ANZIAM J. 61(2019), 305–319
doi:10.1017/S1446181119000129

EXISTENCE AND UNIQUENESS OF WEAK AND CLASSICAL SOLUTIONS FOR A FOURTH-ORDER SEMILINEAR BOUNDARY VALUE PROBLEM

CRISTIAN-PAUL DANET[1]

(Received 12 March, 2019; accepted 9 July, 2019; first published online 19 August 2019)

Abstract

This paper is concerned with the problem of existence and uniqueness of weak and classical solutions for a fourth-order semilinear boundary value problem. The existence and uniqueness for weak solutions follows from standard variational methods, while similar uniqueness results for classical solutions are derived using maximum principles.

2010 *Mathematics subject classification*: primary 35J40; secondary 35J35, 74K20.

Keywords and phrases: fourth-order, maximum principle, plate theory, Paneitz–Branson operator.

1. Introduction

The present work intends to study a semilinear fourth-order equation

$$\Delta^2 u + \varphi(x)\Delta u + \rho(x) f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \quad \text{for all } n \geq 1, \tag{1.1}$$

under the Navier boundary conditions ($u = \Delta u = 0$ on $\partial\Omega$) or the Dirichlet boundary conditions ($u = \partial u/\partial n = 0$ on $\partial\Omega$). Here and throughout the paper, Ω is a bounded domain, $\partial\Omega$ is the boundary of Ω, $\mathrm{diam}(\Omega)$ is the diameter of Ω and $\overline{\Omega} = \Omega \cup \partial\Omega$.

Though by no means exhaustive, we indicate some contexts where equation (1.1) arises. When $n = 1$, $\varphi \equiv k$ (k constant <0), equation (1.1) is known as the Fisher–Kolmogorov equation [2], whereas for $\varphi \equiv k > 0$ it is known as the Swift–Hohenberg equation [23]. For $f(t) = t - t^2$, equation (1.1) arises in the dynamic phase-space analogy of a nonlinearity supported elastic strut [13]. When $f(t) = t^3 - t$, equation (1.1) represents a model for pattern formation in many chemical and biological systems [2].

The case of $n = 2$, $\varphi \equiv 0$ and $f(t) = k_1 t^3 + k_2 t$ with $k_1, k_2 > 0$ arises in bending of cylindrical shells [16]. A uniqueness result for the corresponding boundary value problem under Dirichlet boundary conditions was given by Danet [3]. Equation (1.1)

[1]Department of Applied Mathematics, University of Craiova, Al. I. Cuza St., 13, 200585 Craiova, Romania; e-mail: cristiandanet@yahoo.com.

where $n = 2$, $\varphi \equiv 0$, $\rho \equiv$ constant < 0 and $f(t) = t$ arises in thin plate theory [15]. The case $\varphi \equiv 0$, $f(t) = e^t$, represents a natural higher-order extension of the celebrated Gelfand equation $-\Delta u = 2e^u$, which describes the problems of self-ignition (see Gelfand's article [9]).

Equations that model various aspects of oscillations of suspension bridges are of the type

$$u_{tt} + u_{xxxx} + f(u) = 1.$$

Travelling waves are of interest and are solutions of the form $u = u(x - ct)$ that satisfy

$$u'''' + c^2 u'' + e^u - 1 = 0,$$

where u'' and u'''' denote the second- and fourth-order derivatives, respectively (see the articles [14, 18] and the references cited therein).

There are other various motivations for studying these types of equations. Such equations appear when studying the Paneitz–Branson operator and its generalizations, which have many geometrical properties (in particular, conformal invariance), and are important in mathematical physics (see the articles [7, 8] and the cited references therein).

The paper is organized as follows. In the next section, which is divided in two subsections, we state our main results. The first subsection treats the existence and uniqueness of weak solutions and the next one is dedicated to classical solutions of equation (1.1). We first prove an existence result. In the remaining part of the paper, we prove our main results by defining several P functions which will be used to prove uniqueness for the corresponding boundary value problem, so extending and improving some classical results.

2. Main results

2.1. Existence and uniqueness of weak solutions

Let Ω be a C^2 domain and let $\mathrm{H} = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Here and throughout the paper the symbol $W^{k,p}(\Omega)$ denotes the classical Sobolev spaces (see [1]).

Our first result is concerned with the existence and uniqueness of weak solutions of

$$\begin{cases} \Delta^2 u + \varphi\Delta u + \rho(x) f(u) = 0 & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where φ is a a constant function.

Definition 2.1. A weak solution of (2.1) is a function $u \in \mathrm{H}$ such that

$$\int_\Omega (\Delta u \Delta v - \varphi \nabla u \nabla v + \rho(x) f v)\, dx = 0 \quad \text{for all } v \in \mathrm{H}.$$

A strong solution of (2.1) is a function $u \in \mathrm{W}^{4,2}(\Omega) \cap \mathrm{C}^2(\overline{\Omega})$ that satisfies (2.1) almost everywhere.

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

It is well known (see [17, Ch. 1]) that H becomes a Hilbert space endowed with the scalar product

$$(u, v) \to \int_\Omega \Delta u \Delta v\, dx \quad \text{for all } u, v \in \mathrm{H}.$$

This scalar product induces a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$. Problem (2.1) has a variational structure, and its solutions can be found as critical points of the energy $J : \mathrm{H} \to \mathbb{R}$

$$J(u) = \int_\Omega \Big[\tfrac{1}{2}((\Delta u)^2 - \varphi|\nabla u|^2) + \rho(x)F(u)\Big]\, dx,$$

where $F(s) = \int_0^s f(t)\, dt$.

We may follow one of the author's papers [5] to show that J is weakly lower semicontinuous on the reflexive space H if

$$\rho \geq \rho_0 > 0 \quad \text{in } \Omega, \tag{2.2}$$

$$F(s) \geq -\beta|s|^\alpha \quad \text{where } \beta > 0,\ 1 \leq \alpha < 2. \tag{2.3}$$

Moreover, if $\varphi \leq 0$ and

$$f' \geq 0 \quad \text{in } \mathbb{R}, \tag{2.4}$$

then J is convex.

Note that (2.3) is satisfied if $|f(s)| \leq C|s|^\alpha$, where $\alpha < 1$, $\alpha \neq -1$. We are now able to state the first result.

Theorem 2.2. *Suppose that* (2.2) *and* (2.3) *hold. Then the boundary value problem* (2.1) *has at least one weak solution. If* (2.4) *also holds, then the weak solution is unique.*

If, in addition,

$$F \geq 0 \quad \textit{in } \mathbb{R},$$

then the unique solution is the trivial one, that is, $u \equiv 0$.

If we admit that

$$\partial\Omega \in C^4, \quad f \in C^0(\mathbb{R}), \quad \rho \in C^0(\overline{\Omega}),$$

then the boundary value problem (2.1) *admits a unique strong solution.*

2.2. Existence and uniqueness of classical solutions

2.2.1 *Existence.*

Definition 2.3. A classical solution of:

- equation (1.1) is a function $u \in C^4(\Omega)$;
- the boundary value problem (1.1) under Navier boundary condition is a function $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$;
- the boundary value problem (2.1) is a function $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$.

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

We remark that equation (1.1) is equivalent to the cooperative system

$$\begin{cases} -\Delta u = v & \text{in } \Omega \\ -\Delta v = -\rho(x)f(u) + \varphi(x)v & \text{in } \Omega. \end{cases} \tag{2.5}$$

For $x \in \Omega$, let

$$U(x_1, \dots, x_n) = M \ln(x_1, \dots, x_n) \quad \text{and} \quad V(x_1, \dots, x_n) = N \ln(x_1, \dots, x_n),$$

where the constants $M, N > 0$ are to be determined such that (U, V) is a supersolution of (2.5) (that is, the pair (U, V) satisfies (2.5) with the relation "≥" instead of "=").

Since

$$-\Delta U = \frac{Mn}{(x_1 + \cdots x_n)^2} > V \quad \text{in } \Omega,$$

if we choose $M > Nn(\mathrm{diam}(\Omega) + 1)^2 \ln(n(\mathrm{diam}(\Omega) + 1))$, and since

$$-\Delta V = \frac{Nn}{(x_1 + \cdots x_n)^2} > -\rho f(U) + \varphi(x)V \quad \text{in } \Omega,$$

if $\varphi \leq 0$ and if we choose $N > \alpha n(\mathrm{diam}(\Omega) + 1)^2$ and $|\rho f| \leq \alpha$, we obtain that (U, V) is a supersolution of (2.5).

Since the system (2.5) is cooperative, and $(0, 0)$ and (U, V) are a pair of ordered sub- and supersolutions ($f(0) \leq 0$), respectively, we infer the following existence result.

THEOREM 2.4. *Equation (1.1) has at least one positive classical solution, if ρ is bounded in Ω, $\varphi \leq 0$ in Ω, $f(0) \leq 0$ and f is bounded on bounded intervals in* $\mathbb{R}_+ = [0, \infty)$.

2.2.2 *Uniqueness via maximum principles (the* P *function method* [26]*).* Before we state and prove the uniqueness results, we begin here by proving several maximum principle results for equation (1.1). As a consequence of the maximum principles, we obtain various uniqueness results.

A review of the literature reveals that almost all maximum principles and uniqueness theorems (based on maximum principles) related to equation (1.1) are stated under the restriction $\rho > 0$ and/or $\Delta\rho \leq 0$ and/or (2.4) (see [11, 12, 19, 20, 25, 27, 28]).

The next maximum principles and uniqueness theorems extend classical results of Goyal [11] or Schaefer [25] by treating the uncovered case, ρ, $\Delta\rho$, f' of arbitrary sign. The first P function we introduce below is based on the Miranda function [21] $\mathrm{M} = |\nabla u|^2 - u\Delta u$. In the following, we adopt the notation $u_{,i} = \partial u/\partial x_i$, $u_{,ij} = \partial^2 u/\partial x_i \partial x_j$, and so on. Also, repeated indices are summed from one to n.

THEOREM 2.5. *Let u be a classical solution of (1.1) in $\Omega \subset \mathbb{R}^2$, where $\rho \in C^2(\Omega)$. Suppose that one of the following conditions is satisfied:*

$$sf(s) \geq 0 \quad \text{in } \mathbb{R}, \quad \rho > 0 \quad \text{in } \overline{\Omega}, \quad \varphi \geq 0 \quad \text{in } \Omega, \tag{2.6}$$

$$sf(s) \geq 0 \quad \text{in } \mathbb{R}, \quad \rho < 0 \quad \text{in } \overline{\Omega}, \quad \varphi \leq 0 \quad \text{in } \Omega. \tag{2.7}$$

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

(a) *If, for some* $\alpha \in \mathbb{R}$,

$$\alpha \frac{\rho\Delta\rho + (\alpha - 1)|\nabla\rho|^2}{\rho^2} + \varphi \le 0 \quad \text{in } \Omega, \tag{2.8}$$

then the function

$$\mathrm{P}_\alpha = \frac{|\nabla u|^2 - u\Delta u}{\rho^\alpha} = \frac{\mathrm{M}}{\rho^\alpha}$$

does not attain a nonnegative maximum in Ω, *unless it is constant in* Ω.

(b) *Suppose that either* (2.6) *with* $\varphi \equiv 0$ *or* (2.7) *holds, and let* $i_1, \dots, i_n \in \{1, \dots, n\}$ *be distinct numbers. If one of the following conditions holds:*

(i) $\rho_{,k} \ge 0$ *for all* $k = 1, \dots, n$ *in* Ω*;*
(ii) *there exist(s)* $i_1, \dots, i_q$ $(1 \le q \le n-1)$ *such that* $\rho_{,i_1}, \dots, \rho_{,i_q} \le 0$ *in* Ω *and the rest of the functions* $\rho_{,k}$ *are nonnegative in* Ω*;*
(iii) $\rho_{,k} \le 0$ *for all* $k = 1, \dots, n$ *in* Ω,

then, for a sufficiently small α, *the function*

$$\frac{\mathrm{P}_\alpha}{\psi}$$

does not attain a nonnegative maximum in Ω, *unless it is constant in* Ω. *Here* $\psi(x) = 1 - be^{ax_i} > 0$ *in* $\overline{\Omega}$ *for some* $i \in \{1, \dots, n\}$, *where* $b = \sup_\Omega \gamma(\mathrm{diam}(\Omega))^2/4$, $a = 2/\mathrm{diam}(\Omega)$ *and* γ *is such that* $\sup_\Omega \gamma < 4/e^2(\mathrm{diam}(\Omega))^2$.

PROOF. (a) Since

$$\begin{aligned}\Delta\mathrm{P}_\alpha &= \Delta\Big(\frac{1}{\rho^\alpha}\Big)\mathrm{M} + \nabla\Big(\frac{1}{\rho^\alpha}\Big)\nabla\mathrm{M} + \frac{1}{\rho^\alpha}\Delta\mathrm{M} \\ &= -\alpha\frac{\rho\Delta\rho - (\alpha+1)|\nabla\rho|^2}{\rho^{\alpha+2}}\mathrm{M} - \frac{2\alpha}{\rho^{\alpha+1}}\nabla\rho\nabla\mathrm{M} + \frac{1}{\rho^\alpha}\Delta\mathrm{M},\end{aligned}$$

we get

$$\Delta\mathrm{P}_\alpha + \frac{2\alpha}{\rho}\nabla\rho\nabla\mathrm{P}_\alpha = -\alpha\frac{\rho\Delta\rho - (\alpha+1)|\nabla\rho|^2}{\rho^{\alpha+2}}\mathrm{M} + \frac{1}{\rho^\alpha}\Delta\mathrm{M} - \frac{2\alpha^2|\nabla\rho|^2}{\rho^{\alpha+2}}\mathrm{M}.$$

Hence, P_α satisfies

$$\Delta\mathrm{P}_\alpha + \frac{2\alpha}{\rho}\nabla\rho\nabla\mathrm{P}_\alpha + \alpha\frac{\mathrm{P}_\alpha}{\rho^2}(\rho\Delta\rho + (\alpha-1)|\nabla\rho|^2) = \frac{1}{\rho^\alpha}\Delta\mathrm{M}.$$

Using the well-known inequality

$$u_{,ij}\,u_{,ij} \ge (\Delta u)^2/2$$

and equation (1.1),

$$\Delta\mathrm{M} = 2u_{,ij}\,u_{,ij} - (\Delta u)^2 - u\Delta^2 u \ge \varphi u\Delta u + \rho u f(u),$$

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

which yields

$$\Delta \mathrm{P}_\alpha + \frac{2\alpha}{\rho}\nabla\rho\nabla \mathrm{P}_\alpha + \mathrm{P}_\alpha\Big(\alpha\frac{\rho\Delta\rho + (\alpha-1)|\nabla\rho|^2}{\rho^2} + \varphi\Big) \geq \frac{uf(u)}{\rho^{\alpha-1}} + \frac{\varphi|\nabla u|^2}{\rho^\alpha}. \tag{2.9}$$

If (2.6) is satisfied, we can choose α to be arbitrary and get from (2.9) that

$$\Delta \mathrm{P}_\alpha + \frac{2\alpha}{\rho}\nabla\rho\nabla \mathrm{P}_\alpha + \mathrm{P}_\alpha\Big(\alpha\frac{\rho\Delta\rho + (\alpha-1)|\nabla\rho|^2}{\rho^2} + \varphi\Big) \geq 0 \quad \text{in } \Omega. \tag{2.10}$$

The conclusion can now be inferred from the classical maximum principle [24]. If (2.7) holds, then the conclusion follows by choosing α to be odd.

(b) According to inequality (2.10) and some previous results of the author [6, Theorem 3.2] or [4, Theorem 1.11, page 15], the function P_α/ψ cannot attain a nonnegative maximum in Ω, unless it is a constant, if one of the conditions (i)–(iii) are satisfied and if

$$\alpha\Big(\frac{\rho\Delta\rho + (\alpha-1)|\nabla\rho|^2}{\rho^2}\Big) + \varphi \leq \frac{4}{e^2\,\mathrm{diam}(\Omega)} \quad \text{in } \Omega. \tag{2.11}$$

Since α is arbitrary, we can choose α sufficiently small such that inequality (2.11) always holds.

This completes the proof of the theorem. □

With the aid of Theorem 2.5, we establish two uniqueness results that extend Schafer's classical uniqueness results [25].

Theorem 2.6. *Suppose that either:*

(a) *conditions* (2.6) *and* (2.8) *hold;*
or
(b) *condition* (2.6) ($\varphi \equiv 0$) *and one of the conditions* (i)–(iii) *of Theorem* 2.5 *hold.*

Then the boundary value problem

$$\begin{cases} \Delta^2 u + \varphi(x)\Delta u + \rho(x) f(u) = 0 & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

has no nontrivial classical solution in the C^1 domain Ω.

Proof. Suppose that (b) holds for some α. By Theorem 2.5,

$$\frac{\mathrm{P}_\alpha}{\Psi} \quad \text{does not attain a nonnegative maximum in } \Omega \tag{2.12}$$

unless there exists a constant $k \in \mathbb{R}$ such that

$$\frac{\mathrm{P}_\alpha}{\Psi} \equiv k \quad \text{in } \Omega. \tag{2.13}$$

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

From (2.12) we get

$$\max_{\overline{\Omega}} \frac{\mathrm{P}_\alpha}{\Psi} = \max_{\partial\Omega} \frac{\mathrm{P}_\alpha}{\Psi} = 0,$$

which means that $\mathrm{P}_\alpha \leq 0$ in Ω. From (2.13), the boundary conditions and continuity, we obtain $k = 0$, that is, $\mathrm{P}_\alpha \equiv 0$ in Ω. It follows that $|\nabla u| = 0$ in Ω, so that $u = $ constant, and therefore $u \equiv 0$ in Ω.

Hence we are left to check the case $\mathrm{P}_\alpha < 0$: that is,

$$|\nabla u|^2 - u\Delta u < 0 \quad \text{in } \Omega.$$

After integrating the last inequality over Ω,

$$\int_\Omega |\nabla u|^2 \, dx < 0,$$

which is impossible. Hence, $u \equiv 0$ in Ω could be the only solution (if $f(0) = 0$).

We can argue similarly if we are under hypothesis (a), so this completes the proof. □

Theorem 2.7. *Under hypotheses* (a) *or* (b) *of Theorem 2.6, the boundary value problem*

$$\begin{cases} \Delta^2 u + \varphi(x)\Delta u + \rho(x) f(u) = 0 & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.14}$$

has no nontrivial classical solution in the convex domain Ω.

Proof. Since the proof of part (a) is essentially contained in the proof of part (b), it will be omitted.

By the maximum principle Theorem 2.5, for some small α, the function $\mathrm{M}/\rho^\alpha\Psi$ takes its nonnegative maximum on the boundary $\partial\Omega$ at a point, say, $y_0 = (y_0^1, y_0^2)$, unless, $\mathrm{M}/\rho^\alpha\Psi \equiv$ constant in $\overline{\Omega}$, where $\psi(x) = 1 - be^{ax_2}$.

By introducing normal coordinates in the neighbourhood of the boundary, we write (see [26, page 46, relation 4.3])

$$\Delta u = \frac{\partial^2 u}{\partial n^2} + \frac{\partial^2 u}{\partial s^2} + K\frac{\partial u}{\partial n} \quad \text{on } \partial\Omega, \tag{2.15}$$

where $K > 0$ denotes the curvature of $\partial\Omega$, and $\partial u/\partial s$ denotes the tangential derivative of u. Since $u = \Delta u = 0$ on $\partial\Omega$, equation (2.15) becomes

$$\frac{\partial^2 u}{\partial n^2} = -K\frac{\partial u}{\partial n}. \tag{2.16}$$

Using (2.16) and the convexity assumption,

$$\frac{\partial \mathrm{M}}{\partial n} = -2K\Big(\frac{\partial u}{\partial n}\Big)^2 < 0 \quad \text{on } \partial\Omega. \tag{2.17}$$

By computation and using (2.17),

$$\frac{\partial}{\partial n}\Big(\frac{\mathrm{M}}{\rho^\alpha\Psi}\Big) = \Big(\frac{\partial u}{\partial n}\Big)^2\Big(- 2K\rho^\alpha\Psi - \frac{\partial(\rho^\alpha\Psi)}{\partial n}\Big)\frac{1}{\rho^{2\alpha}\Psi^2},$$

and, by the convexity assumption, we get $-2K\rho^\alpha\Psi < 0$.

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

We now choose at y_0 a principal coordinate system (see [10, page 354] for details). The outer unit normal at y_0 is $n(y_0) = (n^1(y_0), n^2(y_0)) = (0, -1)$. Hence

$$\frac{\partial(\rho^\alpha \Psi)}{\partial n}(y_0) = \alpha\rho^{\alpha-1}(y_0)\frac{\partial\rho}{\partial n}(y_0)\Psi(y_0) + ab\rho^\alpha(y_0)e^{ay_0^2}. \tag{2.18}$$

If $(\partial\rho/\partial n)(y_0) < 0$, then we choose α sufficiently small such that (2.11) holds (that is, the maximum principle holds) and

$$\frac{\partial(\rho^\alpha \Psi)}{\partial n}(y_0) > 0.$$

Consequently,

$$\frac{\partial}{\partial n}\Big(\frac{\mathrm{M}}{\rho^\alpha\Psi}\Big)(y_0) < 0,$$

which is a contradiction to the generalized maximum principle of Hopf [24, Theorem 10, page 73]). It follows from the maximum principle that there exists a constant $C \geq 0$ such that

$$\mathrm{M} = C\rho^\alpha\Psi \quad \text{in } \overline{\Omega}.$$

By (2.18), the case $C > 0$ would imply that

$$\frac{\partial \mathrm{M}}{\partial n}(y_0) > 0,$$

which contradicts (2.17). It follows that $C = 0$, that is, $\mathrm{M} = 0$ in Ω, which gives $|\nabla u|^2 = 0$, and therefore $u \equiv 0$ in Ω.

We are now left to check the case

$$\frac{\mathrm{M}}{\rho^\alpha\Psi} < 0 \quad \text{in } \Omega.$$

It follows that

$$|\nabla u|^2 - u\Delta u < \quad \text{in } \Omega.$$

We use the argument presented by Schaefer [25] to get again $u \equiv 0$ in $\overline{\Omega}$ (see also a simplified argument by Sperb [26, Corollary 10.1, pages 177–178]). □

We shift our attention from the two-dimensional case to the n-dimensional case, and prove four maximum principles and uniqueness results under some relaxed hypotheses (f', ρ or $\Delta\rho$ of arbitrary sign), so extending results of Schaefer [25] and Zhang [28].

Theorem 2.8. *Let u be a classical solution of (1.1) in $\Omega \subset \mathbb{R}^n$, where $\varphi \leq 0$, $\rho \in C^2(\Omega)$ and $f \in C^1(\mathbb{R})$.*

(a) *Assume that*

$$\begin{aligned} &\rho f' > \beta \quad \text{for some } \beta > 0, \\ &(\rho f' - \beta)\Delta\rho F - f^2|\nabla\rho|^2 \geq 0. \end{aligned} \tag{2.19}$$

Then the functional

$$\mathrm{P}_\beta = \frac{(\Delta u)^2}{2} + \beta|\nabla u|^2 + \rho(x)\int_0^u f(s)\,ds$$

assumes its maximum value on $\partial\Omega$.

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

(b) *Assume that the following inequalities hold:*

$$\begin{aligned} &\rho f' > -\gamma, \\ &(\rho f' + \gamma)\Big(\Delta\rho + \frac{\gamma}{\beta}\rho\Big)F - f^2|\nabla\rho|^2 \geq 0 \end{aligned} \tag{2.20}$$

for some constants $\beta > 0$ *and* $\gamma \geq 0$*, such that*

$$\frac{\gamma}{\beta} \leq \frac{\pi^2}{d^2}.$$

Then the function

$$\frac{\mathrm{P}_\beta}{\chi}$$

cannot attain a nonnegative maximum in Ω *unless*

$$\frac{\mathrm{P}_\beta}{\chi} \equiv \text{constant}.$$

Here we suppose that Ω *lies in a strip (of width* d*)* $0 < x_i < d$ *for some* $i = 1, 2, \ldots, n$*, and*

$$\chi(x) = \cos\frac{\pi(2x_i - d)}{2(d+\varepsilon)}\prod_{j=1}^{n}\cosh(\varepsilon x_j) \in C^\infty(\overline{\Omega}),$$

where $\varepsilon > 0$ *is small.*

Proof. (a) A computation shows that

$$\Delta\mathrm{P}_\beta \geq |\nabla(\Delta u)|^2 + 2\beta\nabla u\nabla(\Delta u) + \Delta\rho F(u) + \rho f'(u)|\nabla u|^2 + 2f(u)\rho_{,i}\, u_{,i}\,.$$

Completing the square of the first two terms gives

$$\Delta\mathrm{P}_\beta \geq (\rho f' - \beta)|\nabla\rho|^2 + 2f(u)\rho_{,i}\, u_{,i} + \Delta\rho F(u).$$

Again, we complete the square of the first two terms by adding and subtracting the term $f^2\rho_{,i}\,\rho_{,i}\,/(\rho f' - \beta)$, to get

$$\Delta\mathrm{P}_\beta \geq \Delta\rho F(u) - \frac{f^2\rho_{,i}\,\rho_{,i}}{\rho f' - \beta} \geq 0 \quad \text{in } \Omega,$$

by (2.19), so that the conclusion follows from the maximum principle.

(b) By calculations

$$\Delta\mathrm{P}_\beta + \frac{\gamma}{\beta}\mathrm{P}_\beta \geq F\Big(\Delta\rho + \frac{\gamma}{\beta}\rho\Big) + |\nabla u|^2(\rho f' + \gamma) + 2f\rho_{,i}\, u_{,i}\,.$$

Completing the square of the last two terms and using (2.20) gives

$$\Delta\mathrm{P}_\beta + \frac{\gamma}{\beta}\mathrm{P}_\beta \geq F\Big(\Delta\rho + \frac{\gamma}{\beta}\rho\Big) - \frac{f^2|\nabla\rho|^2}{\rho f' + \gamma} \geq 0 \quad \text{in } \Omega.$$

The conclusion now follows from a previous result [3, Theorem 2.1]. □

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

THEOREM 2.9. *Suppose that we are under hypothesis* (a) *of Theorem* 2.8 *and that* $\rho F \geq 0$ *in* Ω*. Then the only classical solution of the boundary value problem* (2.14) *in the convex domain* $\Omega \subset \mathbb{R}^n$ *is the trivial solution.*

PROOF. According to Theorem 2.8, for some β, the function P_β attains its maximum value on $\partial\Omega$ at a point x_0. From Hopf's lemma [24, Theorem 10, page 73], it follows that

$$\frac{\partial \mathrm{P}_\beta}{\partial n} > 0 \quad \text{at } x_0.$$

A computation shows that

$$\frac{\partial \mathrm{P}_\beta}{\partial n} = \frac{\partial \Delta u}{\partial n}\Delta u + \beta \frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2} + \rho f(u)\frac{\partial u}{\partial n} + \frac{\partial \rho}{\partial n}\int_0^u f(s)\,ds. \tag{2.21}$$

Note that, from $\int_0^u f(t)\,dt \geq 0$ and $f' > 0$, it follows that $f(0) = 0$.

By similar calculations used in the the proof of Theorem 2.7 and using the boundary conditions, equation (2.21) becomes

$$\frac{\partial \mathrm{P}_\beta}{\partial n} = -\beta H\Big(\frac{\partial u}{\partial n}\Big)^2 \leq 0 \quad \text{on } \partial\Omega,$$

where $H > 0$ denotes the mean curvature of Ω. This contradicts Hopf's lemma at the point $x_0 \in \partial\Omega$, where P_β ($\mathrm{P}_\beta \not\equiv$ constant) assumes its maximum value. Hence P_β is constant in $\overline{\Omega}$.

It follows that $\partial \mathrm{P}_\beta/\partial n = 0$ on $\partial\Omega$ and, consequently, $\partial u/\partial n = 0$ on $\partial\Omega$. By the boundary conditions, it follows that $\mathrm{P}_\beta \equiv 0$ in $\overline{\Omega}$. Since we have supposed that $\rho F \geq 0$ in Ω, it follows that $u \equiv 0$ in $\overline{\Omega}$. □

THEOREM 2.10. *Suppose that we are under hypothesis* (b) *of Theorem* 2.8*. Then the boundary value problem* (2.14) *has no nontrivial classical solution in the convex domain* $\Omega \subset \mathbb{R}^n$ *if* $\rho F \geq 0$ *in* Ω*.*

PROOF. According to Theorem 2.8, the function P_β/χ does attain a nonnegative maximum on $\partial\Omega$ at a point x_0, unless

$$\frac{\mathrm{P}_\beta}{\chi} \equiv \text{constant},$$

where

$$\chi(x) = \cos\frac{\pi(2x_n - \operatorname{diam}(\Omega))}{2(\operatorname{diam}(\Omega) + \varepsilon)}\prod_{j=1}^n \cosh(\varepsilon x_j).$$

Without loss of generality, we assume that Ω lies in the strip $0 < x_n < \operatorname{diam}(\Omega)$ and that $x_0 \in \partial\Omega \cap \{\operatorname{diam}(\Omega/2) < x_n < \operatorname{diam}(\Omega)\} = \partial\Omega^1$. A computation shows that

$$\frac{\partial}{\partial n}\Big(\frac{\mathrm{P}_\beta}{\chi}\Big) = -\frac{\beta}{\chi^2}\Big(H\chi + \frac{\partial \chi}{\partial n}\Big) \quad \text{on } \partial\Omega.$$

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

Since
$$0 < \frac{\pi}{2(\operatorname{diam}(\Omega)+\varepsilon)}(2x_n - \operatorname{diam}(\Omega)) < \pi/2,$$
it follows that
$$0 < \sin\frac{\pi}{2(\operatorname{diam}(\Omega)+\varepsilon)}(2x_n - \operatorname{diam}(\Omega)) < 1.$$
Consequently, $\partial\chi/\partial x_n < 0$ on $\partial\Omega^1$.

As in the proof of Theorem 2.7, we choose a principal coordinate system at x_0 and, by the last inequality,
$$\frac{\partial\chi}{\partial n}(x_0) > 0.$$
Hence,
$$\frac{\partial}{\partial n}\Big(\frac{\mathrm{P}_\beta}{\chi}\Big)(x_0) < 0.$$
Now we use the same arguments as in the final part of the proof of Theorem 2.9 to conclude that $\mathrm{P}_\beta \equiv 0$. Finally, since $\rho F \geq 0$, we have $u \equiv 0$ in $\overline{\Omega}$, and the proof is complete. □

Theorem 2.11. *Let u be a classical solution of (1.1) in $\Omega \subset \mathbb{R}^n$, where $\varphi \equiv$ constant and $f(s) = s$. Suppose that one of the following conditions holds:*

(a) $\varphi \in (0,1]$*; or*
(b) $\varphi < 0$*,* $\rho > 0$ *in* Ω *and* $\varphi^2\rho \leq (\rho-1)^2$ *in* Ω.

Then the function
$$\mathrm{P}_1 = \tfrac{1}{2}(\Delta u + \varphi u)^2 + \tfrac{1}{2}(\Delta u)^2 + u^2$$
takes its maximum value on $\partial\Omega$.

Proof. (a) By computing and completing the square, we get
$$\begin{aligned}\Delta \mathrm{P}_1 &\geq -\varphi(\Delta u)^2 - \rho\varphi u^2 - 2(\rho-1)u\Delta u\\ &= -\varphi\Big((\Delta u)^2 + \rho u^2 + 2\frac{\rho-1}{\varphi}u\Delta u\Big)\\ &\geq -\varphi\Big((\Delta u)^2\Big(1-\frac{1}{\varphi^2}\Big) + u^2(-\rho^2+\rho-1)\Big) \geq 0 \quad \text{in } \Omega.\end{aligned}$$

The proof follows from the classical maximum principle.

(b) In a similar manner, we obtain
$$\begin{aligned}\Delta \mathrm{P}_1 &\geq -\varphi(\Delta u)^2 - \rho\varphi u^2 - 2(\rho-1)u\Delta u\\ &= \rho\Big(-\varphi u^2 - 2\frac{\rho-1}{\rho}u\Delta u\Big) - \varphi(\Delta u)^2\\ &\geq (\Delta u)^2\Big(-\varphi + \frac{\rho}{\varphi}\Big(\frac{\rho-1}{\rho}\Big)^2\Big) \geq 0 \quad \text{in } \Omega.\end{aligned}$$

This completes the proof. □

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

We note that the previous result as well as the following uniqueness result (case (a)) hold without any restriction on f. The result follows by a simple application of Theorem 2.11.

Theorem 2.12. *There is at most one classical solution of the boundary value problem*

$$\begin{cases} \Delta^2 u + \varphi(x)\Delta u + f(u) = 0 & \text{in } \Omega \\ u = g_1, \quad \Delta u = g_2 & \text{on } \partial\Omega, \end{cases}$$

where one of the following conditions holds:

(a) *f is an arbitrary function and $\varphi \in (0, 1]$; or*
(b) *$\varphi < 0$, $f' > 0$ in Ω and $\varphi^2 f' \le (f' - 1)^2$ in Ω.*

Remark 2.13. Although the maximum principles and uniqueness results stated above do not apply to the equation that appears in the plate theory [15], $\Delta^2 u - \rho u = 0$ in Ω, where $\rho > 0$ is constant, one can check that the functions

$$\mathrm{P}_2 = |\nabla u|^2 - u\Delta u + \frac{u^2}{2} \quad \text{and} \quad \mathrm{P}_3 = \frac{|\nabla u|^2 - u\Delta u}{\rho} + \frac{u^2}{2}$$

satisfy a maximum principle if $\rho \le 1/2$, $d^2 \le \pi^2$ and $\rho \ge 2$, $\rho d^2 \le \pi^2$, respectively. Hence, we can deduce uniqueness results for the corresponding boundary value problem.

So far, we have only considered uniqueness results under Navier or Dirichlet boundary conditions.The next maximum principle allows us to prove a uniqueness result for a different kind of boundary condition.

Theorem 2.14. *Let u be a classical solution of equation (1.1) in $\Omega \subset \mathbb{R}^n$, where $n \le 4$, $\varphi \equiv$ constant ≤ 0, $\rho \ge 0$ in Omega and $f' \ge 0$ in $\mathbb{R}$. Then the functional*

$$\mathrm{P}_4 = u_{,ij}\, u_{,ij} - \nabla u \nabla(\Delta u) - \frac{\varphi}{2}|\nabla u|^2$$

attains its maximum value on $\partial\Omega$.

Proof. We observe that

$$\begin{aligned} \Delta \mathrm{P}_4 = {} & 2(\Delta u)_{,ij}\, u_{,ij} + 2u_{,ijk}\, u_{,ijk} - |\nabla(\Delta u)|^2 - 2(\Delta u)_{,ij}\, u_{,ij} - \nabla u \nabla(\Delta^2 u) \\ & - \varphi(u_{,ij}\, u_{,ij} + \nabla u \nabla(\Delta u)). \end{aligned}$$

Using Payne's inequality [22] (which holds in n dimensions) [26, inequality (10.3), page 179]

$$u_{,ijk}\, u_{,ijk} \ge \frac{3}{n+2}|\nabla(\Delta u)|^2,$$

and our assumptions, we get

$$\begin{aligned} \Delta \mathrm{P}_4 & \ge -\nabla u \nabla(-\varphi\Delta u - \rho f(u)) - \varphi \nabla u \nabla(\Delta u) \\ & = \rho f' |\nabla u|^2 \ge 0 \quad \text{in } \Omega. \end{aligned}$$

Hence P_4 is subharmonic, and the conclusion follows by the classical maximum principle. □

https://doi.org/10.1017/S1446181119000129 Published online by Cambridge University Press

THEOREM 2.15. *There is no nontrivial classical solution of the boundary value problem*

$$\begin{cases} \Delta^2 u + \varphi\Delta u + \rho(x) f(u) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial^2 u}{\partial n^2} = 0 & \text{on } \partial\Omega \in C^2, \end{cases}$$

where φ, ρ and f satisfy the conditions of Theorem 2.14.

PROOF. Using Theorem 2.14 we deduce that

$$\max_{\overline{\Omega}} \mathrm{P}_4 = \max_{\partial\Omega} \mathrm{P}_4.$$

It can be checked that, on $\partial\Omega$,

$$u_{,ij}\, u_{,ij} = \left(\frac{\partial^2 u}{\partial n^2}\right)^2.$$

Combining the last two equations

$$\mathrm{P}_4 = u_{,ij}\, u_{,ij} - \nabla u \nabla(\Delta u) - \frac{\varphi}{2}|\nabla u|^2 \leq 0 \quad \text{in } \Omega. \tag{2.22}$$

Integrating equation (2.22) over Ω and using the following relation that results from Green's identity [24],

$$-\int_\Omega \nabla u \nabla(\Delta u)\, dx = \int_\Omega (\Delta u)^2\, dx,$$

and the boundary conditions, we get

$$\int_\Omega u_{,ij}\, u_{,ij}\; dx + \int_\Omega (\Delta u)^2\, dx - \frac{\varphi}{2}\int_\Omega |\nabla u|^2\, dx \leq 0.$$

It follows that $\Delta u \equiv 0$ in Ω and hence $\Delta^2 u \equiv 0$ in Ω. Consequently, equation (1.1) becomes $-\rho u \equiv 0$ in Ω, and hence $u \equiv 0$ in $\overline{\Omega}$. □

3. Conclusion

This paper treats the problem of existence and uniqueness of weak and classical solutions for a fourth-order semilinear boundary value problem. The results are derived using standard variational methods as well as maximum principles. Future research will widen these results by proving similar results for a class of sixth or higher-order equations.

Acknowledgements

This research was carried out while the author was visiting the University of North Carolina at Charlotte (UNCC), USA, and was supported by the Horizon 2020–2017 RISE-777911 project.

It is a pleasure to thank Professor Douglas Shafer (UNCC) for his help, hospitality and for several valuable discussions which we had during my stay in Charlotte.

References

[1] R. A. Adams and J. Fournier, *Sobolev spaces*, 2nd edn, Volume 140 of *Pure and Applied Mathematics Series* (Academic Press, Boston, MA, 2003); https://www.elsevier.com/books/sobolev-spaces/adams/978-0-12-044143-3.

[2] D. Bonheure, "Multitransition kinks and pulses for fourth order equations with a bistable nonlinearity", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004) 319–340; doi:10.1016/j.anihpc.2003.03.001.

[3] C.-P. Danet, "Uniqueness in some higher order elliptic boundary value problems in *n* dimensional domains", *Electron. J. Qual. Theory Differ. Equ.* **54** (2011) 1–12; doi:10.14232/ejqtde.2011.1.54.

[4] C.-P. Danet, *The classical maximum principle. Some of its extensions and applications* (Lambert Academic Publishing, Saarbrcken, Germany, 2013); https://www.lap-publishing.com/catalog/details//store/gb/book/978-3-659-40556-3/the-classical-maximum-principle-some-extensions-and-applications.

[5] C.-P. Danet, "On a hinged plate equation of nonconstant thickness", *Differ. Equ. Appl.* **10** (2018) 235–238; doi:10.7153/dea-2018-10-16.

[6] C.-P. Danet and A. Mareno, "Maximum principles for a class of linear equations of even order", *Math. Inequal. Appl.* **16** (2013) 809–822; doi:10.7153/mia-16-61.

[7] Z. Djadli, A. Malchiodi and M. Ahmedou, "Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications", *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **5** (2002) 387–434; https://www.academia.edu/2086163/Prescribing_a_fourth_order_conformal_invariant_on_the_standard_sphere_Part_II_blow-up_analysis_and_applications.

[8] F. Gazolla, H.-C. Grunau and G. Sweers, *Polyharmonic boundary value problems* (Springer, Berlin, Heidelberg, 2010); https://www.springer.com/gp/book/9783642122446.

[9] I. M. Gelfand, "Some problems in the theory of quasilinear equations", *Amer. Math. Soc. Transl.* **29**(2) (1963) 295–381; doi:10.1007/978-3-642-61705-8.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Springer, Berlin, Heidelberg, 2001); https://www.springer.com/gp/book/9783540411604.

[11] V. B. Goyal, "Liouville-type results for fourth order elliptic equations", *Proc. Roy. Soc. Edinburgh* **103A** (1986) 209–213; doi:10.1017/S0308210500018862.

[12] V. B. Goyal and P. W. Schaefer, "On a subharmonic functional in fourth order nonlinear elliptic problems", *J. Math. Anal. Appl.* **83** (1981) 20–25; doi:10.1016/0022-247X(81)90243-2.

[13] G. W. Hunt, H. M. Bolt and J. M. T. Thompson, "Structural localization phenomena and the dynamical phase-space analogy", *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **425** (1989) 245–267; doi:10.1098/rspa.1989.0105.

[14] P. Karageorgis and J. Stalker, "A lower bound for the amplitude of traveling waves of suspension bridges", *Nonlinear Anal.* **75** (2012) 5212–5214; doi:10.1016/j.na.2012.04.037.

[15] B. Kawohl and G. Sweers, "On the differential equation $u_{xxxx} + u_{yyyy} = f$ for an anisotropic stiff material", *SIAM J. Math. Anal.* **37** (2006) 1828–1853; doi:10.1137/050624704.

[16] V. Komkov, "Certain estimates for solutions of nonlinear elliptic differential equations applicable to the theory of thin plates", *SIAM J. Appl. Math.* **28** (1975) 24–34; doi:10.1137/0128003.

[17] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow* (Gordon and Breach Science Publishers, New York, 1969); https://www.worldcat.org/title/mathematical-theory-of-viscous-incompressible-flows/oclc/639875838.

[18] A. C. Lazer and P. J. McKenna, "On traveling waves in a suspension bridge model as the wave speed goes to zero", *Nonlinear Anal.* **74** (2011) 3998–4001; doi:10.1016/j.na.2011.03.024.

[19] A. Mareno, "Maximum principles for some higher-order semilinear elliptic equations", *Glasg. Math. J.* **53** (2010) 313–320; doi:10.1017/S001708951000073X.

[20] A. Mareno, "A maximum principle result for a general fourth order semilinear elliptic equation", *J. Appl. Math. Phys.* **4** (2016) 1682–1686; doi:10.4236/jamp.2016.48176.

[21] C. Miranda, “Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche di due variabili”, *Giorn. Mat. Battaglini* **78** (1948) 97–118.

[22] L. E. Payne, “Some remarks on maximum principles”, *J. Anal. Math.* **30** (1976) 421–433; doi:10.1007/BF02786729.

[23] L. A. Peletier and W. C. Troy, *Spatial patterns. Higher order models in physics and mechanics*, Volume 45 of *Progress in Nonlinear Differential Equations and their Applications* (Birkhäuser Boston Inc., Boston, MA, 2001); https://www.springer.com/gp/book/9780817641108.

[24] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations* (Prentice-Hall Inc., Englewood Cliffs, NJ, 1967); https://www.springer.com/gp/book/9780387960685.

[25] P. W. Schaefer, “On a maximum principle for a class of fourth-order semilinear elliptic equations”, *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977) 319–323; doi:10.1017/S0308210500025233.

[26] R. P. Sperb, *Maximum principles and their applications* (Academic Press, New York, 1981); https://www.elsevier.com/books/maximum-principles-and-their-applications/sperb/978-0-12-656880-6.

[27] S. Tseng and C.-S. Lin, “On a subharmonic functional of some even order elliptic problems”, *J. Math. Anal. Appl.* **207** (1997) 127–157; doi:10.1006/jmaa.1997.5272.

[28] H. Zhang and W. Zhang, “Maximum principles and bounds in a class of fourth-order uniformly elliptic equations”, *J. Phys. A: Math. Gen.* **35** (2002) 9245–9250; doi:10.1088/0305-4470/35/43/318.