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A NOTE ON ITERATIVE PREMIUM CALCULATION PRINCIPLES

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GERBER (1974) has given a characterization of the exponential utility function by proving the fact that a premium calculation principle of zero utility is iterative iff the underlying utility function is linear or exponential. In the present note we prove the more general result that the premium calculation principle introduced by BÜHLMANN, GAGLIARDI, GERBER and STRAUB (1977) is iterative iff the underlying function v is linear or exponential or when the principle is a mean value principle.

1. ON SOME PREMIUM CALCULATION PRINCIPLES

A premium calculation principle is a rule that assigns a nonnegative number P (the premium) to any given risk S (a positive random variable). In BÜHLMANN (1970) four premium calculation principles can be found, namely

a) the net premium principle

$$P = E(S)$$

b) the standard deviation principle

$$P = E(S) + \alpha \sqrt{\text{Var}(S)}, \qquad \alpha > 0$$

c) the variance principle

$$P = E(S) + \beta \,\text{Var}(S), \qquad \beta > 0$$

d) the principle of zero utility.

Let $u(x)$ $(x \in R)$ be a twice differenciable function with $u'(x) > 0$. Then P is defined by the equation

$$E(u(P - S)) = u(0)$$

Without loss of generality one may assume that

$$u(0) = 0, \; u'(0) = 1$$

GERBER (1974) mentioned some additional principles

e) The mean value principle.

Let $v(x)$ $(x \in R^*)$ be a continuous strictly increasing function. Then one defines:

$$P = v^{-1} E(v(S))$$

where v^{-1} denotes the inverse function of v.

* We are very much indebted to Prof. H. Bühlmann for providing an easier proof of the result starting from equation (A'').

f) the maximal loss principle

For $p \geqslant 0$, $q = 1 - p \geqslant 0$ one sets

$$P = pE(S) + q \operatorname{Max}(S)$$

where Max (S) denotes the right end point of the range of S.

BÜHLMANN, GAGLIARDI, GERBER and STRAUB (1977) introduced a certain family of premium calculation principles. For any given risk the premium P is defined as the solution of the equation.

g) $E(v(S - zP)) = v((1 - z)P)$

where $v(t)$ denotes a twice differentiable function with $v'(t) > 0$, $v''(t) \geqslant 0$ for $t \in R^*$. In the sequel we shall focus our attention on this principle and generalize a result of GERBER (1974) who has shown that the principle of zero utility is iterative, iff the underlying utility function is linear or exponential.

For $z = 1$ this principle coincides with a zero utility principle. For $z = 0$ the mean value principle is obtained as a special case.

A premium principle $H(P = H(X))$ is said to be iterative, if $H(X \mid Y)$ is a risk and

$$H(X) = H(H(X \mid Y))$$

for any pair of risks X, Y.

2. ANOTHER CHARACTERIZATION OF SOME PREMIUM CALCULATION PRINCIPLE

Theorem:

The Swiss principle is iterative iff it reduces to a mean value principle or to a zero utility principle with a linear or exponential utility function.

Proof

To some extent our proof proceeds as in GERBER (1974). Clearly the mean value principle and the zero utility principle with a linear or exponential underlying utility function is iterative. For a proof see GERBER (1974). We consider the risk with distribution function

$$F_{X \mid T}(x) = tH(x - a) + (1 - t)\, H(x - b)$$

$X \mid T$ thus denotes a Bernoulli risk:

$$p(X \mid T = a) = t$$
$$p(X \mid T = b) = 1 - t.$$

Let $P(t)$ denote the premium for the given risk, then

$$v((1 - z)\, P(t)) = tv(a - zP(t)) + (1 - t)\, v\,(b - zP(t))$$

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The distribution function of the risk X is given by

$$F_X(x) = mH(x - a) + (1 - m)\, H\,(x - b)$$

where $m = E(T)$.

Hence X denotes a Bernoulli risk

$$p(X = a) = m$$
$$p(X = b) = 1 - m$$

$P(m)$ is determined by the equation

$$v((1 - z)\, P(m)) = mv(a - zP(m)) + (1 - m)\, v\,(b - zP(m)).$$

Iterativity implies

$$P(m) = H(H(X \mid T)),$$

or $P(m)$ satisfies

(1) $$v((1 - z)\, P(m)) = \int_0^1 v(P(t) - zP(m))\, dF_T(t)$$

The only conditions imposed on the distribution $F_T(t)$ are

$$\int_0^1 dF(t) = 1 \qquad \int_0^1 tdF(t) = m$$

The above mentioned equation can be cast into the form

(a) $\int_0^1 [v(P(t) - zP(m)) - v\,((1 - z)\, P(m))]\, dF_T(t) = 0$

together with the conditions:

(b) $\int_0^1 dF_T(t) = 1$

(c) $\int_0^1 tdF_T(t) = m$

which imply that

$$\int_0^1 [v(P(t) - zP(m)) - v\,((1 - z)\, P(m)) - \alpha t - \beta]\, dF_T(t) = 0$$

where α and β are certain constants (independent of t).

Because this relation has to hold for an arbitrary $F_T(t)$ a theorem from the variational calculus implies:

$$v(P(t) - zP(m)) - v\,((1 - z)\, P(m)) - \alpha t - \beta = 0$$

for all $t \in [0, 1]$.

From the equation which determines $P(t)$ one deduces

$$P(0) = b$$
$$P(1) = a$$

Making use of this boundary conditions provides us with the possibility of determining the constants α and β. An elementary calculation yields:

$$(A) \qquad v(P(t) - zP(m)) - v((1 - z)\,P(m)) = [v(a - zP(m)) - v((1 - z)\,P(m))]\,t + [v(b - zP(m)) - v((1 - z)\,P(m))]\,(1 - t)$$

This equation has to be satisfied by $P(m)$ as well as the equivalent one:

$$(A') \qquad v(P(t) - zP(m)) = tv(a - zP(m)) + (1 - t)\,v(b - zP(m))$$

We write it as

$$(A'') \qquad v[P(t) - x] = tv[a - x] + (1 - t)\,v[b - x] \text{ for all } t \in [0, 1] \text{ for all } x \in [zb, za]$$

Differentiating with respect to t we obtain

$$(1) \qquad v'[P(t) - x]\,P'(t) = v[a - x] - v[b - x]$$

Differentiating with respect to x we obtain

$$(2) \qquad v'[P(t) - x] = tv'[a - x] + (1 - t)\,v'[b - x]$$

Hence

$$P'(t) = \frac{v[a - x] - v[b - x]}{tv'[a - x] + (1 - t)\,v'[b - x]}$$

Case 1 $v'[a - x] = v'[b - x]$ $\rightarrow v$ linear

Case 2 Integrating $P'(t)$ and using the boundary condition $P(1) = a$ we arrive at

$$(3) \qquad P(t) = \frac{a + v[a - x] - v[b - x]}{v'[a - x] - v'[b - x]} \ln \frac{tv'[a - x] + (1 - t)\,v'[b - x]}{v'[a - x]}$$

Using the other boundary condition $P(0) = b$ we get

$$(4) \qquad b - a = \frac{v[a - x] - v[b - x]}{v'[a - x] - v'[b - x]} \ln \frac{v'[b - x]}{v'[a - x]}$$

Consider now the special case $b = 0$. As $0 \in [z0, za]$ put also $x = 0$.

$$a = \frac{v(a) - v(0)}{v'(a) - v'(0)} \ln \frac{v'(a)}{v'(0)}$$

or

$$a \times \frac{v'(a) - v'(0)}{v(a) - v(0)} = \ln \frac{v'(a)}{v'(0)} \tag{5}$$

Assuming

$$v(0) = c > 0$$
$$v'(0) = d > 0$$

and writing x instead of a for the variable

$$x \frac{v'(x) - d}{v(x) - c} = \ln \frac{v'(x)}{d} \Leftrightarrow \frac{v'(x)}{d} = e^{x \frac{v'(x) - d}{v(x) - c}} \tag{6}$$

$$v'(x) = d \cdot F[x, v, v']$$

Since

$$\left.\begin{matrix} \frac{\partial F}{\partial a} \\ \frac{\partial F}{\partial v'} \\ \frac{\partial F}{\partial v} \end{matrix}\right\} \text{ all } \neq 0$$

we have exactly one solution $v(x)$ satisfying (6), and the boundary condition

$$v(0) = c$$
$$v'(0) = d.$$

This is, as one easily verifies, the exponential $v(x)$.

REFERENCES

BÜHLMANN, H. (1970), Mathematical Methods in Risk Theory, Berlin: Springer-Verlag.

BÜHLMANN, H., B. GAGLIARDI, H. U. GERBER, and F. STRAUB (1977), Some Inequalities for Stop Loss Premiums, Astin Bulletin **9**, 75-83.

GERBER, H. U. (1974), On Iterative Premium Calculation Principles, Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker **74**, 163-172.