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RESEARCH ARTICLE

# Asymptotics for the sum-ruin probability of a bi-dimensional compound risk model with dependent numbers of claims

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**MSC:** 62P05; 62E10

**Abstract**
This paper studies a bi-dimensional compound risk model with quasi-asymptotically independent and consistently varying-tailed random numbers of claims and establishes an asymptotic formula for the finite-time sum-ruin probability. Additionally, some results related to tail probabilities of random sums are presented, which are of significant interest in their own right. Some numerical studies are carried out to check the accuracy of the asymptotic formula.

## 1. Introduction

### *1.1. Model descriptions*

This paper studies a bi-dimensional continuous-time compound risk model, where an insurance company concurrently engages in two lines of business. Both lines are exposed to severe accidents such as car accidents, fire disasters, or catastrophes such as earthquakes, floods, and hurricanes. It is reasonable to assume that one of these risks may lead to numerous claims for both two lines of business at the same time or for only one line of business (if some catastrophe does not affect any line of business, it is then not considered as a risk). To characterize this scenario, we assume that the arrival times of successive risks are $0 < \sigma_1 \le \sigma_2 \le \cdots$, which constitute a counting process

$$N(t) = \sup\{n \ge 1 : \sigma_n \le t\}, \quad t \ge 0$$

with a mean function $\lambda(t) = EN(t) = \sum_{i=1}^{\infty} P(\sigma_i \le t)$ being finite. We adopt non-negative random vectors $\left(\tau_i^{(1)}, \tau_i^{(2)}\right)$, $i \ge 1$, to represent the numbers of claims resulting from each line of business caused by catastrophic risks. Then, the discounted surplus process of the insurance firm at time $t$ is as follows:

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \int_0^t e^{-rs} dC_1(s) \\ \int_0^t e^{-rs} dC_2(s) \end{pmatrix} - \sum_{i=1}^{N(t)} \begin{pmatrix} \sum_{j=1}^{\tau_i^{(1)}} X_{ij}^{(1)} e^{-r\sigma_i} \\ \sum_{j=1}^{\tau_i^{(2)}} X_{ij}^{(2)} e^{-r\sigma_i} \end{pmatrix}, \; t \ge 0, \tag{1.1}$$

where, for $l = 1, 2$, we denote by $x_l$ the initial reserve, $r \ge 0$ the constant force of interest, $X_{ij}^{(l)}$ the $j$th claim triggered by the $i$th risk from the $l$th line of business, $\{C_l(t), t \ge 0\}$ the premium accumulation process of the $l$th line of business whose paths are non-negative, non-decreasing, right continuous, and $C_l(t) < \infty$ almost surely for any $t > 0$.

In the above bi-dimensional continuous-time compound risk model (1.1), we suppose that

- $\left(\tau_i^{(1)}, \tau_i^{(2)}\right), i \geq 1$, are non-negative, independent and identically distributed random vectors with generic random pair $(\tau_1, \tau_2)$ satisfying $P(\tau_1 = 0) > 0$, $P(\tau_2 = 0) > 0$, and $P(\tau_1 = 0, \tau_2 = 0) = 0$;
- for $l = 1, 2$, $\left\{X_{ij}^{(l)}, i \geq 1, j \geq 1\right\}$ is a non-negative identically distributed array with mutually independent rows and $\left\{X_{ij}^{(l)}, j \geq 1\right\}$, $i \geq 2$ are independent copies of $\left\{X_{1j}^{(l)}, j \geq 1\right\}$;
- $\{(C_1(t), C_2(t)); t \geq 0\}$ and $\{N(t); t \geq 0\}$ are mutually independent, and they are independent of $\left\{X_{ij}^{(1)}, i \geq 1, j \geq 1\right\}$, $\left\{X_{ij}^{(2)}, \ i \geq 1, j \geq 1\right\}$, and $\left\{\left(\tau_i^{(1)}, \tau_i^{(2)}\right), i \geq 1\right\}$, where the later three sequences are arbitrarily dependent.

### *1.2. Brief review*

The asymptotic analysis for ruin probabilities in compound risk models is an enduring topic that can be observed from numerous references.

Tang *et al.* [24] gave the concept of compound risk models first. According to them, the compound risk models are more realistic than the classical risk models and natural extensions of the classical ones. They established the precise large deviations for this kind of model. Since then, many researchers have further explored this topic and advanced the research. Yang and Wang [27] investigated the precise large deviations for dependent dominatedly varying-tailed random variables, and the results were utilized to derive asymptotic bounds for finite-time ruin probabilities in compound renewal risk models. Yang *et al.* [29] and Zong [31] obtained asymptotic formulae for finite-time ruin probabilities of non-standard compound renewal risk models with negatively dependent claims, respectively. The former discussed two cases, the first one considered the distributions of the numbers of claims to be dominatedly varying-tailed and the second one considered the distributions of the numbers of claims to be in the maximum domain of attraction of the Gumbel distribution, and the claim sizes are light-tailed; the later studied the case where both the distributions of the numbers of claims and the claim sizes are heavy-tailed but without a dominating relationship among their tails. Aleškevičiene *et al.* [1] considered the random sums of independent and identically distributed random variables and applied the main result to study the asymptotic behavior of the finite-time ruin probability. Leipus and Šiaulys [19] studied the modified compound discrete-time risk model and obtained the asymptotics of the finite-horizon ruin probability in such a model for a subclass of heavy-tailed claim sizes and numbers of claims. For some recent research for ruin probabilities in compound risk models, the reader is referred to Yang *et al.* [28], who developed a second-order asymptotic formula that provides a more accurate and efficient solution compared to traditional methods for calculating ruin probabilities over an infinite time and to Liu and Gao [21], who analyzed a nonstandard compound renewal risk model incorporating stochastic return on investments and derived asymptotic formulae for finite-time ruin probabilities.

As the insurance industry continues to develop, insurance operations are becoming increasingly diversified. Traditional unidimensional models are sometimes becoming pale in interpreting realistic actuarial issues. Chan *et al.* [4] promoted it by introducing bi-dimensional ones. For bi-dimensional risk models, various kinds of ruin probabilities are defined, and some relevant results are derived. In this paper, we mainly explore the sum-ruin probability, which is defined in the following way.

$$\psi_{\text{sum}}(x_1, x_2; t) = P\left(T_{\text{sum}} \leq t | (U_1(0), U_2(0)) = (x_1, x_2)\right), \tag{1.2}$$

where $T_{\text{sum}}$ is the corresponding time of ruin defined by

$$T_{\text{sum}} = \inf\{s \geq 0 | U_1(s) + U_2(s) < 0\}.$$

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$T_{\text{sum}}$, first defined in Chan *et al.* [4], is a rather appealing choice to measure the time when an insurance firm ruins. This measure is generally considered a vital actuarial quantity in the measurement of the solvency of an insurance company and has natural interpretations. For instance, $\{T_{\text{sum}} \le t\}$ indicates that the sum of $U_1(s)$ and $U_2(s)$ is negative for one or more instances within the time interval $[0, t]$. For the papers investigating $\psi_{\text{sum}}(x_1, x_2; t)$, we refer the reader to Gao and Yang [16] for pairwise strongly quasi-asymptotically independent (QAI) claims, Cheng and Yu [10] for different claim-number processes and strongly subexponential claims, Cheng [9] for uniform asymptotics for $\psi_{\text{sum}}(x_1, x_2; t)$ with two arbitrarily dependent claim-number processes, Chen *et al.* [8] for the claims that follow the dependence structure appearing in Ko and Tang [18], for Chen *et al.* [7] which partly extended the results of Chen *et al.* [8] by introducing the stochastic returns.

### *1.3. Motivation*

Inspired by the papers mentioned above, this paper aims to study the asymptotic formula for $\psi_{\text{sum}}(x_1, x_2; t)$ in compound risk model (1.1) mainly based on the following considerations:

- We find that almost all papers concerning compound risk models are unidimensional and little work focuses on bi-dimensional ones. However, it is meaningful to investigate bi-dimensional risk models because of the increasing complexity of insurance products.
- $\psi_{\text{sum}}(x_1, x_2; t)$ is rarely studied in bi-dimensional risk models for mathematical complexity. This pushes us to investigate $\psi_{\text{sum}}(x_1, x_2; t)$ in a bi-dimensional risk model for it is an important tool to measure the solvency of an insurance company.
- In a major catastrophe, the number of insurance claims may be beyond expectations, and a larger number of claims implies more casualties and also indicates larger claim amounts. This highlights the importance of assuming arbitrary dependency between the claims sequence and the numbers of claims sequence.

Consequently, in this paper, we are going to investigate the asymptotic formula for the sum-ruin probability in a bi-dimensional compound risk model, where the tails of claims are dominated by the tails of the numbers of claims and no extra dependence structures are equipped between the numbers of claims and the claims, which has not been considered as far as we know.

The remainder of this paper is organized as follows. Section 2 shows some preliminaries and presents our main results. Section 3 conducts some numerical studies, and the results are presented by some figures. Section 4 gives some necessary lemmas and the proofs of the main results.

## 2. Preliminaries and main results

Throughout the paper, we denote by $1_A$ the indicator function of a set $A$ and by $x^+ = x1_{(x\ge 0)}$, $x^- = -x1_{(x<0)}$ the positive and negative parts of $x$, respectively. All limit relations hold as $x \to \infty$ or $(x_1, x_2) \to (\infty, \infty)$ unless otherwise noted. For two univariate or bivariate functions $f$ and $g$, we write $f = o(g)$ if $\lim f/g = 0$; $f = O(g)$ if $\limsup f/g < \infty$; $f \asymp g$ if both $f = O(g)$ and $g = O(f)$; $f \lesssim g$ if $\limsup f/g \le 1$ and $f \sim g$ if $\lim f/g = 1$. For two real numbers $a$ and $b$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

### *2.1. Heavy-tailed distributions*

Heavy-tailed distributions are a superior choice to model the claims or the numbers of claims. Recall a random variable $\xi$ or its distribution function $F_\xi$ is said to be heavy-tailed if $\int_{-\infty}^{\infty} e^{ux} F_\xi(dx) = \infty$ for any $u > 0$. We are particularly interested in the class of distribution functions with consistent variation,

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denoted by $\mathscr{C}$. Recall that a random variable $\xi$ or its distribution function $F_\xi$ supported on $(-\infty,\infty)$ is consistently varying-tailed if

$$\lim_{y\uparrow 1}\limsup \frac{\overline{F_\xi}(xy)}{\overline{F_\xi}(x)} = 1 \quad \text{or} \quad \lim_{y\downarrow 1}\liminf \frac{\overline{F_\xi}(xy)}{\overline{F_\xi}(x)} = 1.$$

We need the following subclasses of the heavy-tailed distribution class to continue our research. A random variable $\xi$ or its distribution function $F_\xi$ is called long-tailed, denoted by $\xi \in \mathscr{L}$ or $F_\xi \in \mathscr{L}$, if for any fixed $y > 0$,

$$\overline{F_\xi}(x+y) \sim \overline{F_\xi}(x);$$

is called dominatedly varying-tailed, denoted by $\xi \in \mathscr{D}$ or $F_\xi \in \mathscr{D}$ if

$$\overline{F_\xi}(xy) = O\left(\overline{F_\xi}(x)\right)$$

holds for some $0 < y < 1$; is called regularly varying-tailed, denoted by $\xi \in \mathscr{R}_{-\alpha}$ or $F_\xi \in \mathscr{R}_{-\alpha}$ for some $\alpha > 0$ if

$$\overline{F_\xi}(xy) \sim y^{-\alpha}\overline{F_\xi}(x)$$

for any $y > 0$. In conclusion, we have the following relations,

$$\mathscr{R}_{-\alpha} \subset \mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{L}.$$

For a more comprehensive discussion about the heavy-tailed distribution class or its subclasses and their applications to insurance and finance, the reader is referred to Bingham *et al.* [3], Cline and Samorodnitsky [13], and Embrechts *et al.* [15], among others.

At the end of this subsection, we recall an elementary quantity. We set

$$\overline{F_\xi}_*(y) = \liminf \frac{\overline{F_\xi}(xy)}{\overline{F_\xi}(x)}$$

and define

$$M^+_{F_\xi} = -\lim_{y\to\infty} \frac{\log \overline{F_\xi}_*(y)}{\log y}$$

as the upper Matuszewska index of distribution function $F_\xi$. The reader is referred to Bingham *et al.* [3] for a formal discussion.

### 2.2. *Dependence structures and main results*

To describe the dependency of random variables, various dependence structures have been introduced over the past few decades. To learn more about popular ones, the reader is referred but not limited to the following sources: Ko and Tang [18], Liu [20], Geluk and Tang [17], Chen and Yuen [5], Asimit and Badescu [2], Chen and Yuen [6], and Cheng and Cheng [12]. Among these papers, Chen and Yuen [5] proposed a dependence structure named quasi-asymptotic independence.

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**Definition 2.1.** *For two non-negative random variables $\xi_1$ and $\xi_2$ with distributions $F_1$ and $F_2$, respectively, they are said to be QAI if the following relation*

$$\lim \frac{P(\xi_1 > x, \xi_2 > x)}{\overline{F_1}(x) + \overline{F_2}(x)} = 0 \tag{2.1}$$

*holds.*

Note that (2.1) can be expressed in terms of copula,

$$\lim_{(u,v)\to(0^+,0^+)} \frac{\hat{C}(u,v)}{u+v} = 0, \quad \text{with} \quad u = \overline{F_1}(x) \quad \text{and} \quad v = \overline{F_2}(x),$$

where $\hat{C}(\cdot,\cdot)$ is the survival copula, which connects with the corresponding copula $C(\cdot,\cdot)$ through the equality $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. For a formal and comprehensive discussion about copula, the reader is referred to Nelson [22]. We point out that Definition 2.1 provides us with a wide range of random variables. Many bivariate copulas satisfy (2.1) such as the Frank copula, the Johnson-Kotz iterated Farlie-Gumbel-Morgenstern copula, the Ali-Mikhail-Haq copula, and so on.

In this paper, we assume that the number of claims $\tau_1$ and $\tau_2$ are QAI. Concurrently, we apply extended negative dependence, which first appeared in Liu [20], to characterize the dependency among the claims.

**Definition 2.2.** *For random variables $\{\xi_n, n \geq 1\}$, if there exists a finite constant $M > 0$ such that for each $n \geq 1$ and all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,*

$$P\left(\bigcap_{i=1}^{n} \{\xi_i > x_i\}\right) \leq M \prod_{i=1}^{n} P(\xi_i > x_i), \tag{2.2}$$

*then the random variables $\{\xi_n, n \geq 1\}$ are said to be extended upper negatively dependent (EUND); analogously, if there exists a finite constant $M > 0$ such that for each $n \geq 1$ and all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,*

$$P\left(\bigcap_{i=1}^{n} \{\xi_i \leq x_i\}\right) \leq M \prod_{i=1}^{n} P(\xi_i \leq x_i), \tag{2.3}$$

*then the random variables $\{\xi_n, n \geq 1\}$ are said to be extended lower negatively dependent (ELND). Further, the random variables $\{\xi_n, n \geq 1\}$ are called extended negatively dependent (END) if they are both EUND and ELND.*

The constant $M$ is called the dominating constant. When dealing with END random variables, we always assume that the dominating constants provided in (2.2) and (2.3) are identical. The extended negative dependence structure indeed contains a wide range of random variables. If $M = 1$ in (2.2) and (2.3), then the random variables $\{\xi_n, n \geq 1\}$ are negatively upper dependent and negatively lower dependent, respectively. And they are called positively dependent (PD) if the inequalities (2.2) and (2.3) both hold in the reverse direction when $M = 1$. An ND sequence obviously must be an END sequence. Nevertheless, it is possible for some PD sequences to find a corresponding positive constant $M$ such that both (2.2) and (2.3) hold, {for example, the Ali-Mikhail-Haq copula with parameter $\theta \in (0, 1]$, and Example 4.1 in [20]. The END structure is notably more extensive than the ND structure as it can not only represent a negative dependence structure but also, to a certain extent, a positive one.

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The END structure and some other dependence structures appearing in the papers mentioned at the beginning of this subsection have been extensively studied. These structures hold significance for research due to their breaking of the classical independent hypothesis and thus bridging the gap between theoretical research and practical applications. Moreover, these concepts have demonstrated their value in various other fields, including precise large deviations, probability inequalities, asymptotic analysis on random sums, and so on.

We would like to share our main results in this part.

**Theorem 2.1.** *Consider the bi-dimensional compound risk model (1.1). For* $l = 1, 2$, *let* $\left\{X_{1j}^{(l)}, j \geq 1\right\}$ *be an END sequence and* $F_l$, $G_l$ *be the distribution functions of* $X_{11}^{(l)}$ *and* $\tau_l$, *respectively. Suppose that* $G_l \in \mathscr{C}$, $\mu_l := EX_{11}^{(l)} < \infty$ *and*

$$x\overline{F_l}(x) = o\left(\overline{G_l}(x)\right). \tag{2.4}$$

*Let* $\tau_1$, $\tau_2$ *be QAI. If* $T > 0$ *satisfies* $\lambda(T) > 0$ *and*

$$E\left[N^{\beta+1}(T)\right] < \infty \quad \text{for some} \quad \beta > M_H^+, \tag{2.5}$$

*where H is any distribution whose tail satisfies* $\overline{H}(x) \sim \overline{G_1}\left(\frac{x}{\mu_1}\right) + \overline{G_2}\left(\frac{x}{\mu_2}\right)$. *Then, we have*

$$\psi_{\text{sum}}(x_1, x_2; T) \sim \sum_{l=1}^{2} \int_0^T \overline{G_l}\left(\frac{x_1 + x_2}{\mu_l} e^{rs}\right) \lambda(ds). \tag{2.6}$$

We will give a corollary in the following to provide convenience to numerical studies, which simplifies (2.6).

**Corollary 2.1.** *Let the conditions of Theorem 2.1 be valid, if* $G_1$, $G_2 \in \mathscr{R}_{-\alpha}$ *for some* $\alpha > 0$, *and* $\{N(t), t \geq 0\}$ *is a Poisson process with rate* $\lambda > 0$, *then we have*

$$\psi_{\text{sum}}(x_1, x_2; T) \sim \frac{\lambda\left(1 - e^{-\alpha r T}\right)}{\alpha r} \sum_{l=1}^{2} \mu_l^{\alpha} \overline{G_l}(x_1 + x_2). \tag{2.7}$$

## 3. Numerical examples

This section will give some numerical examples to examine the accuracy of (2.7). Concretely speaking, we will use the crude Monte Carlo method to estimate the ruin probability defined in (1.2) and compare the simulated values with the asymptotic values on the RHS of (2.7). In what follows, for $l = 1, 2$, we present some settings.

- sample size $m = 100,000$;
- $x_1 + x_2$ varies from 20,000 to 40,000 with step 100;
- $C_l(t) = t$, for any $t \geq 0$;
- the rate $\lambda = 1$, $T = 10$, and $r = 0.03$;

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- for each $n = 1, 2, \cdots, P(\tau_1 = n) = P(\tau_2 = n) = \frac{6}{\pi^2 n^2}$ and $\tau_1$ and $\tau_2$ are connected by Frank copula as follows.

$$C(u_1, u_2) = -\log\left(1 + \frac{(e^{-u_1} - 1)(e^{-u_2} - 1)}{e^{-1} - 1}\right),$$

then, in this case, $G_1, G_2 \in \mathscr{R}_{-1}$.

We will then give the details about how we estimate the empirical values of (1.2) in several steps. For any $1 \leq k \leq m$,

- step 1: independently generate a sequence of inter-arrival times $\{\vartheta_{k1}, \vartheta_{k2}, \cdots,\}$ and determine

$$\tilde{N}_k(T) = \sup\left\{n \geq 1 : \sum_{i=1}^{n} \vartheta_{ki} \leq T\right\};$$

- step 2: generate $\tilde{N}_k(T)$ pairs corresponding numbers of claims $\left(\tau_{kj}^{(1)}, \tau_{kj}^{(2)}\right)$, $1 \leq j \leq \tilde{N}_k(T)$;
- step 3: for every pair of $\left(\tau_{kj}^{(1)}, \tau_{kj}^{(2)}\right)$, $1 \leq j \leq \tilde{N}_k(T)$, generate two sequences of claims $\left\{X_{kij}^{(1)}, 1 \leq i \leq \tau_{kj}^{(1)}\right\}$ and $\left\{X_{kij}^{(2)}, 1 \leq i \leq \tau_{kj}^{(2)}\right\}$;
- step 4: compute the quantity $U_1\left(\sigma_{kj}\right) + U_2\left(\sigma_{kj}\right)$ in the following way,

$$U_1\left(\sigma_{kj}\right) + U_2\left(\sigma_{kj}\right) = x_1 + x_2 + 2\left(\frac{1}{r} - \frac{1}{r}e^{-r(\sigma_{kj})}\right) - \sum_{l=1}^{j}\left(\sum_{i=1}^{\tau_{kl}^{(1)}} X_{kij}^{(1)} + \sum_{i=1}^{\tau_{kl}^{(2)}} X_{kij}^{(2)}\right), \tag{3.1}$$

where $\sigma_{kj} = \sum_{i=1}^{j} \vartheta_{ki}$ and $j = 1, 2, \cdots, \tilde{N}_k(T)$.

After $m$ iterations of such loops, we may give empirical values of (1.2) as follows:

$$\tilde{\psi}_{\text{sum}}(x_1, x_2; T) = \frac{1}{m}\sum_{k=1}^{m} 1_{\left(\wedge_{j=1}^{\tilde{N}_k(T)}\left(U_1\left(\sigma_{kj}\right)+U_2\left(\sigma_{kj}\right)<0\right)\right)}.$$

We will subsequently introduce our numerical examples.

### *3.1. Independent claims*

In the following examples, the claims from the same line of business are independent of each other.

#### *3.1.1. Numerical example 1*

Let $F_1(x) = F_2(x) = 1 - e^{-\sqrt{x}}$, $x > 0$, be Weibull distributions, which means that the insurance firm has to face heavy-tailed claims. When the initial reserve varies, we will repeat steps 1–4 for $m$ times and compute the proportion of ruining samples according to (3.1). Figure 1 presents the result of this example.

#### *3.1.2. Numerical example 2*

Let $F_1(x) = F_2(x) = 1 - e^{-x}$, $x > 0$, be exponential distributions, which means that the insurance firm faces light-tailed claims in the sense that the claims have finite exponential moments. In a manner analogous to Numerical example 1, the result is presented in Figure 2.

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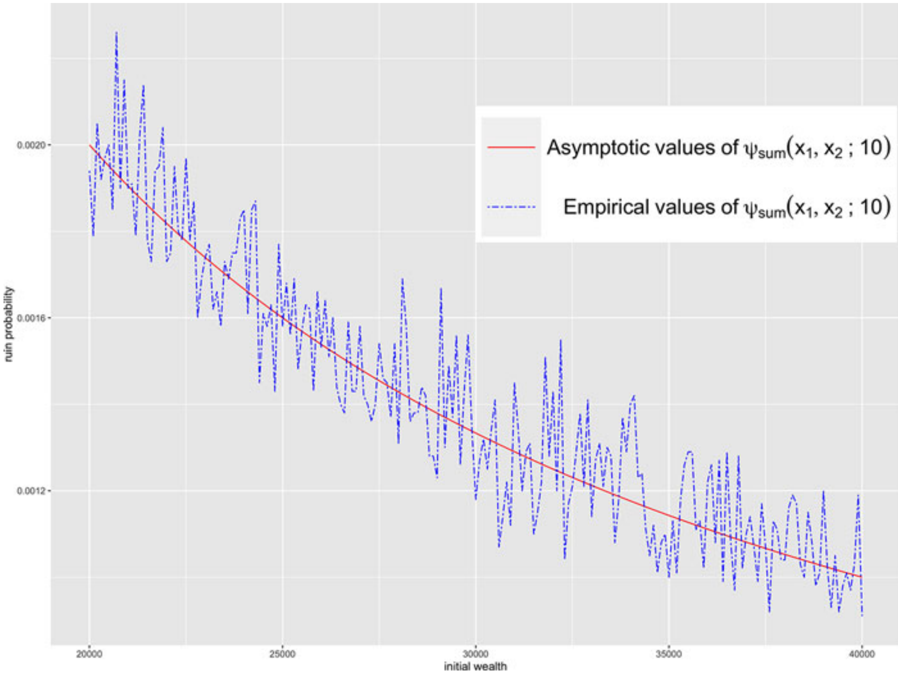


***Figure 1.*** *Asymptotic and empirical values of $\psi_{\mathrm{sum}}(x_1, x_2; T)$ with independent Weibull distributed claims.*

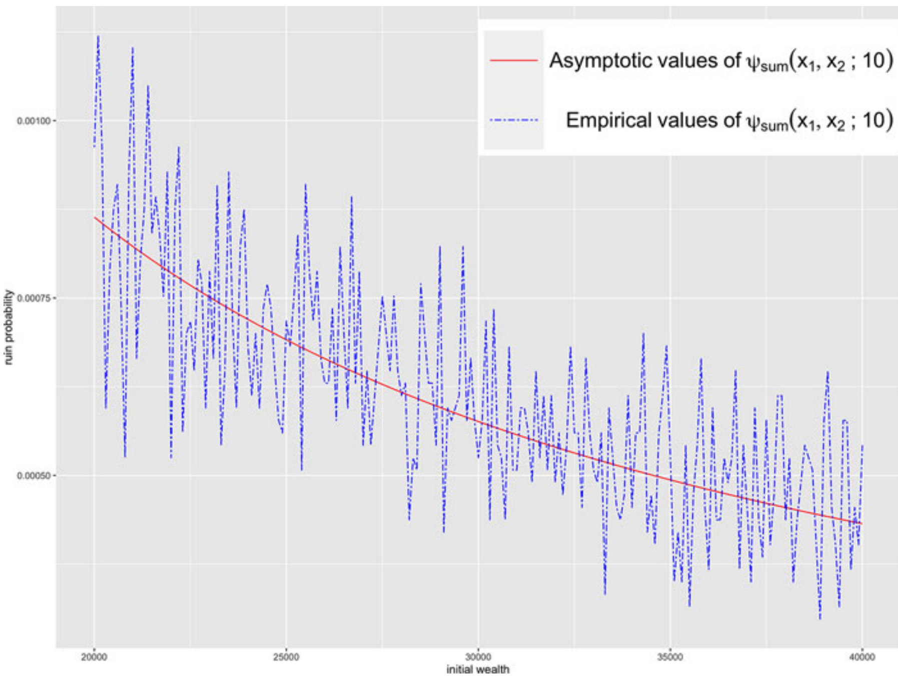


***Figure 2.*** *Asymptotic and empirical values of $\psi_{\mathrm{sum}}(x_1, x_2; T)$ with independent exponentially distributed claims.*

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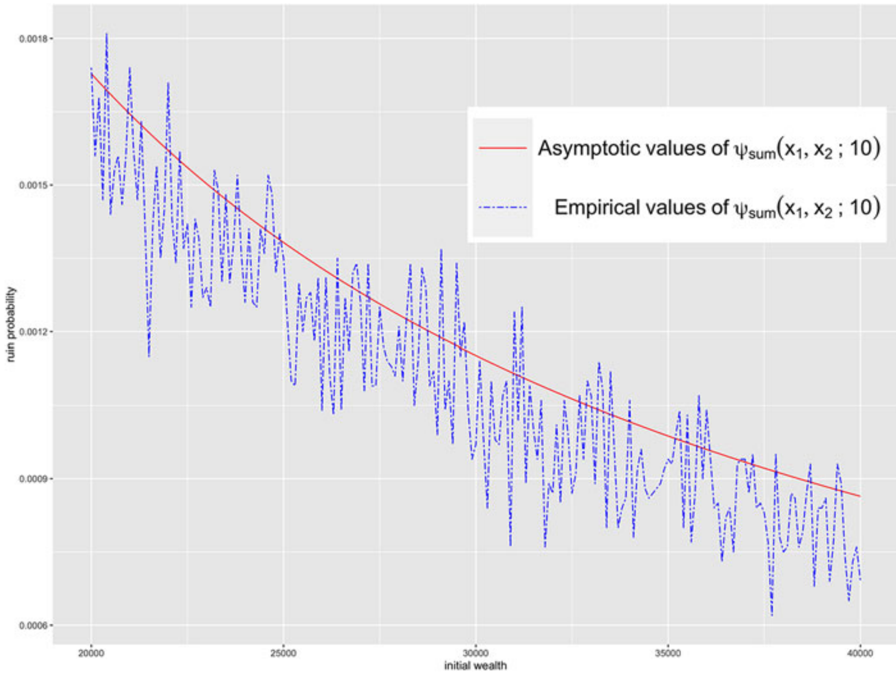


***Figure 3.*** *Asymptotic and empirical values of* $\psi_{\text{sum}}(x_1, x_2; T)$ *with dependent Weibull distributed claims.*

### *3.2. Dependent claims*

In the following examples, the claims from each line of business are jointly distributed by Clayton copula of the form

$$C(u_1, u_2, \cdots, u_n) = \left(u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1} - n + 1\right)^{-1},$$

where $n = \tau_{kj}^{(1)}$ or $\tau_{kj}^{(2)}$ were indicated in the beginning of this section. According to Example 4.2 of [20] and Remark 3.1 of [18], these claims are EDN random variables.

#### *3.2.1. Numerical example 3*

Let $F_1(x) = F_2(x) = 1 - e^{-\sqrt{x}}$, $x > 0$, be Weibull distributions. Similarly, the result is shown in Figure 3.

#### *3.2.2. Numerical example 4*

Let $F_1(x) = F_2(x) = 1 - e^{-x}$, $x > 0$, be exponential distributions, which means that the insurance firm faces light-tailed claims. Figure 4 shows the result of this example.

Through Figures 1–4, we find that the obtained asymptotic results perform well when the initial wealth is large enough. However, we find that the empirical values fluctuate around the asymptotic values, which is not a surprise since the larger the initial wealth is the smaller the ruin probability $\psi_{\text{sum}}(x_1, x_2; T)$ becomes and the more fluctuation the empirical value exhibits.

## 4. Proofs of the main results

### *4.1. Some lemmas*

In preparation for proving Theorem 2.1, we need some necessary lemmas that not only aid in establishing the main results but also have their own merits.

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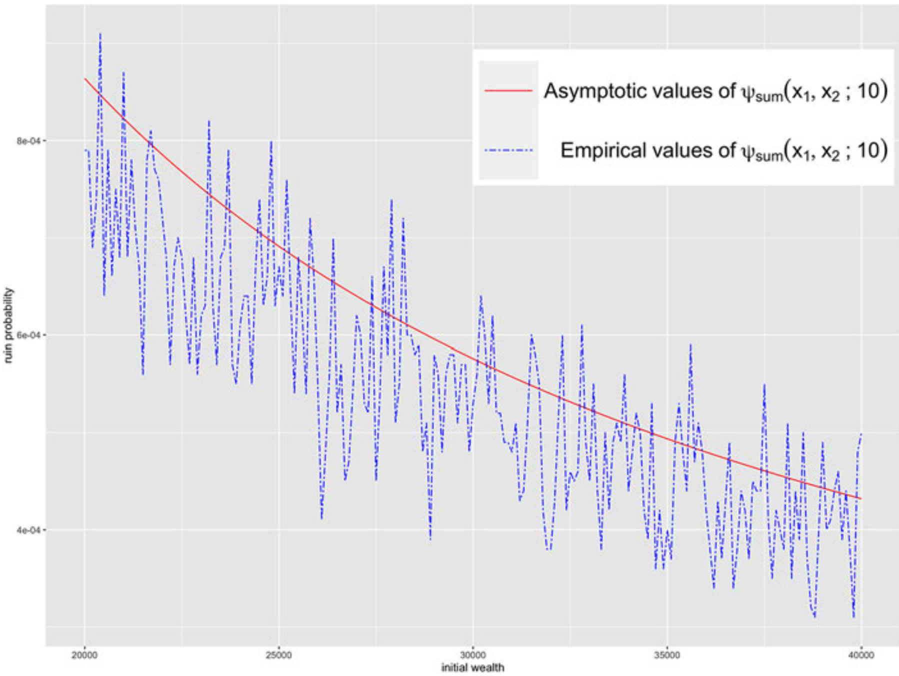


***Figure 4.*** *Asymptotic and empirical values of $\psi_{\text{sum}}(x_1, x_2; T)$ with dependent exponential distributed claims.*

The first lemma obtains a result about tail probabilities of random sums, where the distributions of random numbers of summands dominate the distributions of summands. Aleškevičiene textitet al. [1], Robert and Segers [23], Zhang *et al.* [30], and Cheng [11] have explored this topic in different ways. In contrast to them, the following lemma discusses the random sums of two arbitrarily dependent END sequences, which is powerful in deriving (2.6).

**Lemma 4.1.** *Let $\{\xi, \xi_i, i \geq 1\}$, $\{\eta, \eta_i, i \geq 1\}$ be two sequences of non-negative identically distributed END random variables with positive means, and $\omega_1$, $\omega_2$ be two non-negative integer-valued QAI random variables. If $F_{\omega_1}$, $F_{\omega_2} \in \mathscr{C}$,*

$$x\overline{F_\xi}(x) = o\left(\overline{F_{\omega_1}}(x)\right), \quad x\overline{F_\eta}(x) = o\left(\overline{F_{\omega_2}}(x)\right), \tag{4.1}$$

*then it holds that*

$$P\left(\sum_{i=1}^{\omega_1} \xi_i + \sum_{j=1}^{\omega_2} \eta_j > x\right) \sim \overline{F_{\omega_1}}\left(\frac{x}{E\xi}\right) + \overline{F_{\omega_2}}\left(\frac{x}{E\eta}\right). \tag{4.2}$$

*Proof.* We denote by $H_1$ and $H_2$ the distributions of $\sum_{i=1}^{\omega_1} \xi_i$ and $\sum_{j=1}^{\omega_2} \eta_j$, respectively. According to Theorem 3.1 of [11], we have

$$\overline{H_1}(x) \sim \overline{F_{\omega_1}}\left(\frac{x}{E\xi}\right) \quad \text{and} \quad \overline{H_2}(x) \sim \overline{F_{\omega_2}}\left(\frac{x}{E\eta}\right).$$

This implies that both $H_1$ and $H_2$ belong to $\mathscr{C}$. Therefore, because of Theorem 3.1 of [5], if we want to prove (4.2), it is sufficient to prove that $\sum_{i=1}^{\omega_1} \xi_i$ and $\sum_{j=1}^{\omega_2} \eta_j$ are QAI. For every $0 < \delta < 1$, we do the

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following decompositions.

$$
\begin{aligned}
P\left(\sum_{i=1}^{\omega_1}\xi_i > x, \sum_{j=1}^{\omega_2}\eta_j > x\right) &\le P\left(\sum_{i=1}^{\omega_1}\xi_i > x, \sum_{j=1}^{\omega_2}\eta_j > x, \left\{\omega_1 \le \frac{(1-\delta)x}{E\xi}\right\}\bigcup\left\{\omega_2 \le \frac{(1-\delta)x}{E\eta}\right\}\right) \\
&\quad + P\left(\omega_1 > \frac{(1-\delta)x}{E\xi}, \omega_2 > \frac{(1-\delta)x}{E\eta}\right) \\
&:= I_1 + I_2.
\end{aligned} \tag{4.3}
$$

Since $F_{\omega_1}$ and $F_{\omega_1}$ belong to $\mathscr{C} \subset \mathscr{D}$ and $\omega_1, \omega_2$ are QAI, it is obvious that

$$
\begin{aligned}
I_2 &\le P\left(\omega_1 > \frac{(1-\delta)x}{E\xi \vee E\eta}, \omega_2 > \frac{(1-\delta)x}{E\xi \vee E\eta}\right) \\
&= o\left(\overline{F_{\omega_1}}\left(\frac{(1-\delta)x}{E\xi \vee E\eta}\right) + \overline{F_{\omega_2}}\left(\frac{(1-\delta)x}{E\xi \vee E\eta}\right)\right) \\
&= o\left(\overline{H_1}(x) + \overline{H_2}(x)\right).
\end{aligned} \tag{4.4}
$$

Before we estimate $I_1$, we need to make some preparations. According to (4.1) and Lemma 4.4 of [14], there exists some non-decreasing slowly varying function $L(x)$ with $L(x) \to \infty$ such that $\overline{F_\xi}(x) = o\left(\frac{\overline{F_{\omega_1}}(x)}{xL(x)}\right)$ and $\overline{F_\eta}(x) = o\left(\frac{\overline{F_{\omega_2}}(x)}{xL(x)}\right)$ (a function $L(x)$ is said to be slowly varying if $L(xy) \sim L(x)$ for any $y > 0$). We define two distributions $A_1$ and $A_2$ by

$$
A_l(x) = \left(1 - \frac{\overline{F_{\omega_l}}(x)}{xL(x)} \wedge 1\right) 1_{(x>1)}, \quad l = 1, 2.
$$

It is easy to check that $A_l \in \mathscr{D}$ since $F_{\omega_l} \in \mathscr{C}$, $l = 1, 2$. Note that $\overline{F_\xi}(x) = o\left(\overline{A_1}(x)\right)$ and $\overline{F_\eta}(x) = o\left(\overline{A_2}(x)\right)$, then there exists some positive constant $C$ (irrespective of $x$) such that for large $x$,

$$
\begin{aligned}
I_1 &\le P\left(\sum_{i=1}^{[(1-\delta)x/E\xi]}\xi_i > x\right) + P\left(\sum_{j=1}^{[(1-\delta)x/E\eta]}\eta_j > x\right) \\
&\le P\left(\sum_{i=1}^{[(1-\delta)x/E\xi]}(\xi_i - E\xi) > \frac{\delta}{2}x\right) + P\left(\sum_{j=1}^{[(1-\delta)x/E\eta]}(\eta_j - E\eta) > \frac{\delta}{2}x\right) \\
&\le C\frac{(1-\delta)x}{E\xi}\overline{A_1}\left(\frac{\delta}{2}x\right) + C\frac{(1-\delta)x}{E\eta}\overline{A_2}\left(\frac{\delta}{2}x\right) \\
&= C\frac{(1-\delta)}{E\xi}\frac{\overline{F_{\omega_1}}\left(\frac{\delta}{2}x\right)}{\frac{\delta}{2}L\left(\frac{\delta}{2}x\right)} + C\frac{(1-\delta)}{E\eta}\frac{\overline{F_{\omega_2}}\left(\frac{\delta}{2}x\right)}{\frac{\delta}{2}L\left(\frac{\delta}{2}x\right)} \\
&= o(1)\left(\overline{H_1}(x) + \overline{H_2}(x)\right),
\end{aligned} \tag{4.5}
$$

where the third step holds because of Lemma 5.1 of [11] and the last step holds due to $L(x) \to \infty$ and $F_{\omega_l} \in \mathscr{C} \subset \mathscr{D}$, $l = 1, 2$. Plugging (4.4) and (4.5) into (4.3) yields the conclusion. Thus, (4.2) is proved. □

The following lemma helps prove Theorem 2.1, which can be proved by exploiting Lemma 4.1.

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**Lemma 4.2.** *Let the conditions of Theorem 2.1 be valid, then for any $T$ satisfying $\lambda(T) > 0$ and (2.5), it holds that*

$$P\left(\sum_{i=1}^{N(T)}\sum_{j=1}^{\tau_i^{(1)}} X_{ij}^{(1)} e^{-r\sigma_i} + \sum_{i=1}^{N(T)}\sum_{j=1}^{\tau_i^{(2)}} X_{ij}^{(2)} e^{-r\sigma_i} > x\right) \sim \sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds). \tag{4.6}$$

*Proof.* For simplicity, we write $Y_i^{(1)} = \sum_{j=1}^{\tau_i^{(1)}} X_{ij}^{(1)}$, $Y_i^{(2)} = \sum_{j=1}^{\tau_i^{(2)}} X_{ij}^{(2)}$, $i \geq 1$. It is easy to see that $\left\{Y_i^{(1)}, i \geq 1\right\}$ and $\left\{Y_i^{(2)}, i \geq 1\right\}$ are two sequences of independent and identically distributed random variables. Because of Lemma 4.1, it is not hard to check that for $i \geq 1$, the survival function

$$P\left(Y_i^{(1)} + Y_i^{(2)} > x\right) \sim \overline{G_1}\left(\frac{x}{\mu_1}\right) + \overline{G_2}\left(\frac{x}{\mu_2}\right), \tag{4.7}$$

which suggests that the distribution of $Y_i^{(1)} + Y_i^{(2)}$ belongs to the class $\mathscr{C}$. Relation (4.6) amounts to the conjunction of

$$P\left(\sum_{i=1}^{N(T)} Y_i^{(1)} e^{-r\sigma_i} + \sum_{i=1}^{N(T)} Y_i^{(2)} e^{-r\sigma_i} > x\right) \lesssim \sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds) \tag{4.8}$$

and

$$P\left(\sum_{i=1}^{N(T)} Y_i^{(1)} e^{-r\sigma_i} + \sum_{i=1}^{N(T)} Y_i^{(2)} e^{-r\sigma_i} > x\right) \gtrsim \sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds). \tag{4.9}$$

On the one hand, for any fixed integer $n_0 > 0$, we split the probability into two parts.

$$\begin{aligned} &P\left(\sum_{i=1}^{N(T)} Y_i^{(1)} e^{-r\sigma_i} + \sum_{i=1}^{N(T)} Y_i^{(2)} e^{-r\sigma_i} > x\right) \\ &= \left(\sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{\infty}\right) P\left(\sum_{i=1}^{n} Y_i^{(1)} e^{-r\sigma_i} + \sum_{i=1}^{n} Y_i^{(2)} e^{-r\sigma_i} > x, N(T) = n\right) \\ &:= I_1 + I_2. \end{aligned} \tag{4.10}$$

We write $\Omega_n = \{(y_1, y_2, \cdots, y_{n+1}) : 0 < y_1 < y_2 < \cdots < y_n \leq T < y_{n+1}\}$, then we use the law of total probability to obtain

$$\begin{aligned} I_1 &= \sum_{n=1}^{n_0}\int_{\Omega_n} P\left(\sum_{i=1}^{n}\left(Y_i^{(1)} + Y_i^{(2)}\right)e^{-ry_i} > x\right) P\left(\bigcap_{i=1}^{n+1}\{\sigma_i \in dy_i\}\right) \\ &\sim \sum_{n=1}^{n_0}\int_{\Omega_n}\sum_{i=1}^{n}\left(\overline{G_1}\left(\frac{xe^{ry_i}}{\mu_1}\right) + \overline{G_2}\left(\frac{xe^{ry_i}}{\mu_2}\right)\right) P\left(\bigcap_{i=1}^{n+1}\{\sigma_i \in dy_i\}\right) \\ &\leq \sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\sum_{l=1}^{2} P\left(\tau_l e^{-r\sigma_i} > \frac{x}{\mu_l}, N(T) = n\right) \\ &= \sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds), \end{aligned} \tag{4.11}$$

where (4.7) and Proposition 5.1 of [25] are used in the second step. As for the estimate of $I_2$, by utilizing Lemma 2.4 of [26] and (4.7), we know for the $\beta$ presented in (2.5), there exists some constant $C>0$ such that

$$\begin{aligned} I_2 &\le \sum_{n=n_0+1}^{\infty} P\left(\sum_{i=1}^{n}\left(Y_i^{(1)}+Y_i^{(2)}\right)>x\right)P(N(T)=n) \\ &\lesssim C\sum_{n=n_0+1}^{\infty} n^{\beta+1}\left(\overline{G_1}\left(\frac{x}{\mu_1}\right)+\overline{G_2}\left(\frac{x}{\mu_2}\right)\right)P(N(T)=n) \\ &\le C\left(\overline{G_1}\left(\frac{x}{\mu_1}\right)+\overline{G_2}\left(\frac{x}{\mu_2}\right)\right)E\left[N^{\beta+1}(T)1_{(N(T)>n_0)}\right] \\ &= o\left(\sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds)\right), \quad \text{as} \quad n_0\to\infty, \end{aligned} \tag{4.12}$$

where $G_1, G_2 \in \mathscr{C}$ and (2.5) are used in the last step. Plugging (4.11) and (4.12) into (4.10) yields (4.8). On the other hand, in a manner analogous to deriving (4.11), it holds that

$$\begin{aligned} &P\left(\sum_{i=1}^{N(T)} Y_i^{(1)}e^{-r\sigma_i}+\sum_{i=1}^{N(T)} Y_i^{(2)}e^{-r\sigma_i}>x\right) \\ &\gtrsim \left(\sum_{n=1}^{\infty}-\sum_{n=n_0+1}^{\infty}\right)\int_{\Omega_n}\sum_{i=1}^{n}\left(\overline{G_1}\left(\frac{xe^{ry_i}}{\mu_1}\right)+\overline{G_2}\left(\frac{xe^{ry_i}}{\mu_2}\right)\right)P\left(\bigcap_{i=1}^{n+1}\{\sigma_i\in dy_i\}\right) \\ &:= J_1 - J_2, \end{aligned}$$

where $J_1=\sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds)$ is direct. While $J_2$ can be bounded from above as follows:

$$\begin{aligned} J_2 &\le \left(\overline{G_1}\left(\frac{x}{\mu_1}\right)+\overline{G_2}\left(\frac{x}{\mu_2}\right)\right)E\left[(N(T))\,1_{(N(T)>n_0)}\right] \\ &= o\left(\sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x}{\mu_l}e^{rs}\right)\lambda(ds)\right), \quad \text{as} \quad n_0\to\infty, \end{aligned}$$

where the last step is obtained similarly to (4.12). Then, (4.9) is proved and so is (4.6). This completes the proof of Lemma 4.2. □

### *4.2. Proofs of the main results*

*Proof of Theorem 2.1* The asymptotic upper bound of $\psi_{\text{sum}}(x_1,x_2;T)$ is immediately obtained by Lemma 4.2, now we aim to prove the asymptotic lower bound of $\psi_{\text{sum}}(x_1,x_2;T)$, i.e.,

$$\psi_{\text{sum}}(x_1,x_2;T)\gtrsim \sum_{l=1}^{2}\int_0^T \overline{G_l}\left(\frac{x_1+x_2}{\mu_l}e^{rs}\right)\lambda(ds). \tag{4.13}$$

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Let $B$ be a positive constant. By utilizing Lemma 4.2, it is not hard to verify that

$$\begin{aligned}\psi_{\text{sum}}(x_1,x_2;T) &\geq P\left(\sum_{l=1}^{2}\left(\sum_{i=1}^{N(T)}\sum_{j=1}^{\tau_i^{(l)}}X_{ij}^{(l)}e^{-r\sigma_i}-\int_0^T e^{-rs}C_l(ds)\right)>x_1+x_2\right)\\ &\gtrsim \sum_{l=1}^{2}\int_0^T\overline{G_l}\left(\frac{x_1+x_2+2B}{\mu_l}e^{rs}\right)\lambda(ds)P\left(\int_0^T e^{-rs}C_1(ds)\leq B,\int_0^T e^{-rs}C_2(ds)\leq B\right)\\ &\sim \sum_{l=1}^{2}\int_0^T\overline{G_l}\left(\frac{x_1+x_2}{\mu_l}e^{rs}\right)\lambda(ds)P\left(\int_0^T e^{-rs}C_1(ds)\leq B,\int_0^T e^{-rs}C_2(ds)\leq B\right)\\ &\sim \sum_{l=1}^{2}\int_0^T\overline{G_l}\left(\frac{x_1+x_2}{\mu_l}e^{rs}\right)\lambda(ds),\quad \text{as}\quad B\to\infty,\end{aligned}$$

where the third step holds due to $G_l \in \mathscr{C} \subset \mathscr{L} \cap \mathscr{D}$, $l = 1, 2$. This confirms (4.13) and thus (2.6) is proved. □

*Proof of Corollary 2.1.* It is not hard to verify that (2.7) immediately follows from Theorem 2.1 and Theorem 1.5 of [3]. To save space, we omit the details. □

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