation (GASHE)

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{2\alpha}{y}\frac{\partial v}{\partial y}+k^2 v=0, \quad k\geqq 0, \tag{47}$$

which are analytic about the origin is given by

$$\begin{aligned} v_n(r,\theta) &= a_n C_n^\alpha(\cos\theta) I_k(\alpha,0) r^n \\ &= a_n(\alpha+n) C_n^\alpha(\cos\theta) r^{-\alpha}\int_0^r J_0\{k\sqrt{(r^2-ru)}\} u^{\alpha+n-1}\,du \\ &= A_n C_n^\alpha(\cos\theta) r^{-\alpha} J_{n+\alpha}(kr), \quad n=0,1,2,\ldots, \end{aligned} \tag{48}$$

where the A_n are constants.

(*b*) As a final example we show that the operators can be used to obtain a formal derivation of the inversion formula for the Kontorovich–Lebedev transform of the function $f(x)$, $0\leqq x<\infty$, which is defined by

$$F(s)=\int_0^\infty K_s(kx)x^{-1}f(x)\,dx, \quad \mathrm{Re}\,(s)<0, \quad k\geqq 0, \tag{49}$$

where $K_s(kx)$ is the modified Bessel function of the second kind.

Multiplying both sides of the above equation by

$$2[\Gamma(-s)]^{-1}\left(\frac{kt}{2}\right)^{-s}, \quad t\geqq 0$$

and applying the Mellin inversion formula we get

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2F(s)}{\Gamma(-s)}\left(\frac{kt}{2}\right)^{-s} ds &= \frac{-1}{2\pi i}\int_0^\infty x^{-1}f(x)\,dx\int_{c-i\infty}^{c+i\infty}\frac{2s}{\Gamma(1-s)}K_s(kx)\left(\frac{kt}{2}\right)^{-s} ds \\ &= \frac{t}{2\pi i}\frac{\partial}{\partial t}\int_0^\infty x^{-1}f(x)\,dx\int_{c-i\infty}^{c+i\infty}\frac{2}{\Gamma(1-s)}K_s(kx)\left(\frac{kt}{2}\right)^{-s} ds. \end{aligned} \tag{50}$$

Making use of the result

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2}{\Gamma(1-s)}K_s(kx)\left(\frac{kt}{2}\right)^{-s} ds = J_0\{k\sqrt{(xt-x^2)}\}H(t-x), \quad \mathrm{Re}\,(s)<0, \tag{51}$$

where $H(x)$ is the Heaviside unit function, we find that equation (50) can be written as

$$I_{ik}^{-1}(0,0)f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2}{\Gamma(-s)}F(s)\left(\frac{kt}{2}\right)^{-s}ds, \tag{52}$$

where $I_{ik}^{-1}(0,0)$ is the inverse operator defined by equation (42) when $\eta=0$.

Applying the operator $I_{ik}(0,0)$, defined by equation (40), to both sides of the above equation we get

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{2}{\Gamma(-s)}F(s)I_{ik}(0,0)\left(\frac{kx}{2}\right)^{-s}ds. \tag{53}$$

Finally, on using the result

$$I_{ik}(0,0)\left(\frac{kx}{2}\right)^{-s}=\Gamma(1-s)I_{-s}(kx), \quad \mathrm{Re}\,(s)<0,$$

we see that an inversion formula for the integral transform (49) is given by

$$f(x)=\frac{i}{\pi}\int_{c-i\infty}^{c+i\infty}sF(s)I_{-s}(kx)\,ds, \quad \mathrm{Re}\,(s)<0 \tag{54}$$

and this agrees with a result given in [**2**].

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