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THE P^n-INTEGRAL

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1. **Introduction.** In the definition of the P^n-integral [2] there is a difficulty with the condition B_{n-2} ([2], p. 150) since it is not linear on the set of major and minor functions. As a result, the proof of Lemma 5.1 [2] fails since the difference $Q(x)-q(x)$ need not satisfy the conditions of Theorem 4.2, [2].

This note shows that a very simple modification of the definition of major and minor functions avoids this difficulty and leads to a definition of an integral which is strong enough to solve the coefficient problem in trigonometric series under the same conditions as posited by James in [3].

2. **Definitions and Notations.** Let $F(x)$ be a real-valued function defined on the bounded interval $[a, b]$. If there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_r$ which depend on x_0 only and not on h, such that

$$F(x_0+h)-F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r), \quad \text{as} \quad h \to 0, \tag{2.1}$$

then α_k, $1 \leq k \leq r$, is called the Peano derivative of order k of F at x_0 and is denoted by $F_{(k)}(x_0)$. If F possesses derivatives $F_{(k)}(x_0)$, $1 \leq k \leq r-1$, we write

$$\frac{h^r}{r!}\gamma_r(F; x_0, h) = F(x_0+h)-F(x_0)-\sum_{k=1}^{r-1}\frac{h^k}{k!}F_{(k)}(x_0). \tag{2.2}$$

If there exists constants $\beta_0, \beta_2, \ldots, \beta_{2r}$ which depend on x_0, and not on h, such that

$$\frac{F(x_0+h)+F(x_0-h)}{2} = \sum_{k=0}^{r} \beta_{2k}\frac{h^{2k}}{(2k)!} + o(h^{2r}), \quad \text{as} \quad h \to 0,$$

then β_{2k}, $0 \leq k \leq r$ is called the de la Vallée Poussin derivative of order $2k$ of F at x_0 and is denoted by $D_{2k}F(x_0)$.

If F has derivatives $D_{2k}F(x_0)$, $0 \leq k \leq r-1$, we write

$$\frac{h^{2r}}{(2r)!}\theta_{2r}(F; x_0, h) = \frac{F(x_0+h)+F(x_0-h)}{2} - \sum_{k=0}^{r-1}\frac{h^{2k}}{(2k)!} D_{2k}F(x_0)$$

and define

$$\Delta^{2r}F(x_0) = \limsup_{h\to 0} \theta_{2r}(F; x_0, h)$$

$$\delta^{2r}F(x_0) = \liminf_{h\to 0} \theta_{2r}(F; x_0, h).$$

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All the above symbols are defined similarly for odd-numbered indices (see, for example, [2], pp. 163–164).

The function F will be said to satisfy condition $A_n^*(n \geq 3)$ in $[a, b]$ if it is continuous in $[a, b]$, if, for $1 \leq k \leq n-2$, each $F_{(k)}(x)$ exists and is finite in (a, b) and if

$$\lim_{h \to 0} h\theta_n(x, h) = 0,$$

for all $x \in (a, b)-E$ where E is countable.

THEOREM 2.1. *If F satisfies condition $A_{2m}^*(A_{2m+1}^*)$ in $[a, b]$, then $F_{(2k)}(x) = D_{2k}F(x)(F_{(2k+1)}(x) = D_{(2k+1)}(x))$ does not have an ordinary discontinuity in (a, b) for $0 \leq k \leq m-1$.*

Proof. This is Lemma 8.1 [2].

NOTE: Condition A_{2m}^* is a stronger form of James' condition A_{2m}, [2], in that it replaces the requirement that $D_{2k}F(x)$ exist and be finite for $1 \leq k \leq m-1$ by the same condition on the Peano derivatives. Theorem 2.1 then shows that A_{2m}^* also implies James' condition B_{2m-2}, [2].

THEOREM 2.2. *If F satisfies condition A_n^*, $n \geq 2$, in $[a, b]$ and*

(a) $\Delta^n F(x) \geq 0$, $x \in (a, b)-E$, $|E| = 0$,

(b) $\Delta^n F(x) > -\infty$, $x \in (a, b)-S$, S *a scattered set,*

(c) $\limsup_{h \to 0} h\theta_n(F; x, h) \geq 0 \geq \liminf_{h \to 0} h\theta_n(F; x, h)$, $x \in S$, *then F is n-convex.*

Proof. In [1, Theorem 16] Bullen proves a similar result which implies this theorem. In place of condition A_n^* he uses a condition C_n which is just A_n together with B_{n-2}, but as was noted above these are implied by A_n^*.

DEFINITION 2.1. Let $f(x)$ be a function defined in $[a, b]$ and let a_i, $i=1, 2, \ldots, n$, be fixed points such that $a=a_1<a_2<\cdots<a_n=b$. The functions $Q(x)$ and $q(x)$ are called major and minor functions respectively of $f(x)$ over $(a_i)=(a_1, a_2, \ldots, a_n)$ if

(2.1.1) $Q(x)$ and $q(x)$ satisfy condition A_n^*;

(2.1.2) $Q(a_i) = q(a_i) = 0$, $i = 1, 2, \ldots, n$;

(2.1.3) $\partial^n Q(x) \not\leq f(x) \geq \Delta^n q(x)$, $x \in [a, b]-E$, $|E| = 0$;

(2.1.4) $\partial^n Q(x) \neq -\infty$, $\Delta^n q(x) \neq +\infty$, $x \in [a, b]-S$, S a scattered set;

(2.1.5) $\limsup h\theta_n(F; x, h) \geq 0 \geq \liminf_{h \to 0} h\theta_n(F; x, h)$, $x \in S$, $F = Q, q$.

LEMMA. *For every pair $Q(x)$ and $q(x)$ the difference $Q(x)-q(x)$ is n-convex in $[a, b]$.*

Proof. Follows from Theorem 2.2 above.

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DEFINITION 2.2. For each major and minor function of $f(x)$ over (a_i) the functions defined by

$$Q^*(x) = (-1)^r Q(x),\ q^*(x) = (-1)^r q(x),\ a_r \leq x < a_{r+1}$$

are called associated major and minor functions respectively of $f(x)$ over (a_i).

LEMMA. *For every pair of associated major and minor functions of $f(x)$ over (a_i),*

$$Q^*(x) - q^*(x) \geq 0$$

for all x in $[a, b]$.

Proof. This is Lemma 5.2 of [2].

DEFINITION 2.3. Let c be a point in (a_1, a_n) such that $c \neq a_i$, $i=1, \ldots, n$. If for every $\varepsilon > 0$ there is a pair $Q(x)$, $q(x)$ such that

$$|Q(c) - q(c)| < \varepsilon$$

then $f(x)$ is said to be P^n-integrable over $(a_i; c)$.

The remainder of the theory goes through as in James [2].

3. **Application to trigonometric series.**

THEOREM 3.1. *Suppose the series*

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} a_n(x) \tag{3.1}$$

is summable (C, k) to a finite function $f(x)$ for all $x \in [0, 2\pi] - E$, where E is at most countable, and let $f(x)=0$, $x \in E$. If $A_n^{k-1}(x) = o(n^k)$ for $x \in E$ and $B_n^{k-1}(x) = o(n^k)$ for $x \in [0, 2\pi]$ then $f(x)$, $f(x)\cos px$, $f(x)\sin px$ are each P^{k+2}-integrable.

Proof. The series obtained by integrating (3.1) formally term-by-term $k+2$ times converges uniformly to a continuous function $F(x)$. It follows from the proof of Theorem 3.2 [3] that $F_{(r)}(x)$, $0 \leq r \leq k$ exists for each $x \in [0, 2\pi]$.

Let

$$Q(x) = F(x) - \sum_{i=1}^{k+2} \lambda(x; \alpha_i) F(\alpha_i)$$

where

$$\lambda(x; \alpha_i) \equiv \prod_{j \neq i} (x - \alpha_j)/(\alpha_i - \alpha_j).$$

Then

$$h\theta_{k+2}(Q; x, h) = h\theta_{k+2}(F; x, h),$$

$$Q(\alpha_i) = 0, \qquad i = 1, 2, \ldots, k+2,$$

and

$$D_{k+2}Q(x) = D_{k+2}F(x) = f(x), \qquad x \in [0, 2\pi] - E.$$

That condition (2.1.4) above is satisfied follows from Theorem 5.1 [3]. The function $Q(x)$ therefore satisfies all the conditions for both a major and minor function of $f(x)$ and $f(x)$ is therefore P^{k+2}-integrable over $(\alpha_i; x)$.

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Following the notation and proof of Theorem 4.2 [3] we have

$$\sum_{n=0}^{\infty} u_n(x) = f(x) \cos px, \qquad (C, k) \tag{3.2}$$

for $x \in [0, 2\pi]-E$ and $u_n = o(n^k)$, $U_n^{k-1}(x) = o(n^k)$, for all x, where $u_r(x)$ is the nth term of the series which is the formal product of series (3.1) and $\cos px$, and $U_n^{k-1}(x)$ is the $(k-1)$st Cesàro mean of the same series.

The series obtained by integrating (3.2) formally term-by-term $k+2$ times converges uniformly to a continuous function $G(x)$ such that

$$\lim \theta_{k+2}(G; x, h) = 0 \quad \text{as} \quad h \to 0 \quad \text{for all} \quad x,$$

and

$$\frac{u_0 x^{2r}}{2(2r)!} + (-1)^r \sum_{n=1}^{\infty} \frac{u_n(x)}{n^{2r}} = D_{k+2-2r} G(x), \qquad (C, k-2r),$$

for $0 \le r \le (k+1)/2$ and $x \in [0, 2\pi]-E$.

We show next that the kth unsymmetric derivative of $G(x)$ exists everywhere and then as in the first part of this proof, it follows that $f(x) \cos px$ is P^{k+2}-integrable.

Consider the series

$$\tfrac{1}{2} u_0 x - \sum_{n=1}^{+\infty} w_n(x),$$

where $w_n(x) = v_n(x)/n$ and $v_n(x)$ is the nth term of the formal product of the series conjugate to (3.1) and $\cos px$. It follows from Theorem 2.1 [3] that $V_n^{k-1}(x) = o(n^k)$ which in turn implies that $W_n^{k-2}(x) = o(n^{k-1})$. Then again, as in the proof of Theorem 3.2 [3], it follows that $G_{(k)}(x)$ exists everywhere.

Corollary. *If* $k = 2m-2$, *let* $\gamma_k = (2m)!/(m!)^2$ *and let* (α_i) *be the set*

$$(-2m\pi, \ldots, -2\pi, 2\pi, \ldots, 2m\pi).$$

If $k = 2m-1$, *let* $\gamma_k = (2m+1)!/m!(m+1)!$ *and let* (α_1) *be the set*

$$(-2m\pi, \ldots, -2\pi, 2\pi, \ldots, (2m+2)\pi).$$

Then under the hypothesis of Theorem 3.1 *the coefficients of the series* (2.1) *are given by*

$$a_p = \frac{\gamma_k}{2^{k+1}\pi^{k+2}} \int_{(\alpha_i)}^{0} f(x) \cos px\, d_{k+2} x$$

$$b_p = \frac{\gamma_k}{2^{k+1}\pi^{k+2}} \int_{(\alpha_i)}^{0} f(x) \sin px\, d_{k+2} x.$$

Note. The definition of the $\mathscr{P}^k$-integral [2] may be modified to allow exceptional sets which are of measure 0 and countable in the conditions on $\mathscr{P}^k$-major and minor functions corresponding to (2.1.3) and (2.1.4) above respectively. Then it follows that the $\mathscr{P}^k$-integral is included in the P^k-integral.

Also Theorem 3.1 is valid using the $\mathscr{P}^k$-integral if (3.1) is replaced by

$$\sum_{n=0}^{+\infty} c_n e^{inx}. \tag{3.3}$$

This result is apparently less general than Theorem 3.1 since the summability of (3.3) is equivalent to the summability of both (3.1) and its conjugate.

References

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