A NOTE ON FINITE-DIMENSIONAL DIFFERENTIABLE MAPPINGS

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Let E be a real infinite-dimensional Banach space. Let $\mathscr{L}$ be the Banach algebra of all continuous linear mappings of E into itself with the topology defined by the norm:

$$||l|| = \sup_{||x||=1} ||l(x)|| \qquad (l \in \mathscr{L}).$$

A mapping f of E into itself is said to be (Fréchet)-*differentiable* if, for each $a \in E$, there exists an $l \in \mathscr{L}$ such that

$$\lim_{||x|| \to 0} \frac{1}{||x||} ||f(a+x)-f(a)-l(x)|| = 0.$$

The linear mapping l is determined uniquely for each a. We denote it by $f'(a)$ and call it the *derivative of f at a*. The set of all differentiable mappings of E into itself is denoted by $\mathscr{D}$. Following definitions were given in [2].

$$d(f) = \{f'(x) | x \in E\} \qquad \text{for } f \in \mathscr{D},$$

$$d(M) = \bigcup_{f \in M} d(f) \qquad \text{for } M \subset \mathscr{D},$$

and

$$d^{-1}(N) = \{f \in \mathscr{D} | d(f) \subset N\} \qquad \text{for } N \subset \mathscr{L}.$$

A mapping f is said to be *finite-dimensional* if the range $R(f)$ is contained in a finite-dimensional subspace of E. The set of all finite-dimensional mappings of E into itself is denoted by $\mathscr{F}$.

The purpose of this paper is to show that

(1) $$\mathscr{F} \cap \mathscr{D} \subsetneqq d^{-1}(\mathscr{F} \cap \mathscr{L})$$

and

(2) *$f \in \mathscr{F} \cap \mathscr{D}$ if and only if there exists a finite-dimensional subspace E_0 of E such that $\bigcup_{x \in E} R(f'(x)) \subset E_0$.*

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1. Proof of (1)

Let f be in $\mathscr{F} \cap \mathscr{D}$. Since $f \in \mathscr{F}$, there exists a finite-dimensional subspace E_0 of E such that $R(f) \subset E_0$. Therefore, from

$$f'(a)(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [f(a+\varepsilon x) - f(a)]$$

it follows that $R(f'(a)) \subset E_0$, which means that $f \in d^{-1}(\mathscr{F} \cap \mathscr{L})$.

To show that the equality does not hold, we consider the case when $E = L^2[-\pi, \pi]$, the Banach space of all square-integrable measurable real functions on the interval $[-\pi, \pi]$. Then, the mapping f:

$$f(x)(t) = \sin(x(t))$$

defined for all $x \in L^2[-\pi, \pi]$. The fact that $f \in \mathscr{D}$ and

$$f'(a)(x)(t) = \cos(a(t))x(t)$$

follows from Theorem 3.3 and Theorem 20.2 of [1]. Next, take the function $e(t)$ such that

$$e(t) = t \text{ for } t \in [-\pi, \pi],$$

and consider the one-dimensional linear mapping

$$l(x) = (x, e)e,$$

where (x, e) denotes the inner product of x and e. Then, for the mapping

$$g(x) = f(l(x)),$$

we have

$$g'(a)(x) = f'(l(a))l(x),$$

from which it follows that

$$\begin{aligned} R(g'(a)) &= \{f'(l(a))(x, e)e \mid x \in E\} \\ &= \{\xi f'(l(a))e \mid -\infty < \xi < \infty\}, \end{aligned}$$

which is obviously a one-dimensional subspace of E.

On the other hand, $R(g)$ is not finite-dimensional, because

$$g\left(n \frac{e}{||e||^2}\right)(t) = f(ne)(t) = \sin(ne(t)) = \sin nt$$

for $n = 1, 2, \cdots$ and $\{\sin nt\}$ is an orthogonal system of $L^2[-\pi, \pi]$.

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2. Proof of (2)

We assume that $f \in \mathscr{D}$ and $\bigcup_{x \in E} R(f'(x)) \subset E_0$ for some finite-dimensional subspace E_0. Take an arbitrary $\bar{x} \in \bar{E}$ (the conjugate space of E) such that $\bar{x}(y) = 0$ for $y \in E_0$. Then, by Lemma 3.2 of [1], we have, for every $x \in E$,

$$\bar{x}(f(x) - f(0)) = \bar{x}(f'(\tau x)(x)) \text{ for some number } \tau,$$

from which it follows that

$$\bar{x}(f(x) - f(0)) = 0.$$

This means that the set

$$\{f(x) - f(0) | x \in E\}$$

is contained in E_0. Therefore, $R(f)$ is contained in a finite-dimensional subspace that is generated by E_0 and $f(0)$. The other half was proved in the previous section.

3. Remark

As in [2], let us regard $\mathscr{D}$ as a near-ring. The example given in the section 1 shows that $\mathscr{F} \cap \mathscr{D}$ is not an ideal of $\mathscr{D}$. (We only consider two-sided ideals.) On the other hand, by the fact proved in [2], $d^{-1}(\mathscr{F} \cap \mathscr{L})$ is a d-ideal of $\mathscr{D}$. (An ideal I of the near-ring $\mathscr{D}$ is said to be a *d-ideal* if $d^{-1}d(I) = I$.) Moreover, we can prove that

(3) *$d^{-1}(\mathscr{F} \cap \mathscr{L})$ is the second smallest d-ideal of $\mathscr{D}$,*

and

(4) *$d^{-1}(\mathscr{F} \cap \mathscr{L})$ is the smallest among d-ideals I such that $d(I)$ is not the zero-ideal of the Banach algebra $\mathscr{L}$.*

Proof. It has been shown in [2] that the set $I(E)$ of all constant mappings is the smallest d-ideal of $\mathscr{D}$ and $d(I(E)) = (0)$. Let us take an arbitrary d-ideal I. If $d(I) = (0)$, we have

$$I = d^{-1}d(I) = d^{-1}((0)) = I(E).$$

If $d(I) \neq (0)$, since $d(I)$ is a non-zero-ideal of $\mathscr{L}$, we have $d(I) \supset \mathscr{F} \cap \mathscr{L}$, from which it follows that

$$I = d^{-1}d(I) \supset d^{-1}(\mathscr{F} \cap \mathscr{L}).$$

References

[1] M. M. Vainberg, *Variational methods for the study of non-linear operators*, translated by A. Feinstein. (Holden-Day, Inc. 1964).

[2] Sadayuki Yamamuro, 'A note on *d*-ideals in some near-algebras', *Journ. Australian Math. Soc.*, 7 (1967), 129–134.

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