It may be interesting to write down a few cases for the example already considered, namely $(1+x)^{1/3}$;

the first result is $\dfrac{3+2x}{3+\ x}$

the second $\dfrac{1+7x/6+7x^2/27}{1+5x/6+5x^2/54}$

the third $\dfrac{1+5x/3+7x^2/9+7x^3/81}{1+4x/3+2x^2/9+2x^3/81}$.

In Gauss's notation the identity is

$$(1+x)^n = \frac{F(-n-r,\ -r,\ -2r,\ -x)}{F(\ n-r,\ -r,\ -2r,\ -x)},$$

of which a more general case will be found in his *Methodus nova integralium valores per approximationem inveniendi*, §§ 40-42.

On the history and degree of certain geometrical approximations.

Part I.

By A. J. Pressland, M.A.

According to Luther's translation 1 Kings vi. 31 should read— "At the entrance of the choir he made two doors with pentagonal door-posts." This is probably a wrong translation, for nowhere on Asiatic monuments of this time has a pentagon been found. Prof. A. Merx in Cantor's *Vorlesungen über Geschichte der Mathematik*, Vol. i., p. 91.

It is probable that the construction of the regular pentagon is due to the Pythagorean school as a consequence of I. 47. (*Ibid*, p. 151.)

Various approximations to the arc of a regular pentagon in a circle have been given. Thus (fig. 8) a diameter AB of a circle is divided into five equal parts at E, D, etc. On AB an equilateral triangle is erected giving the vertex C. CD is drawn to cut the circumference in P, and AP is assumed to be the arc of the regular inscribed pentagon. By dividing AB into n equal parts and joining C to the second of these from A, an approximation to the arc of the regular n-gon is obtained.

The first mention of this construction is in *Les fortifications* of Chevalier Antoine de Ville (achevé d'imprimer, 1er Août, 1628). In this, however, the point C is joined to the point E and the resulting arc doubled. On page 29 the following occurs: "Ce probleme ne

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se demonstre point : et au calcul il ne revient pas tout à fait precisément juste principalement aux figures qui ont grand nombre de costez . . . il est à estimer pour la facilité et justesse plus grande que de tous les autres qui ont esté escrits pour ce sujet."

Bion in his *Instruments*, pp. 19, 20 (1703) gives this method, and ascribes it to De Ville. He says that by De Ville's method the angle at the centre is 44′ too large for the pentagon and 5′ too large for the heptagon, from which point the difference gradually increases. He follows a variation given by Bosse in his *Traité des pratiques géométrales et perspectives*, p. 62 (1665) (privilège 16th May 1653), in which C is joined to D. This improvement, he says, makes the angle at the centre 2′ too small for the pentagon and 6′ too large for the heptagon.

The following note occurs in Christian Wolf's *Elementa Matheseos Universae* (1730) :—

Pentagoni Decagoni et Quindecagoni constructionem tradunt Euclides IV. 11, 16 : XIII., 10 et Ptolomaeus Almag. lib. 1 *c* 9 *fm* 8 : equidem et heptagoni enneagoni et hendecagoni constructiones Geometricae passim apud autores practices imprimis occurrunt : sed a rigore demonstrationis abhorrent. Joh. Carolus Renaldinus lib. 2 de Resolut. et Composit. Mathem. *f* 367 omnium polygonorum describendorum regulam catholicam praescribit, passim geometriis practicis insertam [De Ville's method].

It is possible that Wolf was misled by Sturm, for on page 233 of Sturm's *Mathesis Juvenilis* (1702), De Ville's method is ascribed to Renaldinus, as also on p. 38 of his *Mathesis Enucleata*.

This solution appears in John Ward's *Young Mathematician's Guide*, 1706. T. S. Davies, in his edition of Hutton's Course of Mathematics, ascribes it to Thomas Malton, in whose *Royal Road to Geometry* it occurs, Part II., p. 40 (1793).

In Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*, 6^{me} *Édit.*, p. 279, a short history is given, and the following formula for the length of AP in a circle of radius unity

$$x^2 = 2 - \frac{n-4}{2} \cdot \frac{3n + \sqrt{n^2 + 16n - 32}}{(n-1)^2 + 3} .$$

In the *Nouvelles Annales de Mathématiques*, tome XII., p. 77 (1853), M. Housel gives the following table :—

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Number of Sides.	Approximate Angle at Centre.	Correct Angle at Centre.
3	120°	120°
4	90°	90°
5	71° 57′ 12″	72°
6	60°	60°
7	51° 31′ 5″	51° 25′ 43″
8	45° 11′ 14″	45°
9	40° 16′ 40″	40°
10	36° 21′ 24″	36°
11	33° 8′ 52″	32° 43′ 38″
12	30° 29′ 45″	30°
13	28° 12′ 30″	27° 41′ 32″
14	26° 15′ 48″	25° 42′ 52″
15	24° 34′ 30″	24°
16	23° 5′ 54″	22° 30′
17	21° 47′ 12″	21° 10′ 35″

A modification of the construction is to be found in Hawney's *Mensuration*, and in Adams' *Mathematical Instruments* (1803). In this the perpendicular is made 1·75 times the radius. The centre angle so obtained is 72° 1′ 7·9″.*

In Hutton's *Miscellanea Mathematica*, pp. 311-12 (1775) a discussion of this problem is given by Geometricus, in which the problem is presented in the following form :—

Is it possible to find a point in OC so that CDP being drawn the line DO shall bear a constant ratio to the arc PQ?

§ 2. The construction given in fig. 9 is due to Albrecht Dürer being given in his *Underweysung der Messung mit dem Zirckel*, fig. 16 (1525). He also gives the construction from the Almagest. This construction is effected with a single opening of the compasses, a condition observed by various writers and afterwards amplified by Tartaglia. The four points H, A, B, F are concyclic. If a be the centre of this circle the side AB subtends 71° 38′ 6·3″ at a.

With regard to this construction, Daniel Schwenter, in his *Deliciae Mathematicae*, Part III., page 181 (1636), says: "Dieses

* The very close approximations are marked with an asterisk.

Fünfeck berechnet Benedict in Epist. f. 349 und erweiset dass es ganz richtig ist."

But in the new edition of his *Geometria practica* (Nürnberg 1677) edited by George Andreas, the following note occurs: "Es ist zu mercken dass dasselbige nur gleiche Seiten hat aber nicht gleiche Winckel, wie solches vom M. J. P. Seligen und nach ihme von dem Clavio erwiesen worden."

In a list of authors given in the *Deliciae Mathematicae* reference is made to M. Johannes Praetorius, the only author having the above initials.

In Clavius's *Opera Mathematica*, Vol. II., pp. 310–11 (1611) the angles A and B are calculated to be 108° 22′, F and H 107° 2′ and the remaining angle 109° 12′.

Chasles in his *Aperçu historique*, p. 530, says:

Durer apprend à construire un pentagone régulier sur un côté donné, et sa construction a cela de remarquable qu'elle se fait avec une seule ouverture de compas; mais elle n'est qu'approximative, et la figure qui a conservé le nom de pentagone de Durer n'a pas tous ses angles égaux ainsi que l'ont démontré J. B. de Benedictis* et Clavius† dans le siècle suivant.

§ 3. In Schwenter's *Deliciae Mathematicae* the following construction occurs (fig. 10).

Take a line CD and on it describe an equilateral triangle CBD. Let CE, ED be two sides of a square having CD for diagonal. Bisect EB at F and assume F as the centre of the regular pentagon on CD as base.

He further remarks:—By taking off along AB produced parts equal to BF, the centres of regular polygons, on CD as side, of from 7 to 12 sides may be obtained.

The angle subtended by CD at F is 72° 24′ 43″·4.

§ 4. A method ascribed by Schwenter to Augustine Hirschvogel (fig. 11) consists in describing three sides of a regular hexagon *adec*. With *c* as centre and *ca* as radius describe an arc cutting *ed* produced in *g*. Assume *bg* the side of the pentagon inscribed in *adec*.

* *Diversarum speculationum mathematicarum et physicarum Liber* (1585)

† *Geometria practica*, lib. VIII. prop. 29.

The angle subtented at the centre by a chord equal to *bg* is 72° 22′ 32″·4.

As Daniel Schwenter was professor of Oriental languages as well as of Mathematics, it is possible that some of these solutions are taken from Oriental sources.

§ 5. Cantor says that from Arabic remains it seems that Archimedes wrote a book on the inscription of the heptagon in a circle which is lost.

The method of inscribing a regular heptagon in a circle by taking half the side of the inscribed equilateral triangle for the side of the heptagon is ascribed by Cantor, p. 640, to Abû'l Wafâ, a Persian of Khorassan (940-998). See Wöpcke *Journal Asiatique*, Feb. and March 1855, pp. 329, 332. This method gives the value 51° 19′ 4″ for the angle subtended at the centre instead of 51° 25′ 43″.

De Ville's method makes the centre angle 51° 31′ 5′.

A method ascribed by Clavius to Carolus Marianus Cremonensis is to produce a radius CB of a circle one-fourth of its length to A, (fig. 12). With A as centre and radius equal to the radius of the given circle, describe an arc cutting the given circle in D, and assume BD the side of the regular inscribed heptagon.

The length BD can be proved equal to half the side of the inscribed equilateral triangle (Pappus III. 5) and depends on the close approximation of $\cos 2\pi/7$ to 5/8.

If the method of § 3 be followed, the angle subtended by CD at the assumed heptagon centre would be 50° 58′ 3″.

It is to be noted that the regular heptagon can be inscribed in a circle by means of Peaucellier-cells. See *Educational Times Reprint*, Vol. LV., Question 10865, p. 61, by Mr A. A. Robb.

§ 6. The following with slight alterations is given by Albrecht Dürer in the folio of 1525 (fig. 13).

Describe a circle and draw three radii AB, AC, AD inclined to each other at 120°. Trisect AB at *h* and *k*. At *h* draw *ehf* perpendicular to AB meeting the arc BAD in *f* and the arc BAC in *e*. With centre A radius A*f* describe a circle giving the points *g*, *l*, *m*, *n*.

Bisect the arcs *fg*, *lm*, *nc* at *p*, *q*, *r*.

gl subtends 39° 35′ 38″·6 at A, as do *ef* and *mn*;
while *fp*, *pg*, *lq*, *qm*, *nr*, *re* subtend 40° 12′ 10″·7 at A.

The construction given by Schwenter for inscribing a regular nonagon consists in taking half the side of the inscribed square as the side of the nonagon. The angle subtended by such a chord at the centre is 41° 24′ 34″.

The following approximations are also given by Schwenter :—

For the side of the regular 11-gon take 9/32 of the diameter.
" " " 13-gon " 1/4 " "
" " " 17-gon " 11/60 " "
" " " 19-gon " 1/6 " "
" " " 18-gon " 1/5 of the side of the equilateral triangle.

§ 7. The following methods are given in Le Clerc's *Pratique de la Géométrie sur le Papier et sur le Terrain* (1668).

On a given straight line to describe any polygon from a hexagon to a dodecagon (fig. 14).

Let AB be the given straight line. Bisect it at O and on BA describe an equilateral triangle BAC. Divide the arc AC into six equal parts at M, N, P, Q, R. With C as centre and radii CM, CN, CP, CQ, CR describe arcs cutting OC produced in D, E, F etc. Assume D as the centre of the regular heptagon on AB, E as the centre of the regular octagon and so on.

Taking AB as unity this construction gives for the radius of the inscribed circle of the regular figure on AB

of 7 sides the value $\frac{\sqrt{3}}{2}+2\sin 5°$

8 " " $\frac{\sqrt{3}}{2}+2\sin 10°$

. . . .

12 " " $\frac{\sqrt{3}}{2}+2\sin 30°$

and from this the following table results :—

No. of Sides.	Approximate Value.	True Value of Radius.
7	1·0403368	1·0382608
8	1·2133218	1·2071068
9	1·3836634	1·3737387
10	1·5500656	1·5388418
11	1·7112620	1·7028440
12	1·8660254	1·8660254

The method can be applied as follows to obtain the regular polygons of from 12 to 24 sides on the line AB (fig. 15).

Describe the equilateral triangle as before. Divide the arc AC into twelve equal portions. Take as many of these as with twelve added will give the number of sides of the required figure.

Taking three to get the quindecagon, describe from the point C with radius CE an arc cutting IC produced in O. With O as centre and OB as radius obtain the point F, which is the required centre.

The distance FI for the regular figure of n sides where $24 > n > 12$ is

$$\frac{\sqrt{3}}{2} + 2\sin\frac{n-12}{72}\pi + \sqrt{\frac{1}{4} + \left(\frac{\sqrt{3}}{2} + 2\sin\frac{n-12}{72}\pi\right)^2}$$

giving the following values of FI.

No. of Sides.	Approximate Value.	True Value.
13	2·030	2·0281
14	2·194	2·1905
15	2·360	2·3523
16	2·526	2·5126
17	2·691	2·6748
18	2·855	2·8356
19	3·017	2·9950
20	3·179	3·1569
21	3·337	3·3173
22	3·494	3·4775
23	3·647	3·6340
24	3·798	3·7979

§ 8. A method of construction of the regular nonagon is given by Le Clerc. It consists, (fig. 16), in taking AB, AC, AD three radii joining the centre to consecutive angular points of an inscribed hexagon. CD is then produced, and on it an equilateral triangle, EGF having one vertex at E and a side equal to AB, is described; AG cuts the arc DB in H, the assumed vertex of a regular nonagon. The angle DAH is 39° 53′ 46″.

§ 9. The following approximation to the regular hendecagon is given by Le Clerc (fig. 17).

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Take a radius of a circle AB. Bisect it in C, and on AC describe the equilateral triangle ACD. Mark off the chord AL equal to AC. With centre L and radius LD describe an arc cutting the circumference in O. Then CO is approximately the side of the hendecagon. The angle subtended by CO at the centre is 32° 44′ 29″ instead of 32° 43′ 38″. It is to be noted that the angle which AO subtends at B is 21° 12′ 42″. The angle which the side of the regular 17-sided figure would subtend at B is 21° 10′ 35″.

The calculation of this result suggested the following approximation. The length of the side of the 11-gon is equal to one-fifth the diagonal of the circumscribing square.

This gives an angle 32° 51′ 35″.

On the history and degree of certain geometrical approximations.

PART II.*

By A. J. PRESSLAND, M.A.

§ 1. Since the former paper on this subject was read, Prof. Cantor has published the second volume of his history of Mathematics. This has necessitated various additions to the paper, which can perhaps be best given as an appendix.

On page 413 Prof. Cantor says that the construction of Dürer's pentagon is found in a book called *Geometria deutsch*, which was lately discovered in the town library at Nürnberg, and gives 1487 as the upper limit to its date. The construction is said to be "mit unverrücktem Zirckel," the same expression that Schwenter applies to Dürer's solution.

Lionardo da Vinci (1452-1519) gave several methods of accurate and of approximate construction. Thus in fig. 18. the arc $ba = 1/6$, the arc $bc = 1/3$, the arc $cf = 1/8$, and the arc $af = 1/24$ of the circumference (p. 271).

Two constructions for the pentagon are also given by Lionardo. In fig. 19 the arcs are all of the same radius and the arc *am* is approximately 1/5 of the circumference, the value on calculation being found to be 72° 25′ (p. 272).

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* This was read at the Sixth Meeting, 8th April, 1892.