BASIC p-GROUPS: HIGHER COMMUTATOR STRUCTURE

Dedicated to the memory of Hanna Neumann

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1. Introduction

The classification of groups according to the varieties they generate requires the study of a class of indecomposable elements. Such a class is the class of *basic* groups which have been studied in [4], [5] and [6]. A group is called *basic* if it is indecomposable *qua* group; that is, it is critical and indecomposable *qua* variety; that is, its variety is join-irreducible. In this note we consider the higher commutator structure of basic p-groups. Our main theme is the relation between the formal weight of the higher commutator subgroups and the class of the group. We obtain information about the power-commutator structure of a basic p-group, the kinds of laws that can hold in such a group and the varietal structure of groups of the form: Center-extended-by-X.

We conclude this section with some notation and elementary definitions.

If A and B are subgroups of a group G, then $A \subseteq B$ means that A is a subgroup of B while $A \subset B$ means that A is a proper subgroup of B. If $\{a_1, \cdots, a_r\}$ is a set of elements of the group G, then $(a_1, a_2) = a_1^{-1} a_2^{-1} a_1 a_2$ is a simple commutator of weight 2 on $\{a_1, a_2\}$. A simple commutator of weight n on $\{a_1, \cdots, a_n\}$ is defined inductively by: $(a_{\sigma 1}, \cdots, a_{\sigma n}) = ((a_{\sigma 1}, \cdots, a_{\sigma(n-1)}), a_{\sigma n})$ with σ a permutation on $\{1, \cdots, n\}$. The rth commutator subgroup, G_r, is the subgroup generated by simple commutators of weight r on the elements of G. We say the class of G is c, $c(G) = c$, if $G_c \neq 1$ while $G_{c+1} = 1$. If A and B are subgroups of G, then $(A, B) = \langle \{(a, b) \mid a \in A,\, b \in B\} \rangle$, the subgroup generated by all commutators of that form. Similarly, if A, B, C are normal subgroups of G, then $(A,B,C) = ((A,B),C)$. If $f(x_1, \cdots, x_n)$ is a word on the letters $x_1, \cdots, x_n$ and $A_1, \cdots, A_n$ are subgroups of G, then $f(A_1, \cdots, A_n) = \langle \{ f(a_1, \cdots, a_n) \mid a_i \in A_i,\ i = 1, \cdots, n\} \rangle$. The exponent of G is denoted by $e(G)$. For each positive integer x, $(G)^x = \langle \{ g^x \mid g \in G\} \rangle$. The center of G is $Z(G)$.

If $u(x_1, \cdots, x_n)$ is a word on $\{x_1, \cdots, x_n\}$ then we denote by $u(x_i \to a)$ the word $u(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n)$. A finite p-group G is called *regular* if for any pair $a, b \in G$ and any positive integer α, $(ab)^{p^\alpha} = a^{p^\alpha} b^{p^\alpha} c^{p^\alpha}$, $c \in \langle a, b \rangle_2$.

For the basic terminology concerning varieties of groups, the reader is referred to the book by Neumann [1]. We remind the reader that a group G is called *critical* if the variety generated by the proper subgroups and quotient groups of G does not contain the group G.

All groups considered are finite.

2. Higher commutator subgroups of basic p-groups: f-class

Let $f(x_1, \cdots, x_n)$ be a commutator (not necessarily simple) on the letters $x_1, \cdots, x_n$ of weight n. Thus each letter appears exactly once. For example, $f(x_1, x_2, x_3, x_4) = ((x_4, x_2), (x_3, x_1))$ is a commutator of weight 4 on x_1, x_2, x_3, x_4. Let $\Lambda = \{\alpha_1, \cdots, \alpha_n\}$ be a sequence of positive integers. Then $f(G_{\alpha_1}, \cdots, G_{\alpha_n})$, sometimes denoted by $f(G, \Lambda)$ is called an *f-commutator subgroup* of G, or simply a higher commutator subgroup of G. Thus if $f(x) = x$ (a commutator of weight 1) $f(G_\alpha) = G_\alpha$ the αth commutator subgroup of G. Now suppose that $f(x_1, \cdots, x_n)$ and $\Lambda = \{\alpha_1, \cdots, \alpha_n\}$ are as above. In analogue with the lower central series of a group we consider the groups $f(G_{\alpha_1}, \cdots, G_{\alpha_{-1}}, G_{\alpha_i+1}, G_{\alpha_{i+1}}, \cdots, G_{\alpha_n})$ for each $i = 1, \cdots, n$. We denote the union of these groups by $f(G, \Lambda + 1)$ and refer to each as a component of $f(G, \Lambda + 1)$. Of particular interest is the case $f(G, \Lambda) \neq 1$ and $f(G, \Lambda + 1) = 1$ since this resembles the last term of the lower central series. (It should be noted that there exist finite p-groups G such that $f(G, \Lambda) = f(G, \Lambda + 1) \neq 1$ for suitable f and Λ). The "weight" of $f(G, \Lambda)$ is given by $\alpha = \sum_{i=1}^n \alpha_i$ and it is natural to ask whether the integer α depends on Λ for a particular f and how it is related to the class of the group G. It is not difficult to construct examples showing that for a fixed p-group G and fixed commutator $f(x_1, \cdots, x_n)$ there exist two sequences $\Lambda_1 = \{\alpha_1, \cdots, \alpha_n\}$ and $\Lambda_2 = \{\beta_1, \cdots, \beta_n\}$ with $\sum_{i=1}^n \alpha_i \neq \sum_{i=1}^n \beta_i$ such that $f(G, \Lambda_1) \neq 1$, $f(G, \Lambda_2) \neq 1$ while $f(G, \Lambda_1 + 1) = f(G, \Lambda_2 + 1) = 1$. Our first result (2.5) will show that this cannot happen if G is a basic p-group of small class c ($c < p$) and can happen only under special circumstances if G is basic with no restrictions on its class. In general the integer α is closely related to the class of G.

DEFINITION 2.1. Let $g(y_1, \cdots, y_r)$ be a word on the r letters $y_1, \cdots, y_r$ and let G be a group. We say that g is *G-multilinear* if for each $i = 1, \cdots, r$ and all $y_1, \cdots, y_r$, $a \in G$, $g(y_i \to y_i a) = g \cdot g(y_i \to a)$.

LEMMA 2.2. *Let $f(x_1, \cdots, x_n)$ be a commutator of weight n and $\Lambda = \{\alpha_1, \cdots, \alpha_n\}$ a sequence of poisitve integers. Then for any group G, $(f(G, \Delta), G) \subseteq f(G, \Lambda + 1)$.*

PROOF. The proof is by induction on n. If $n = 1$, then $\Lambda = \{\alpha_1\}, f(x_1) = x_1$ and so $f(G, \Lambda) = G_{\alpha_1}$. Hence $(f(G, \Lambda), G) = G_{\alpha_1 + 1} = f(G, \Lambda + 1)$. Now assume

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that the lemma is true for all integers k, $1 \leqq k < n$. Since f is a commutator we may assume, by reindexing if necessary, that

$$f(x_1, \cdots, x_n) = (u(x_1, \cdots, x_r), v(x_{r+1}, \cdots, x_n)),$$

u and v being commutators of weights r and $n-r$ respectively. We now use the "three-subgroup-lemma" of P. Hall [2, Theorem 3.4.7] which states that if A, B and C are normal subgroups of the group G, then each of the subgroups: (A, B, C), (B, C, A) and (C, A, B) is contained in the subgroup generated by the other two. Thus $(f(G, \Lambda), G) = (u(G, \Lambda_1), v(G, \Lambda_2), G)$ is contained in the join of $(v(G, \Lambda_2), G, u(G, \Lambda_1))$ and $(G, u(G, \Lambda_1), v(G, \Lambda_2))$ where $\Lambda_1 = \{\alpha_1, \cdots, \alpha_r\}$ and $\Lambda_2 = \{\alpha_{r+1}, \cdots, \alpha_n\}$. Now $(v(G, \Lambda_2), G, u(G, \Lambda_1)) = (u(G, \Lambda_1), (v(G, \Lambda_2), G))$ and $(G, u(G, \Lambda_1), v(G, \Lambda_2)) = ((u(G, \Lambda_1), G), v(G, \Lambda_2))$ and by the induction assumption $(u(G, \Lambda_1), G) \subseteq (G, \Lambda_1 + 1)$ while $(v(G, \Lambda_2), G) \subseteq v(G, \Lambda_2 + 1)$. Hence $(f(G, \Lambda), G) \subseteq (u(G, \Lambda_1), (f(G, \Lambda_2 + 1)) \cdot (u(G, \Lambda_1 + 1), v(G, \Lambda_2))$ and clearly both factors of this product are in $f(G, \Lambda + 1)$. Hence the lemma follows by induction.

LEMMA 2.3. *Let $f(x_1, \cdots, x_n)$ be a commutator of weight n and $\Lambda = \{\alpha_1, \cdots, \alpha_n\}$ a sequence of positive integers. Then for a fixed group G, the commutator $f(y_1, \cdots, y_n)$ with $y_i = (y_{i1}, \cdots, y_{i\alpha_i})$, $i = 1, \cdots, n$ is G-multilinear on the variables $\{y_{ij_i} | i = 1, \cdots, n, j_i = 1, \cdots, \alpha_i\}$ modulo $f(G, \Lambda + 1)$.*

PROOF. The proof is by induction on n. If $n = 1$, then $f(x_1) = x_1$, $f(y_1) = (y_{11}, \cdots, y_{1\alpha_1})$ while $f(G, \Lambda) = G_{\alpha_1}$ and $f(G, \Lambda + 1) = G_{\alpha_1 + 1}$. Clearly $f(y_1)$ is G-multilinear modulo G_{α_1+1}. Now assume the lemma for all k, $1 \leqq k < n$ and $f(x_1, \cdots, x_n) = (u(x_1, \cdots, x_r), v(x_{r+1}, \cdots, x_n))$ (by reindexing if necessary). Choose a fixed pair j, k, $1 \leqq j \leqq r$, $1 \leqq k \leqq \alpha_j$ and consider $f(y_{jk} \to y_{jk}z) = (u(y_{jk} \to y_{jk}z), v)$. By induction $u(y_{jk} \to y_{jk}z) = u \cdot u(y_{jk} \to z)b$ with $b \in u(G, \Lambda_1 + 1)$, $\Lambda_1 = \{\alpha_1, \cdots, \alpha_r\}$ and hence $f(y_{jk} \to y_{jk}z) = (u \cdot u(y_{jk} \to z)b, v)$. Using the well-known identities

$$(cd, e) = (c, e)(c, e, d)(d, e)$$

and

$$(c, de) = (c, e)(c, d)(c, d, e)$$

we obtain

$$f(y_{jk} \to y_{jk}z) = f \cdot (f, b) \cdot (f, u(y_{jk} \to z)) \cdot (f, u(y_{jk} \to z), b) \cdot f(y_{jk} \to z) \cdot (f(y_{jk} \to z), b) \cdot (b, v).$$

Now it follows from 2.2 that each commutator containing f or $f(y_{jk} \to z)$ properly is in $f(G, \Lambda + 1)$, while $(b, v) \in (u(G, \Lambda_1 + 1), v) \subseteq f(G, \Lambda + 1)$. Hence $f(y_{jk} \to y_{jk}z) = f \cdot f(y_{jk} \to z)$ modulo $f(G, \Lambda + 1)$. Now, if we choose the pair j, k so that $r + 1 \leqq j \leqq n$ we repeat the above argument for v. Hence the lemma follows by induction.

LEMMA 2.4. *Let $f(x_1,\cdots,x_n)$ be a commutator of weight n and $\Lambda=\{\alpha_1,\cdots,\alpha_n\}$ a sequence of positive integers. Let G be a p-group such that $f(G,\Lambda)$ is regular. Then $f^{p^\beta}(y_1,\cdots,y_n)$ with $y_i=(y_{i1},\cdots,y_{i\alpha_i})$ $i=1,\cdots,n$ is G-multilinear on the variables $\{y_{ij_i}\mid i=1,\cdots,n, j_i=1,\cdots,\alpha_i\}$ modulo $f^{p^\beta}(G,\Lambda+1)$ for any positive integer β.*

PROOF. It follows from 2.3 that $f(y_1,\cdots,y_n)$ is G-multilinear modulo $f(G,\Lambda+1)$. Thus $f^{p^\beta}(y_{jk}\to y_{jk}z)=(f\cdot f(y_{jk}\to z)\cdot b)^{p^\beta}$ in the notation of the proof of 2.3. Now since $f(G,\Lambda)$ is regular it follows that $(f\cdot f(y_{jk}\to z)\cdot b)^{p^\beta}$ $f^{p^\beta}\cdot f^{p^\beta}(y_{jk}\to z)\cdot b^{p^\beta}\cdot c^{p^\beta}$ with c in the commutator subgroup of $f(G,\Lambda)$ and hence by 2.2 in $f(G,\Lambda+1)$. This completes the proof.

We now answer the question, raised earlier, about the "weight" of $f(G,\Lambda)$, its dependence on Λ and its relation to the class of G.

THEOREM 2.5. *Let G be a p-group of class c such that* var G *is join-irreducible. Let $f(x_1,\cdots,x_n)$ be a commutator of weight n and $\Lambda=\{\alpha_1,\cdots,\alpha_n\}$ a sequence of positive integers such that $f(G,\Lambda)\neq 1$* but *$f(G,\Lambda+1)=1$.* Then if *$\alpha=\sum_{i=1}^n\alpha_i$, $\alpha\equiv c(\bmod p-1)$.*

PROOF. Let F be the relatively free group in var G of the same rank as that of G. F therefore generates var G and satisfies: $f(F,\Lambda+1)=1$. Now consider $f(F,\Lambda)\cap F_c$. Since $f(G,\Lambda)\neq 1$, then also $f(F,\Lambda)\neq 1$. Clearly $F_c\neq 1$. Since var $G=$ var F is join-irreducible it follows from [6, Theorem 1.6] that $f(F,\Lambda)\cap F_c\neq 1$. Hence there is $d\in f(F,\Lambda)$ and $h\in F_c$ such that $d=h\neq 1$ and we may assume from the multilinearity of $f(y_1,\cdots,y_c)$ and $(x_1,\cdots,x_c)$ that both d and h may be expressed as products of powers of elements of the form $f((y_{11},\cdots,y_{1\alpha_1}),\cdots,(y_{n1},\cdots,y_{n\alpha}))$ and $(w_1,\cdots,w_c)$ respectively in a set of free generators of F. Hence, since F is relatively free this relation is a law in F and hence in G. Thus if we replace each generator z_i by z_i^l, l is a positive integer and use the fact that $f(F,\Lambda)$ is central in F (or that $f(G,\Lambda)$ is central in G) we obtain the equations

$$d=h \quad\text{and}\quad d^{l^\alpha}=h^{l^c}.$$

Therefore

$$h^{l^c-l^\alpha}=1$$

and since h is not trivial, $p\mid l^c-l^\alpha=l^\alpha(l^{c-\alpha}-1)$. Now we insert the requirement that l be a primitive root of p, whence $p\mid l^{c-\alpha}-1$ and $p-1\mid c-\alpha$; that is

$$\alpha\equiv c(\bmod p-1).$$

This completes the proof.

It thus follows that, modulo $p-1$, the integer α is independent of Λ and in fact of f itself, as long as the condition: $f(G,\Lambda)\neq 1$, $f(G,\Lambda+1)=1$ is satisfied in a basic p-group G.

The next result is an analogue of [6, Theorem 2.5].

COROLLARY 2.6. *Let G be a p-group of class c such that* var G *is join-irreducible. Let* $f(x_1,\cdots,x_n)$ *be a commutator of weight n and* $\Lambda = \{\alpha_1,\cdots,\alpha_n\}$ *is a sequence of positive integers such that* $f(G,\Lambda)$ *is regular and nontrivial. If* $e(f(G,\Lambda)) = p^{\gamma+1} > e(f(G,\Lambda+1))$ *and* $\alpha = \sum_{i=1}^{n}\alpha_i$, *then* $\alpha \equiv c(\text{mod}\, p-1)$.

PROOF. Consider f^{p^γ}. Clearly $f^{p^\gamma}(G,\Lambda+1) = 1$ and 2.4 applies. Hence we may repeat the proof of 2.5 replacing f by f^{p^γ} and the result follows.

3. Center-extended-by-f groups

As a result of 2.5 we can now state a decomposition theorem for p-groups which satisfy $f(G,\Lambda) \neq 1$, $f(G,\Lambda+1) = 1$.

THEOREM 3.1. *Let G be a p-group of class c,* $f(x_1,\cdots,x_n)$ *a commutator of weight n and* $\Lambda = \{\alpha_1,\cdots,\alpha_n\}$ *a sequence of positive integers with* $\alpha = \sum_{i=1}^{n}\alpha_i$. *If* $f(G,\Lambda) \neq 1$ *while* $f(G,\Lambda+1) = 1$, *then* var G = var$\{A,B\}$ *with* $f(A,\Lambda) = 1$ *and* $c(B) \equiv \alpha\,(\text{mod}\, p-1)$.

PROOF. Since G is a finite p-group, var G is generated by the finite set of basic groups it contains, each satisfying $f(G,\Lambda+1) = 1$. Let A be the direct product of all basic groups H in var G which satisfy $f(H,\Lambda) = 1$. Each remaining basic group M in var G satisfies $f(M,\Lambda) \neq 1$ and $f(M,\Lambda+1) = 1$. Thus by 2.5 $c(M) \equiv \alpha(\text{mod}\, p-1)$. Hence the class of the direct product B of all such basic groups will likewise satisfy the same congruence.

It follows from 2.2 that a group G which satisfies $f(G,\Lambda) \neq 1$ and $f(G,\Lambda+1) = 1$ is a central extension of a group A such that $f(A,\Lambda) = 1$. Unfortunately, the condition that $f(G,\Lambda+1) = 1$ in 3.1 cannot be replaced by the condition that $f(G,\Lambda)$ is central in G. For the one hand there are easy examples of groups G for which both $f(G,\Lambda)$ and $f(G,\Lambda+1)$ are central and non-trivial. Moreover A. G. R. Stewart [3] has given an example of a nonmetabelian, center-extended-by-metabelian group of exponent p ($p > 5$) which is basic and of class 5. Such a group G satisfies $(G_2,G_2) = f(G,\Lambda) \neq 1$ and central with $f = (x_1,x_2)$ and $\Lambda = \{2,2\}$ but clearly fails to satisfy the conclusion of 3.1. The difficulty is that while $(G_2,G_2) \neq 1$ and central, $(G_2,G_2) = (G_3,G_2) = f(G,\Lambda+1) \neq 1$. (Perhaps such groups should be called "center-extended-by-$f(G,\Lambda)$" groups, giving the "maximal" Λ possible.)

If we add the condition $f(G,\Lambda) \supset f(G,\Lambda+1)$ to the requirement that $f(G,\Lambda)$ be central we can obtain a similar congruence on α. The reader should note that in the proof that follows we utilize for the first time the fact that a basic group is critical. In fact we need only the weaker condition that G is monolithic.

COROLLARY 3.2. *Let G be a basic p-group of class c. Let* $f(x_1,\cdots,x_n)$ *be a*

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commutator of weight n and $\Lambda = \{\alpha_1, \cdots, \alpha_n\}$ a sequence of positive integers such that $f(G, \Lambda)$ is central in G and $f(G, \Lambda) \supset f(G, \Lambda + 1)$. Then $\alpha \equiv c (\mathrm{mod}\, p-1)$ with $\alpha = \sum_{i=1}^{n} \alpha_i$.

PROOF. Since G is basic it is critical and hence $Z(G)$ is cyclic. $f(G, \Lambda)$ is a subgroup of $Z(G)$ and so $e(f(G, \Lambda)) > e(f(G, \Lambda + 1))$ since $f(G, A) \supset f(G, \Lambda + 1)$. Clearly $Z(G)$ is regular and so 2.8 applies. Thus $\alpha \equiv c(\mathrm{mod}\, p-1)$.

4. Higher commutator laws of basic p-groups

In this section we investigate some consequences of laws of the form $f(G, \Lambda) = 1$ in basic p-groups. We begin with the simplest non-trivial case: $f(x_1, x_2) = (x_1, x_2)$ and $\Lambda = \{\alpha_1, \alpha_2\}$, $\alpha_1, \alpha_2 \geqq 2$.

THEOREM 4.1. *Let G be a basic p-group of class c. If $(G_r, G_s) = 1$ with*

$$2 \leqq r \leqq s \leqq \frac{c}{2} < \frac{p + 2r - 1}{2},$$

then $(G_r, G_r) = 1$.

PROOF. Assume that $(G_r, G_r) \neq 1$. Now let i, j be chosen so that $i, j \geqq r$, $(G_i, G_j) \geqq 1$ and $i + j$ is maximal with these properties. Thus if $f(x_1, x_2) = (x_1, x_2)$ and $\Lambda = \{i, j\}$ it follows that $f(G, \Lambda) \neq 1$ while $f(G, \Lambda + 1) = 1$. Hence we can apply 2.5 and conclude that $i + j \equiv c(\mathrm{mod}\, p-1)$. Now $i + j \leqq c$ and so either $i + j = c$ or else $2r \leqq i + j \leqq c - (p-1)$. But the second alternative implies that $c \geqq 2r + p - 1$, a contradiction. Thus $i + j = c$. Therefore, either $i \geqq c/2$ or $j \geqq c/2$. We may choose either possibility since $(G_i, G_j) = (G_j, G_i)$. Thus assume that $j \geqq c/2$. Then $G_j \subseteq G_s$ and $G_i \subseteq G_r$ by assumption, and so $(G_i, G_j) \subseteq (G_r, G_s) = 1$, a contradiction. Hence it follows that $(G_r, G_r) = 1$.
The theorem can be restated in terms of laws as follows:

COROLLARY 4.2. *Let G be a basic p-group of class c. If $((x_1, \cdots, x_r), (y_1, \cdots, y_s)) = 1$ is a law of G with $2 \leqq r \leqq s \leqq c/2 < (p + 2r - 1)/2$, then $((x_1, \cdots, x_r), (y_1, \cdots, y_r)) = 1$ is a law of G.*

Generalizations of 4.1 can be carried out in a number of different directions. We give one example in the case: $f(x_1, x_2, x_3) = (x_1, x_2, x_3)$.

THEOREM 4.3. *Let G be a basic p-group of class c. If $(G_r, G_s, G_2) = (G_r, G_2, G_s) = 1$, $2 \leqq r \leqq s \leqq (c-2)/2 < (2r + p - 1)/2$, then $(G_r, G_r, G_2) = (G_r, G_2, G_r) = 1$.*

PROOF. Since (G_s, G_2, G_r) is contained in the subgroup generated by (G_2, G_r, G_s) and (G_r, G_s, G_2), and since $(G_2, G_r, G_s) = (G_r, G_2, G_s)$ it follows that $(G_s, G_2, G_r) = 1$. Thus under all permutations of $2, r, s$ the triple commutator subgroup composed of (G_2, G_r, G_s) is trivial.

Now consider the set $S = \{(G_u, G_v, G_w) |$ one of u, v or w is 2 and the others are $\geqq r\}$. Let $(G_i, G_j, G_k) \in S$ such that $(G_i, G_j, G_k) \neq 1$, and $i+j+k$ is maximal with this property. Let $f(x_1, x_2, x_3) = (x_1, x_2, x_3)$ and $\Lambda = \{i, j, k\}$. Then $f(G, \Lambda) \neq 1$ and $f(G, \Lambda + 1) = 1$. Thus it follows from 2.5 that $i+j+k \equiv c(\bmod p-1)$. But $i+j+k \leqq c$ and so either $i+j+k = c$ or else $2+2r \leqq i+j+k \leqq c-(p-1)$. But the second alternative implies that $c \geqq 2r+p+1$, a contradiction, and so $i+j+k = c$. Hence since one of i, j, k is 2 it follows that the sum of the remaining subscripts is $c-2$ and hence that one of them is $\geqq (c-2)/2 \geqq s$. Thus, for example, if $j = 2$ and $i \geqq (c-2)/2$, $k \geqq r$ and hence $(G_i, G_j, G_k) \subseteq (G_s, G_2, G_r) = 1$, a contradiction. In this way we are able to conclude that the set S consists of the trivial subgroup, and hence that $(G_r, G_r, G_2) = 1$.

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