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A MULTIPLE SEQUENCE ERGODIC THEOREM

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ABSTRACT. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $\{T_1, \ldots, T_k\}$ a set of linear operators of $L_p(X, \mathcal{F}, \mu)$, some p, $1 \leq p \leq \infty$. If

$$\lim_{n_1, \ldots, n_k \to \infty} \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f$$

exists a.e. for all $f \in L_p$, we say that the multiple sequence ergodic theorem holds for $\{T_1, \ldots, T_k\}$. If $f \geq 0$ implies $Tf \geq 0$, we say that T is *positive*. If there exists an operator S such that $|Tf(x)| \leq S\,|f|\,(x)$ a.e., we say that T is dominated by S. In this paper we prove that if $T_1, \ldots, T_k$ are dominated by positive contractions of $L_p(X, \mathcal{F}, \mu)$, p fixed, $1 < p < \infty$, then the multiple sequence ergodic theorem holds for $\{T_1, \ldots, T_k\}$.

1. **Introduction** Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, $\{T_1, \ldots, T_k\}$ a set of linear operators of $L_p(X, \mathcal{F}, \mu)$, some p, $1 \leq p \leq \infty$. If

$$\lim_{n_1, \ldots, n_k \to \infty} \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f$$

exists and is finite a.e. for all $f \in L_p$, we say that the *multiple sequence ergodic theorem holds for* $\{T_1, \ldots, T_k\}$. Multiple sequence ergodic theorems arise from the study of random ergodic theorems, i.e. the study of the existence of

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T_{a(i)} \cdots T_{a(0)} f$$

where $\{a(i)\}$ is a sequence with values from $\{1, \ldots, k\}$, in the case the operators $T_1, \ldots, T_k$ commute. In the present paper, however, we do not require that $T_1, \ldots, T_k$ commute.

If $f \geq 0$ implies $Tf \geq 0$, we say that T is *positive*. If $\|T\|_p \leq 1$, then we say that T is a contraction of $L_p(X, \mathcal{F}, \mu)$. If there exists an operator S such that $|Tf(x)| \leq S\,|f(x)|$ a.e., we say that T *is dominated* by S, or that S *dominates* T.

We note that if T is dominated by S, then S is necessarily positive. Examples

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of such operators are the positive operators themselves, of course, Dunford-Schwartz operators (i.e.: simultaneously contractions of L_1 and L_∞) and the Lamperti operators considered by Kan [4]. In the case that the operators $T_1, \ldots, T_k$ are simultaneously contractions of $L_1(X, \mathcal{F}, \mu)$ and $L_\infty(X, \mathcal{F}, \mu)$ and p is fixed, $1<p<\infty$, Dunford and Schwartz ([2]), have proved that the multiple sequence ergodic theorem holds for $\{T_1, \ldots, T_k\}$. Recently, Sato ([6]) has proved a related result in the case $T_1, \ldots, T_k$ are commuting positive contractions of $L_1(X, \mathcal{F}, \mu)$ satisfying the L_1-mean ergodic theorem. In the present paper, we use Sato's techniques together with an extension of the dominated estimate for positive contractions obtained by Akcoglu ([1], see Section 2 for details) to prove that if $T_1, \ldots, T_k$ are linear operators of $L_p(X, \mathcal{F}, \mu)$, p fixed, $1<p<\infty$, dominated by the positive contractions, $S_1, \ldots, S_k$ of $L_p(X, \mathcal{F}, \mu)$, then the multiple sequence ergodic theorem holds for $\{T_1, \ldots, T_k\}$.

2. **A dominated estimate.** Let $\{T_1, \ldots, T_k\}$ be linear operators of $L_p(X, \mathcal{F}, \mu)$, $1<p<\infty$. Define the operator $M_{T_1,\ldots,T_k}(f)$ by

$$(M_{T_1,\ldots,T_k}f)(x) = \sup_{n_1,\ldots,n_k} \left| \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f(x) \right|.$$

Then Akcoglu has proved the following:

THEOREM 2.1. *Let T be a positive contraction of $L_p(X, \mathcal{F}, \mu)$, p fixed, $1<p<\infty$. Then*

$$\int (M_T f)^p \, du \le \left(\frac{p}{p-1}\right) \int |f|^p \, du.$$

Proof. See [1].

We extend this result as follows.

THEOREM 2.2. *Let $T_1, \ldots, T_k$ be contractions of $L_p(X, \mathcal{F}, \mu)$, p fixed, $1<p<\infty$, such that each T_i is dominated by a positive contraction of $L_p(X, \mathcal{F}, \mu)$. Then*

$$\int (M_{T_1,\ldots,T_k} f)^p \, du \le \left(\frac{p}{p-1}\right)^{kp} \int |f|^p \, du.$$

Proof. We first note that since

$$\left| \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f \right| \le \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} S_1^{m_1} \cdots S_k^{m_k} |f| \text{ a.e.}$$

where each T_i is dominated by the positive contraction S_i, we can and do assume that f and each T_i is positive.

We proceed by induction on k, noting that the case $k=1$ follows from Theorem 2.1. Assuming that

$$\int (M_{T_2,\ldots,T_k}(f))^p \, du \le \left(\frac{p}{p-1}\right)^{p(k-1)} \int f^p \, du,$$

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we note that

$$\frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f \le \frac{1}{n_1} \sum_{m_1=0}^{n_1-1} T_1^{m_1} M_{T_2,\ldots,T_k} f$$

(the induction is on the number of operators involved), so

$$\begin{aligned}\int \sup_{n_1 \cdots n_k} \left| \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f \right|^p \\ \le \int \left| \sup_{n_1} \sum_{m_1=0}^{n_1-1} T_1^{m_1} M_{T_2,\ldots,T_k} f \right|^p du \\ \le \left(\frac{p}{p-1} \right)^p \int |M_{T_2,\ldots,T_k} f|^p \, du \\ \le \left(\frac{p}{p-1} \right)^{kp} \int |f|^p \, du\end{aligned}$$

again by Theorem 2.1, and the proof is completed.

3. **Result**

THEOREM 3.1. *Let $T_1, \ldots, T_k$ be linear operators of $L_p(X, \mathcal{F}, \mu)$, p fixed, $1 < p < \infty$, each of which is dominated by a positive L_p contraction. Then the multiple sequence ergodic theorem holds for $T_1, \ldots, T_k$.*

Proof. We proceed by induction, noting that the theorem is true for $k = 1$ by Akcoglu's theorem ([1]). Next suppose that the multiple sequence ergodic theorem holds for any set of $k-1$ operators bounded by L_p contractions, let $\{T_1, \ldots, T_k\}$ be a set of such operators, and let $f = h + g - T_k g$ where $T_k h = h$ and $g \in L_p(X, \mathcal{F}, \mu)$, noting that the set of such f's are dense in $L_p(X, \mathcal{F}, \mu)$ by the mean ergodic theorem ([5], Theorem 9-1). Then

$$\begin{aligned}\frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} f \\ = \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_k=0}^{n_k-1} T_1^{m_1} \cdots T_k^{m_k} (h + g - T_k g) \\ = \frac{1}{n_1 \cdots n_{k-1}} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_{k-1}=0}^{n_{k-1}-1} T_1^{m_1} \cdots T_{k-1}^{m_{k-1}} h \\ + \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_{k-1}=0}^{n_{k-1}-1} T_1^{m_1} \cdots T_{k-1}^{m_{k-1}} g \\ - \frac{1}{n_1 \cdots n_k} \sum_{m_1=0}^{n_1-1} \cdots \sum_{m_{k-1}}^{n_{k-1}-1} T_1^{m_1} \cdots T_{k-1}^{m_{k-1}} T_k^{n_k} g.\end{aligned}$$

Now the first two terms on the right of the last equality converge a.e. as $n_1, \ldots, n_k \to \infty$ (the second to zero) by the induction hypothesis, so we consider only the last.

$$\left|\frac{1}{n_1\cdots n_k}\sum_{m_1=0}^{n_1-1}\cdots\sum_{m_{k-1}}^{n_{k-1}-1}T_1^{m_1}\cdots T_{k-1}^{m_{k-1}}T_k^{n_k}g(x)\right|\le\frac{1}{n_k}M_{S_1,\ldots,S_{k-1}}(S_k^{n_k}|g|)(x)\quad\text{a.e.}$$

where each T_i is dominated by the positive contraction S_i and the operator $M_{S_1,\ldots,S_{k-1}}$ is as defined in Section 2, and

$$\int\left|\frac{1}{n_k}M_{S_1,\ldots,S_{k-1}}(S_k^{n_k}|g|)\right|^p du\le\frac{1}{n_k^p}\left(\frac{p}{p-1}\right)^{p(k-1)}\int|S_k^{n_k}g|^p\,du$$

$$\le\frac{1}{n_k^p}\left(\frac{p}{p-1}\right)^{p(k-1)}\int|g|^p\,du$$

by Theorem 2.2 and since S_k is a contraction. Therefore,

$$\sum_{n_k=1}^{\infty}\int\left(\frac{1}{n_k}M_{S_1,\ldots,S_{k-1}}(S_k^{n_k}|g|)\right)^p du$$

is finite, so

$$\lim_{n_k\to\infty}\frac{1}{n_k}M_{S_1,\ldots,S_{k-1}}(S_k^{n_k}|g|)=0\quad\text{a.e.},$$

and

$$\overline{\lim}_{n_1,\ldots,n_k\to\infty}\left|\frac{1}{n_1\cdots n_k}\sum_{m_1=0}^{n_1-1}\cdots\sum_{m_{k-1}=0}^{n_{k-1}-1}T_1^{m_1}\cdots T_{k-1}^{m_{k-1}}T_k^{n_k}g\right|$$

$$\le\lim_{n_k\to\infty}\frac{1}{n_k}M_{S_1,\ldots,S_k}(S_k^{m_k}|g|)=0\quad\text{a.e.},$$

so

$$\lim_{n_1,\ldots,n_k\to\infty}\frac{1}{n_1\cdots n_k}\sum_{m_1=0}^{n_1-1}\cdots\sum_{m_{k-1}=0}^{n_{k-1}-1}T_1^{m_1}\cdots T_{k-1}^{m_{k-1}}T_k^{n_k}g=0\quad\text{a.e.}$$

Now we have

$$\lim_{n_1,\ldots,n_k\to\infty}\frac{1}{n_1\cdots n_k}\sum_{m_1=0}^{n_1-1}\cdots\sum_{m_k=0}^{n_k-1}T_1^{m_1}\cdots T_k^{m_k}f$$

exists a.e. for f in a dense subset of L_p, and

$$\sup_{n_1,\ldots,n_k}\left|\frac{1}{n_1\cdots n_k}\sum_{m_1=0}^{n_1-1}\cdots\sum_{m_k=0}^{n_k-1}T_1^{m_1}\cdots T_k^{m_k}f\right|<\infty$$

a.e. for all $f\in L_p(X,\mathcal{F},\mu)$ by Theorem 2.2. This is sufficient to imply the desired result by [2], Theorem IV.11.3 (for more on Theorem IV.11.3's application in this case, see the proof of Theorem VIII.6.9 in [2], p. 681). This completes the proof of the theorem.

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Added in Proof. It has come to my attention since submission of the manuscript for this article that the main result, Theorem 3.1, is essentially contained in Theorem 3 of S. A. McGrath, *Some ergodic theorems for commuting L_1 contractions*, Studia Mathematica, T. LXXX (1981), pp. 153–160. The statement of the theorem is essentially the same, although McGrath requires $T_1, \ldots, T_k$ to be positive. This requirement is not significant, however, and McGrath's proof is essentially the same as the proof of Theorem 3.1 of the present paper, and essentially includes the proof of Theorem 2.2.

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