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THE SPECTRUM OF PERIODIC JACOBI MATRICES WITH SLOWLY OSCILLATING DIAGONAL TERMS

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Abstract We study the spectrum of periodic Jacobi matrices. We concentrate on the case of slowly oscillating diagonal terms and study the behaviour of the zeros of the associated orthogonal polynomials in the spectral gap. We find precise estimates for the distance from single eigenvalues of truncated matrices in the spectral gap to the diagonal entries of the matrix. We include a brief numerical example to show the exactness of our estimates.

Keywords: Jacobi matrices; orthogonal polynomials; spectrum

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1. Introduction

In this paper we study the spectrum of Jacobi matrices, that is, matrices of the form

$$\begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_4 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.1}$$

where $b_n \in \mathbb{R}$ and $a_n > 0$ for $n \in \mathbb{Z}^+$. We will assume that both $\sup_{n\geqslant 0}|a_n|$ and $\sup_{n\geqslant 0}|b_n|$ are finite. These matrices are viewed as operators acting on the Hilbert space

$$l^2(\mathbb{Z}^+) = \left\{ (f_n)_{n=1}^{\infty} \;\middle|\; f_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |f_n|^2 < \infty \right\}$$

with the natural norm.

If T is the operator defined by (1.1), we observe that

$$Tf(n) = \begin{cases} a_{n-1}f_{n-1} + b_n f_n + a_n f_{n+1} & \text{if } n > 1, \\ b_1 f_1 + a_1 f_2 & \text{if } n = 1, \end{cases}$$

and it is easy to see that T is a bounded operator with norm $\|T\| \leqslant k$, where $k = \sup_{n\geqslant 0}\{|a_n|+|b_n|+|a_{n-1}|\}$. It is also clear that T is self-adjoint and hence, $\operatorname{Spec}(T) \subset \mathbb{R}$ and, indeed, $\operatorname{Spec}(T) \subset [-k, k]$.

There exists a one-to-one correspondence between Jacobi matrices and orthogonal polynomials. In this case, the orthogonal polynomials associated with T obey the three-term recurrence relation given by

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{1.2}$$

for $n \geqslant 1$, where $p_0 \equiv 1$ and the a_n and b_n are those which appear in (1.1).

These polynomials are obtained via the Gram–Schmidt process. Let δ_j be the vector in $l^2(\mathbb{Z}^+)$ with components $\delta_{jn} = 1$ if $n = j$, or 0 otherwise, and let μ be the spectral measure. Applying the Gram–Schmidt process to the sequence $\{x^n\}$, we obtain orthogonal polynomials p_n such that

$$\int p_n(x)p_m(x)\,\mathrm{d}\mu(x) = \delta_{nm}$$

and such that they satisfy (1.2). We refer the reader to [**2**] for a more detailed account of this, but observe that what we have presented here provides all the background needed for our analysis.

In [**1**], Simon and Denisov affirmatively answer a question raised by Nevai: is it possible for the single possible zero of each polynomial p_n in a gap (α, β) of the support of $\mathrm{d}\mu$ to yield all of the points of (α, β) as limit points as n varies? To do this, they concentrated on a particular Jacobi matrix with 2-periodic off-diagonal entries. We extend this idea to consider a wider class of off-diagonal entries and slowly oscillating diagonal entries and finally extend the methods to general even periods. We also find estimates for the distance from single eigenvalues in a gap of a truncated matrix to the diagonal elements of the matrix. Theorems 2.10 and 3.9 are our strongest results in this sense and we refer the reader to [**3**], where earlier versions of these results can be found.

2. Jacobi matrices with period 2

In this section we will study matrices of the form

$$\begin{pmatrix} b_1 & a & 0 & 0 & 0 & \cdots \\ a & b_2 & b & 0 & 0 & \cdots \\ 0 & b & b_3 & a & 0 & \cdots \\ 0 & 0 & a & b_4 & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In fact, let A be the Jacobi matrix defined by the sequences (a_n) and (b_n), where

$$a_{2n-1} = a, \quad a_{2n} = b \quad \text{for } n \in \mathbb{Z}^+,$$

and (b_n) satisfies the conditions given above. We will show that, under certain conditions imposed on the entries of the matrix, the operator defined by this matrix has a spectral

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gap and that, in this gap, the set of zeros of the associated orthogonal polynomials, $(p_n(x))$, is dense. Simon and Denisov have shown a particular case of this in [**1**], where they have taken the sequence (b_n) to be the sequence $\{0, -\frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, -\frac{7}{8}, \dots\}$ and have set $a = 3$ and $b = 1$.

Before presenting our most general result in this sense, we include the following particular case, which is more general than the example found in [**1**], as we believe it makes our arguments and presentation easier to follow.

Let us consider a more general sequence, $(\gamma_n)_{n=1}^{\infty}$, in place of $(b_n)_{n=1}^{\infty}$. In fact, let $(\gamma_n)_{n=1}^{\infty} = (\sin n^{\omega})_{n=1}^{\infty}$ with $0 < \omega < 1$. This sequence oscillates infinitely many times between -1 and 1 and does so in such a way that each oscillation takes longer and longer as $x \to \infty$.

Consider now the Jacobi matrix B, given by

$$\left.\begin{aligned} a_{2n-1} &= a \\ a_{2n} &= b \\ b_n &= \gamma_n \end{aligned}\right\} \quad \text{for } n \in \mathbb{Z}^+, \tag{2.1}$$

with $a > b$ and $|a - b| = 2$. The latter condition serves only to make calculations more simple. In reality what we require is $|a - b| = 2r$, where r is the supremum of the b_n.

We need the following results.

Lemma 2.1. *Let B_0 be the Jacobi matrix obtained by replacing the diagonal entries of B with 0, and let B_∞ be the doubly infinite matrix that coincides with B_0 on $\mathbb{Z}^+$ and which is extended in a similar way to $\mathbb{Z}$. Then*

$$\operatorname{Spec}(B_\infty) = [-(b + a), -|b - a|] \cup [|b - a|, b + a].$$

Proof. Consider the equation $B_\infty f = \lambda f$, where $f = (f_n)_{n\in\mathbb{Z}} \in l^2$. $\operatorname{Spec}(B_\infty)$ coincides with the set of λ for which this equation has a bounded solution. These bounded functions are of the form $f_{2n} = \alpha e^{-in\theta}$ and $f_{2n+1} = \beta e^{-in\theta}$, where $\theta \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$.

Thus, if we consider such an f, we have that

$$(B_\infty f)_{2n} = b f_{2n+1} + a f_{2n-1} \quad \text{and} \quad (B_\infty f)_{2n+1} = a f_{2n+2} + b f_{2n}$$

and thus we obtain the following equations:

$$\begin{aligned} \lambda\alpha e^{-in\theta} &= b\beta e^{-in\theta} + a\beta e^{-i(n-1)\theta}, \\ \lambda\beta e^{-in\theta} &= a\alpha e^{-i(n+1)\theta} + b\alpha e^{-in\theta}, \end{aligned}$$

or, equivalently,

$$\frac{\lambda}{b + a e^{i\theta}} = \frac{\beta}{\alpha} = \frac{b + a e^{-i\theta}}{\lambda}$$

and hence $\lambda^2 = a^2 + b^2 + 2ab\cos\theta$, so

$$|b - a| \leqslant |\lambda| \leqslant b + a$$

provides the required solutions, and thus

$$\operatorname{Spec}(B_\infty) = [-(b+a), -|b-a|] \cup [|b-a|, b+a].$$

□

Corollary 2.2. *$\operatorname{Spec}(B_0)$ coincides with $\operatorname{Spec}(B_\infty)$ except for possible eigenvalues in the gap $(-|b-a|, |b-a|)$.*

Proof. This follows from the fact that Ess $\operatorname{Spec}(B_0) =$ Ess $\operatorname{Spec}(B_\infty)$. □

We now analyse when such eigenvalues exist. We claim that if $a > b$, then B_0 has no eigenvalues in the gap.

Lemma 2.3. *If λ is an eigenvalue of B_0, then so is $-\lambda$.*

Proof. Consider the unitary operator U defined by $(Uf)_n = (-1)^{n-1} f_n$ for $f = (f_n) \in l^2(\mathbb{Z}^+)$. It is easy to see that $U^{-1}B_0U = -B_0$ and our claim follows. □

Corollary 2.4. *B_0 has only one possible eigenvalue in $(-|b-a|, |b-a|)$.*

Theorem 2.5. *The point spectrum of B_0 contains 0 if and only if $b > a$.*

Proof. The result follows when considering $B_0(f_n) = 0$, as we obtain that

$$f_{2n} = 0,$$
$$f_{2n+1} = -\left(\frac{a}{b}\right)^n f_1.$$

□

We are now in a position to deal with the spectrum of our original matrix B.

Theorem 2.6. *If 0 is not an eigenvalue of B_0, then*

$$\operatorname{Spec}(B) = [-(b+a)-1, -|b-a|+1] \cup [|b-a|-1, b+a+1];$$

on the other hand, if 0 is an eigenvalue of B_0, then

$$\operatorname{Spec}(B) = [-(b+a)-1, -|b-a|+1] \cup \{\delta\} \cup [|b-a|-1, b+a+1]$$

for some $\delta \in \mathbb{R}$.

Proof. If 0 is not an eigenvalue of B_0, then

$$\operatorname{Spec}(B_0) = [-(b+a), -|b-a|] \cup [|b-a|, b+a]$$

and since $\|B - B_0\| = 1$ it follows that $\operatorname{Spec}(B) \subset [-1,1] + \operatorname{Spec}(B_0)$. And now, as the closure of $\bigcup\{\beta_n : n \in \mathbb{Z}^+\} + \operatorname{Spec}(B_0)$ is contained in $\operatorname{Spec}(B)$, we have that

$$\operatorname{Spec}(B) = [-(b+a)-1, -|b-a|+1] \cup [|b-a|-1, b+a+1].$$

On the other hand, if 0 is an eigenvalue of B_0, the result follows by considering the effect of compact perturbations on single eigenvalues. □

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We are now ready to prove the following result.

Theorem 2.7. *We have*

(i) $\mathrm{Spec}(B) = [-(b+a)-1, -|b-a|+1] \cup [|b-a|-1, b+a+1]$, *and*

(ii) $\{x \in (-1,1) \mid q_n(x) = 0$ *for some* $n\}$ *is dense in* $[-1,1]$, *where* $(q_n(x))$ *are the orthogonal polynomials associated with* B.

Proof. The first claim follows from the above results. In order to prove our second claim we will consider a particular case first; take $\omega = \frac{1}{2}$.

We have that $(\gamma_n)_{n=1}^{\infty} = (\sin n^{1/2})_{n=1}^{\infty}$ and we need to estimate the size of

$$\frac{\|(B_{(j)} - \gamma_j I)\varphi_j\|^2}{\|\varphi_j\|^2} = \frac{\sum_{m=0}^{j-1} |\gamma_m - \gamma_j|^2 |\varphi_{m,j}|^2}{\sum_{m=0}^{j-1} |\varphi_{m,j}|^2}, \tag{2.2}$$

where φ_j is the vector

$$\left(1, 0, \left(-\frac{a}{b}\right), 0, \left(-\frac{a}{b}\right)^2, 0, \dots, \left(-\frac{a}{b}\right)^{j-1}\right)$$

and $B_{(n)}$ is the $n \times n$ matrix obtained by taking the first n rows and columns of B, as we know that the zeros of $q_n(x)$ are the eigenvalues of $B_{(n)}$ (see [**4**]).

Taking into account the recurrence relation that the orthogonal polynomials have to satisfy at 0, i.e.

$$q_{2n+1} = 0 \quad \text{for all } n \qquad \text{and} \qquad q_{2n}(0) = \left(-\frac{a}{b}\right)^n, \tag{2.3}$$

it is easy to see that $(B_{0(j)})\varphi_j = 0$ whenever j is odd, and for any such j let us take $(B_{(j)} - \gamma_n I)\varphi_j$.

We will now split (2.2) into two sums,

$$\frac{\sum_{m=0}^{j-1} |\gamma_m - \gamma_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} = \frac{\sum_{m=0}^{i-1} |\gamma_m - \gamma_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} + \frac{\sum_{m=i}^{j-1} |\gamma_m - \gamma_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}},$$

and our task now is to find an appropriate index i. Let $0 < \varepsilon < 1$.

For j odd, let $i = j - [j^{\varepsilon}]$ and consider the first of these two sums. We obtain

$$\frac{\sum_{m=0}^{i-1} |\gamma_m - \gamma_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} \leqslant 4\frac{(a/b)^{2i} - 1}{(a/b)^{2j} - 1} = 4\frac{(a/b)^{2(j-[j^{\varepsilon}])} - 1}{(a/b)^{2j} - 1} \approx 4(a/b)^{-2[j^{\varepsilon}]},$$

which tends to zero rapidly as a negative power of a/b.

Now, as $|\sin x - \sin y| \leqslant |x - y|$ for all real numbers x and y, we have that

$$\begin{aligned}\frac{\sum_{m=i}^{j-1} |\gamma_m - \gamma_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} &= \frac{\sum_{m=i}^{j-1} |\sin m^{1/2} - \sin j^{1/2}|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} \\ &\leqslant \frac{\sum_{m=i}^{j-1} |m^{1/2} - j^{1/2}|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}}\end{aligned}$$

and we need to observe that $|m^{1/2}-j^{1/2}| = |m-j|/|m^{1/2}+j^{1/2}|$. Hence, it follows that

$$\frac{\sum_{m=j-[j^\varepsilon]}^{j-1}|m^{1/2}-j^{1/2}|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} = \frac{\sum_{m=j-[j^\varepsilon]}^{j-1}|m-j|^2/|m^{1/2}+j^{1/2}|^2(a/b)^{2m}}{((a/b)^{2j}-1)/(a/b)^{2j}}.$$

We note that $|m^{1/2}-j^{1/2}|$ maximizes for $m = j-[j^\varepsilon]$ and hence we consider the term

$$\frac{|(j-[j^\varepsilon])-j|^2}{|(j-[j^\varepsilon])^{1/2}+j^{1/2}|^2} = \frac{[j^\varepsilon]^2}{|(j-[j^\varepsilon])^{1/2}+j^{1/2}|^2}. \tag{2.4}$$

We obtain $j^{1/2} \leqslant (j-[j^\varepsilon])^{1/2}+j^{1/2} \leqslant 2j^{1/2}$ and thus

$$\begin{aligned}\sum_{m=j-[j^\varepsilon]}^{j-1}\frac{|m-j|^2}{|m^{1/2}+j^{1/2}|^2}\left(\frac{a}{b}\right)^{2m} &\leqslant \sum_{m=j-[j^\varepsilon]}^{j-1}\frac{[j^\varepsilon]^2}{|(j-[j^\varepsilon])^{1/2}+j^{1/2}|^2}\left(\frac{a}{b}\right)^{2m}\\ &\leqslant \sum_{m=j-[j^\varepsilon]}^{j-1}[j^{\varepsilon-1/2}]^2\left(\frac{a}{b}\right)^{2m}\\ &= [j^{\varepsilon-1/2}]^2\sum_{m=j-[j^\varepsilon]}^{j-1}\left(\frac{a}{b}\right)^{2m}.\end{aligned}$$

In other words,

$$\begin{aligned}\frac{\sum_{m=j-[j^\varepsilon]}^{j-1}|m^{1/2}-j^{1/2}|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} &\leqslant [j^{\varepsilon-1/2}]^2\frac{\sum_{m=j-[j^\varepsilon]}^{j-1}(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}}\\ &= [j^{\varepsilon-1/2}]^2\frac{(a/b)^{2j}-(a/b)^{2(j-[j^\varepsilon])}}{(a/b)^{2j}-1}\end{aligned}$$

and hence, for $0<\varepsilon<\frac{1}{2}$, this tends to zero as $j\to\infty$. We thus see that $\{x\in(-1,1) \mid q_n(x)=0 \text{ for some } n\}$ is dense in $[-1,1]$, as the zeros of $(q_n(x))$ are the eigenvalues of $B_{(n)}$.

We now consider the general case $(\gamma_n)_{n=1}^\infty = (\sin n^\omega)_{n=1}^\infty$ with $\omega<1$. We have that

$$\frac{\sum_{m=0}^{j-1}|\gamma_m-\gamma_j|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} = \frac{\sum_{m=0}^{i-1}|\gamma_m-\gamma_j|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} + \frac{\sum_{m=i}^{j-1}|\gamma_m-\gamma_j|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}},$$

and for the first of these two sums we have

$$\frac{\sum_{m=0}^{i-1}|\gamma_m-\gamma_j|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} \leqslant 4\frac{(a/b)^{2i}-1}{(a/b)^{2j}-1} \approx 4\left(\frac{a}{b}\right)^{2(i-j)}.$$

If we take $i = j-[\log j]$, again for j odd, then

$$\frac{\sum_{m=0}^{j-1}|\gamma_m-\gamma_j|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} \approx 4\left(\frac{a}{b}\right)^{-[\log j]}.$$

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To deal with the second sum we observe that, for $0 < m \leqslant j$,

$$j^{\omega} - m^{\omega} = \int_m^j \omega x^{\omega-1}\,\mathrm{d}x \leqslant \omega(j-m)m^{\omega-1},$$

and as $j - [\log j] \geqslant \frac{1}{2}j$ we have that

$$j^{\omega} - m^{\omega} \leqslant \omega[\log j](\tfrac{1}{2}j)^{\omega-1} = 2^{1-\omega}\omega[\log j]j^{\omega-1}$$

for $m = j - [\log j]$.

Now,

$$\begin{aligned}\frac{\sum_{m=j-[\log j]}^{j-1}|\gamma_m - \gamma_j|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} &\leqslant \frac{\sum_{m=j-[\log j]}^{j-1}|j^{\omega} - m^{\omega}|^2(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}} \\ &\leqslant 2^{1-\omega}\omega[\log j]j^{\omega-1}\frac{\sum_{m=j-[\log j]}^{j-1}(a/b)^{2m}}{\sum_{m=0}^{j-1}(a/b)^{2m}}\end{aligned}$$

and hence, as $0 < \omega < 1$, this tends to zero as $j \to \infty$.

Thus, as the zeros of $(q_n(x))$ are precisely the eigenvalues of B_n, we see that $\{x \in (-1,1) \mid q_n(x) = 0 \text{ for some } n\}$ is dense in $[-1,1]$. □

Corollary 2.8. *For odd j, $B_{(j)}$ has an eigenvalue, λ_j, close to γ_j. In fact,*

$$|\lambda_j - \gamma_j| = O([\log j]j^{\omega-1}).$$

We now present the following more general result.

Theorem 2.9. *Let $f : \mathbb{R} \to [-1,1]$ be a function that oscillates infinitely many times between 1 and -1 and such that $f'(x) \to 0$ as $x \to \infty$. Consider the sequence $(\rho_n)_{n=1}^{\infty}$, defined by*

$$\rho_1 = f(1), \quad \rho_2 = f(2), \quad \dots, \quad \rho_n = f(n), \quad \dots,$$

and consider the Jacobi matrix C with

$$a_{2n-1} = a, \quad a_{2n} = b, \quad b_n = \rho_n \quad \text{for } n \in \mathbb{Z}^+,$$

where $a > b$ and $|a - b| = 2$. Then

$$\mathrm{Spec}(C) = [-(b+a)-1, -|b-a|+1] \cup [|b-a|-1, b+a+1]$$

and $\{x \in (-1,1) \mid r_n(x) = 0 \text{ for some } n\}$ is dense in $[-1,1]$, where $(r_n(x))$ are the orthogonal polynomials associated with C.

Proof. The claim about the spectrum of C follows directly as before from Theorem 2.6 and thus we just need to prove the density result.

For any $j \in \mathbb{N}$, let $\delta_r = \sup\{|f'(s)| : r \leqslant s \leqslant j\}$. By hypothesis, if we consider large j, δ_r decreases to zero when we consider r sufficiently large, and by the mean-value theorem we obtain

$$\begin{aligned}|\rho_m - \rho_j| &\leqslant |m-j| \sup\{|f'(s)| : m \leqslant s \leqslant j\}\\ &= |m-j|\delta_m.\end{aligned}$$

Now, let $\varepsilon_r = \max\{\delta_r, 1/r^\alpha\}$, for $0 < \alpha < 1$, which again tends to zero as $r \to \infty$.

From this definition of ε_r we see that $j - [\varepsilon_{j/2}^{-\alpha}] \geqslant \frac{1}{2}j$ and we will use this fact to estimate

$$\frac{\sum_{m=0}^{j-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}}.$$

To do so, we again consider the following two sums:

$$\frac{\sum_{m=0}^{j-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} = \frac{\sum_{m=0}^{i-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} + \frac{\sum_{m=i}^{j-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}},$$

where, as in the previous cases, i depends on j.

In fact, let $i = j - [\varepsilon_{j/2}^{-\alpha}]$. Then

$$\frac{\sum_{m=0}^{i-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} \leqslant 4 \frac{\sum_{m=0}^{i-1} (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} \approx 4 \left(\frac{a}{b}\right)^{-2[\varepsilon_{j/2}^{-\alpha}]},$$

which tends to zero as $j \to \infty$.

Now, to estimate the size of

$$\frac{\sum_{m=i}^{j-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}},$$

we observe that

$$\begin{aligned}\frac{\sum_{m=i}^{j-1} |\rho_m - \rho_j|^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}} &\leqslant \frac{\sum_{m=i}^{j-1} |m-j|^2 \varepsilon_m^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}}\\ &\leqslant \frac{\sum_{m=i}^{j-1} |i-j|^2 \varepsilon_i^2 (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}}\\ &\leqslant [\varepsilon_{j/2}^{-\alpha}]^2 \varepsilon_{j-[\varepsilon_{j/2}^{-\alpha}]}^2 \frac{\sum_{m=j-[\varepsilon_{j/2}^{-\alpha}]}^{j-1} (a/b)^{2m}}{\sum_{m=0}^{j-1} (a/b)^{2m}}\\ &\leqslant [\varepsilon_{j/2}^{-\alpha}]^2 \varepsilon_{j/2}^2\\ &\approx \varepsilon_{j/2}^{2(1-\alpha)},\end{aligned}$$

which also tends to zero as $j \to \infty$, and hence the required result follows. □

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Table 1. $|\lambda_j| > 0.9$

j	λ_j	$\lvert\lambda_j - \sin(j^{1/2})\rvert$
23	$-0.995\,14$	$0.001\,39$
61	$0.997\,03$	$0.002\,01$
123	$-0.995\,93$	$0.000\,43$
201	$0.999\,15$	$0.000\,04$
299	$-0.999\,75$	$0.000\,17$

Table 2. $|\lambda_j| < 0.1$

j	λ_j	$\lvert\lambda_j - \sin(j^{1/2})\rvert$
23	$-0.995\,14$	$0.001\,39$
61	$0.997\,03$	$0.002\,01$
123	$-0.995\,93$	$0.000\,43$
201	$0.999\,15$	$0.000\,04$
299	$-0.999\,75$	$0.000\,17$

Summarizing these results, we obtain the following theorem, which not only gives a rate of convergence of the eigenvalues of $C_{(j)}$ in $(-1,1)$ in terms of $|f'(x)|$ (which tends to zero as $x \to \infty$), but allows us to see that the error is particularly small for j such that f is almost stationary in the interval (i,j).

Theorem 2.10. *Let λ_j be the single eigenvalue of $C_{(j)}$ in the interval $(-1,1)$. Then, using the notation introduced in the proof of Theorem 2.9, we have that*

$$|\rho_j - \lambda_j| \leqslant 4\left(\frac{a}{b}\right)^{-2[\varepsilon_{j/2}^{-\alpha}]} + \varepsilon_{j/2}^{2(1-\alpha)}$$

for odd j.

Tables 1 and 2 show a sample of the results obtained by testing this theorem numerically for the matrix B defined in (2.1) with $a = 3$ and $b = 1$, where λ_j is the single eigenvalue of $B_{(j)}$ in the interval $(-1,1)$. Table 1 shows values of $|\lambda_j| > 0.9$ where the error is minimal and Table 2 shows values of $|\lambda_j| < 0.1$ where the error is maximal.

Having presented these results, it is worth noting that, although throughout this section we have restricted ourselves to the study of matrices of period 2, many of our results hold in the more general case. We devote the next section to this case.

3. Jacobi matrices with general even periods

To deal with the general case, we observe that there is an intrinsic difference between even and odd periods. Jacobi matrices with odd periods fall slightly outside the context we wish to work in, as their spectrum contains an odd number of bands and, given that it is symmetrical about the origin, this implies that the origin is an element of the spectrum. We will thus restrict our study to matrices with even periods, but first we prove the following general theorem. This allows us to show that, even though we are dealing with rank 2 perturbations, given the nature of the problem, they will produce at most one eigenvalue in each gap. Note that such eigenvalues still occur in pairs, i.e. if λ is an eigenvalue, then so is $-\lambda$. This result (which we state formally as our next theorem) is true owing to the nature of tri-diagonal matrices and it does not in fact depend on the periodicity of the matrices.

Theorem 3.1. *Let M_∞ be a doubly infinite tri-diagonal self-adjoint matrix and let M_0 be the restriction of M_∞ to $\mathbb{Z}^+$; in other words M_0 is of the form*

$$\begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_4 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we let P denote the projection onto $\mathbb{Z}^+$, then

$$M_0^2 = PM_\infty^2P - R,$$

where R is a rank 1 perturbation. Hence, the restriction of M_∞ to M_0 can create at most one eigenvalue per gap.

Proof. The result follows from the direct computation of M_0^2 and PM_∞^2P. □

Now, given this result, let us proceed with our study of periodic Jacobi matrices. Consider A_0, a Jacobi matrix of period 4 defined as follows:

$$a_{4n-3} = a, \quad a_{4n-2} = b, \quad a_{4n-1} = c, \quad a_{4n} = d \quad \text{and} \quad b_n = 0 \quad \text{for } n \in \mathbb{Z}^+,$$

and, keeping the notation used thus far, let A_∞ be the doubly infinite matrix that coincides with A_0 on $\mathbb{Z}^+$.

Lemma 3.2. $\mathrm{Spec}(A_\infty)$ *coincides with the set* $\{\lambda \mid |p(\lambda)| \leqslant 1\}$, *where $p(\lambda)$ is the fourth-order polynomial defined by*

$$p(\lambda) := \frac{\lambda^4 - (a^2+b^2+c^2+d^2)\lambda^2 + a^2c^2 + b^2d^2}{2abcd}. \tag{3.1}$$

We observe that $\mathrm{Spec}(B_\infty)$ consists of at most four bands which lie symmetrically on the real line with respect to the origin.

Corollary 3.3. $\mathrm{Spec}(A_0)$ *coincides with* $\mathrm{Spec}(A_\infty)$ *except for the occurrence of one eigenvalue in each of the gaps.*

Theorem 3.4. *Given the operator A_0, 0 is the only possible eigenvalue in the central gap and, furthermore, 0 is an eigenvalue of A_0 if and only if $ac < bd$.*

Proof. If we set $A_0 f = 0$, then

$$\begin{aligned} af_2 &= 0, \\ af_1 + bf_3 &= 0, \\ bf_2 + cf_4 &= 0, \\ cf_3 + df_5 &= 0, \\ &\vdots \end{aligned}$$

and this implies that $f_{2n} = 0$. Furthermore, if we set $f_1 = 1$, then

$$f_3 = -\frac{a}{b}, \quad f_5 = \frac{ac}{bd}, \quad f_7 = -\frac{a^2c}{b^2d}, \quad f_9 = \frac{a^2c^2}{b^2d^2}, \quad f_{11} = -\frac{a^3c^2}{b^3d^2}, \quad \dots$$

and the result follows. □

Theorem 3.5. *If $bc < ad$,m then $-\sqrt{a^2+b^2}$ and $\sqrt{a^2+b^2}$ are the eigenvalues of A_0 in each of the two lateral gaps.*

Proof. This can be worked out easily by solving $B_0 f = \pm\sqrt{a^2+b^2}f$ as before. □

Let us now assume that 0 is not an eigenvalue of A_0.

Theorem 3.6. *Let A_0 be an operator as defined above, such that zero is not an eigenvalue and $\operatorname{Spec}(A_0)$ consists of exactly four bands. Let us denote the central gap by $(-K, K)$ and let $f : \mathbb{R} \to [-K, K]$ be a function that oscillates infinitely many times between the extremes of this gap, and such that $f'(x) \to 0$ as $x \to \infty$. Consider the sequence $(\beta_n)_{n=1}^{\infty}$ defined as follows:*

$$\beta_1 = f(1), \quad \beta_2 = f(2), \quad \dots, \quad \beta_n = f(n), \quad \dots,$$

and consider the Jacobi matrix A which coincides with A_0 everywhere but on the diagonal, where we replace the zeros with the sequence $(\beta_n)_{n=1}^{\infty}$. Then $\{x \in (-K, K) \mid p_n(x) = 0 \text{ for some } n\}$ is dense in $[-K, K]$, where $(p_n(x))$ are the orthogonal polynomials associated with A.

Proof. Consider the orthogonal polynomials of A_0 at 0. These polynomials, $(p_n(x))$, obey the following recurrence relations:

$$p_{2n+1} = 0 \quad \text{for all } n$$

and

$$p_{4n}(0) = \left(\frac{ac}{bd}\right)^n, \qquad p_{4n-2}(0) = -\left(\frac{a}{b}\right)^n \left(\frac{c}{d}\right)^{n-1}.$$

The remainder of the proof follows exactly as in that of Theorem 2.9. □

We now state the theorems for the general case.

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Theorem 3.7. *Let $(a_n)_{n=1}^{\infty}$ be a $2k$-periodic sequence for some $k \in \mathbb{N}$, and let G_0 be the Jacobi matrix with zeros along the diagonal and the sequence $(a_n)_{n=1}^{\infty}$ on the off-diagonals. Then $\operatorname{Spec}(G_0)$ consists of at most $2k$ bands together with at most one eigenvalue in each of the gaps.*

For the next result, consider the matrix G_0 and suppose that $\operatorname{Spec}(G_0)$ consists of exactly $2k$ bands and zero is not an eigenvalue. Let us again denote the central gap by $(-K, K)$.

Theorem 3.8. *Given G_0, let $g : \mathbb{R} \to [-K, K]$ be a function that oscillates infinitely many times between $-K$ and K, and such that $g'(x) \to 0$ as $x \to \infty$. Consider the sequence $(\rho_n)_{n=1}^{\infty}$ defined as follows:*

$$\rho_1 = g(1), \quad \rho_2 = g(2), \quad \dots, \quad \rho_n = g(n), \quad \dots,$$

and the Jacobi matrix G which coincides with G_0 everywhere but on the diagonal, where we replace the zeros with the sequence $(\rho_n)_{n=1}^{\infty}$. Then $\{x \in (-K, K) \mid r_n(x) = 0 \text{ for some } n\}$ is dense in $[-K, K]$, where $(r_n(x))$ are the orthogonal polynomials associated with G.

Proof. The proof of this result employs the methods used in the proof of Theorem 3.6. One has only to determine the orthogonal polynomials associated with G_0 at zero and construct an appropriate approximate eigenfunction. The orthogonal polynomials at zero in this case obey the same type of recurrence relations as before, namely

$$p_{2n+1} = 0 \quad \text{for all } n$$

and

$$\begin{aligned} p_{2kn}(0) &= \left(\frac{a_1 a_3 \dots a_{2k-1}}{a_2 a_4 \cdots a_{2k}}\right)^n, \\ p_{2kn-2}(0) &= -\left(\frac{a_1}{a_2}\right)^n \left(\frac{a_3 \dots a_{2k-1}}{a_4 \cdots a_{2k}}\right)^{n-1}, \\ p_{2kn-4}(0) &= -\left(\frac{a_1 a_3}{a_2 a_4}\right)^n \left(\frac{a_5 \cdots a_{2k-1}}{a_6 \cdots a_{2k}}\right)^{n-1}, \\ &\ \vdots \end{aligned}$$

and $\varphi_j = (p_0(0), p_1(0), \dots, p_{j-1}(0))$ produces the required result. □

Finally, we conclude this section by restating this result in terms of the eigenvalues of the truncated matrices $G_{(j)}$ (the $j \times j$ truncations of G).

Theorem 3.9. *Let λ_j be the single eigenvalue of G_j in the interval $(-K, K)$. Then, given $\varepsilon > 0$, there exists an odd $j \in \mathbb{N}$ such that $|\rho_j - \lambda_j| < \varepsilon$.*

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