COUPLED-OSCILLATORS

Y. TANAKA
Faculty of Education, Ibaraki University, Bunkyo, Mito 310, Japan

Abstract. Modal coupling oscillation models for the stellar radial pulsation and coupled-oscillators are reviewed. Coupled-oscillators with the second-order and third-order terms seemed to behave non-systematically. Using the equation by Schwarzschild and Savedoff (1949) with the dissipation term of van del Pol's type which is third-order, we demonstrate the effect of each term. The effects can be understood by the terms of the nonlinear dynamics, which is recently developing, that is, phase-locking, quasi-periodicity, period doubling, and chaos. As the problem of stellar pulsation, especially of double-mode cepheids on the period-ratio, we examine the dependence on the stellar structure from which the coupling constants in the second-order terms are derived. Eigen functions for adiabatic pulsations had been used for the calculation of the constants. It is noted that only two set of the constants are available, that is, for the polytrope model with $n = 3$ and a cepheid model without convection. Some examples of nonlinear dynamical effects will be shown.

It is shown that if the constants were suitable values, the period-ratio of double-mode cepheids is probably realized. The possibility is briefly suggested.

1. Introduction

Studies of the influence of second and higher order terms in the stellar pulsation have been continued for several decades. Woltjer (1935) shows equations of adiabatic stellar pulsation with a second order term can be expressed as the differential equation of time-dependent coefficients that the solution of stellar pulsation is expanded in terms of eigen functions of linear pulsation. He also applies it to Cepheid-variation with nonadiabatic perturbation and discuss the stationary state of periodic pulsation (Woltjer, 1937).

Resseland (1943) developed the mathematical theory of anharmonically pulsating gas spheres in the same way as Woltjer (1935, 1937) and applies it to the pulsation of homogeneous star, only taking the single mode into account. Bhatnager and Kothari (1944) give an exact solution for the pulsation of homogeneous modes for unperturbed stars.

Schwarzschild and Savedoff (1949) study anharmonic pulsation of the standard model which is more realistic model than the homogeneous model. The standard model is constructed by Schwarzschild (1941) with polytropic index 3 and various values of ratio of specific heats. Following Rosseland, they derive the equation of time-dependent amplitude for the fundamental and first overtone modes and compute it numerically. They result in that for the amplitude of characteristic cepheids, anharmonic pulsations yields the same period as harmonic pulsations. Anharmonics shows an appreciable skewness in the velocity-curve, but is still smaller than that observed.

Prasad (1949a, b) studies the interaction of multi-modes with calculations of coupling constants. The solutions are obtained by the use of Fourier series.

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After her, there are many studies of multi-mode coupled oscillator models, including the modes of non-radical oscillations.

Recently, the effects of nonadiabatic term and more realistic coupling constants of modes are studied by Takeuti and Aikawa (1981). Their work is mainly two properties. Following the mode coupling models, they use more realistic cepheid model and obtain the coupling constants. For studying to nonadiabatic effects, they introduce nonlinear damping terms of the van der Pol or the Krogdahl's type, which is introduced by Krogdahl (1955) to the stellar pulsation theory.

In the famous review of Ledoux and Walraven (1958, Fig. 46), the result by Krogdahl (1955) can be seen in comparison with the observational data and with the computed result by Schwarzschild and Savedoff (1949). The orbit in the radial-velocity space by Krogdahl does not fit to that observed at his period. However, this term is very compact and seen convenient for describing the effect of nonlinear dissipation, which may consist of many complicated terms of the nonlinear nonadiabatic effect. Using the equations which is the same ones as we will use later, Takeuti and Aikawa (1981) studied the behaviors of mode-coupling model analytically.

After them, analytical studies are developed by many authors such as Aikawa (1983, 1984), Dziembowski and Kovacs (1984), and Takeuti (1984, 1985, 1986).

2. Structure of Equations of Modal Coupling Model

The motion of simple pendulum is represented by

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x \tag{1}$$

where x and y is the displacement from the equilibrium and the velocity, respectively. The solution of Eq. (1) is the sinusoidal functions such as $x = A\sin(t)$ which depends on the initial values. It should be noted that the equation of motion for the forced oscillation is written as follows:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x^3 + \mu(1 - x^2)y + B\cos(\sigma t), \tag{2}$$

where the second term is so-called van der Pol's damping term and the third is the external force. The system of Eq. (2) is given by Ueda and Akamatsu (1981), which shows the Japanese attractor. This is a hybrid of the Duffing and van der Pol oscillator. If we rewrite Eq. (2) as

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x^3 + \mu(1 - x^2)y + z, \tag{3}$$

$$z = B\cos(\sigma t), \tag{4}$$

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or

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x^3 + \mu(1 - x^2)y + z, \tag{5}$$

$$\frac{dz}{dt} = a, \quad \frac{da}{dt} = -\sigma^2 z, \tag{6}$$

we can see the system is a coupled oscillator that there is no reaction from the first one to the second. The coefficient B in Eq. (2) is dependent on the initial values of Eq. (6) which lead chaotic behaviors of the system.

Now we discuss the equations of the mode coupled oscillator by Takeuti and Aikawa (1981). That is written as follows:

$$\frac{dx_1}{dt} = y_1, \tag{7}$$

$$\frac{dy_1}{dt} = -\sigma_1^2 x_1 + \sigma_1^2\{[\frac{1}{2}C_{111}x_1 + C_{112}x_2]x_1 + \frac{\mu_1}{\sigma_1^2}(1 - \alpha_1^2 x_1^2)y_1 + \frac{1}{2}C_{122}x_2^2\}, \tag{8}$$

$$\frac{dx_2}{dt} = y_2, \tag{9}$$

$$\frac{dy_2}{dt} = -\sigma_2^2 x_2 + \sigma_2^2\{[\frac{1}{2}C_{222}x_2 + C_{212}x_1]x_2 + \frac{\mu_2}{\sigma_2^2}(1 - \alpha_2^2 x_2^2)y_2 + \frac{1}{2}C_{211}x_1^2\}, \tag{10}$$

where C_{ijk} are the coupling constants between the modes 1 and 2.

If we suppose μ_2 and all C_{2jk} equal to zero, and C_{111} and C_{112} also zero, the system becomes the nonlinearly forced van der Pol oscillator. That is, the mode 1 with van der Pol's damping is forced to oscillate by mode 2 through the term of $C_{122}(A\sin\sigma_2 t + B\cos\sigma_2 t)^2$. Two sets of coupling constants between the fundamental and first overtone for stellar models have been evaluated with linear adiabatic eigen functions (Takeuti, 1985). Before analyzing the coupled oscillator for stellar models, we study oscillators with rather simple method.

It should be worthwhile to note that the forced oscillation can be reduced into a sine circle map (Chirikov, 1979), which shows the Arnold's tongues for phase locking and the devil's staircase. The discrete system seems rather convenient to study the rough behavior of differential equation system. The two-dimensional maps such as Henon's and Mira's attractors may also be useful (Gumowski and Mira, 1980).

3. Rough Behaviors of the Equations

The values of C_{1ij} and C_{2ij} obtained by Takeuti (1985) for cepheid models nearly equals 2∼4, and 6, respectively. If we simplify constant values for coupling constants such as $C_{111} = C_{112} = C_{121} = C_1$ and so on, we can decrease the number of parameters whose dependence should be examined. The brief method and results are given by Nakahara and Tanaka (1992) and more detailed discussion can be seen in Tanaka et al. (1992 a, b). Here we sketch their results.

First, fixing the values of $C_2 = 6$ and $\mu_1/\sigma_1^2 = 0.1$, they examine the dependence of solutions on other parameters. They show various phase locking of period ratios from 2/3 to 1/1 (Tanaka et al., 1992a). Between them, we can see the phase locking of period ratios in Farey series where the ratio of 5/7 is included. The ratio of 5/7, however, can be seen in very narrow region of parameter space. The regions between the phase locking are not distinguished whether higher ratios of phase locking, chaotic or quasi-periodic states. They also compute the Lyapunov exponents for examining quasi-periodicity, phase locking and chaos (Nakahara and Tanaka, 1992). But they do not recognize the chaotic states within their computed results.

Tanaka et al. (1992b) show the dependence of C_1 and C_2 in the manner of the Mandelbrot set. They select the color for each point in (C_1, C_2) plane when the solution has diverged (Fig. 1). In Fig. 2, the enlargement of Fig. 1 in the first quarter is shown. We can see complicated patterns in the divergent region and the complex boundary. In Fig. 3, we see the structure of the non-divergent region. The phase locking of lower period ratios such as 3/4, 5/7 etc. are shown. The shape of typical phase locked regions seems similar to so-called Arnold's tongues in the sine maps. It is noted that the non-divergent region is located $C_2 < -1.5C_1 + 15$, depending on the other parameters. As their results is very preliminary ones, further examples are desired.

4. Oscillation of the "Standard Model"

Equations used by Schwarzschild and Savedoff (1949) are the same as those of us without nonadiabatic term and give values of coupling constants between the fundamental and first overtone modes for the standard model. It is noted that their coupling constants are the first one for realistic stellar models. Following Takeuti and Aikawa (1981), Nakahara and Tanaka (1993) introduce the van der Pol's term as the nonadiabatic effect to the equations by Schwarzchild and Savedoff (1949) and demonstrate the behavior of the oscillation of polytropic stars.

They report that in the polytropic stellar pulsation the fundamental and first overtone mode synchronize each other for most values of parameters of the damping terms, while phase locking of 4/5 and 3/4 rarely appears. It is found that for the period ratio of 1/1 which located near the boundary between the divergent and non-divergent regions, the period doubling of 1/1 is observed, which will lead to chaos. Since the mesh of parameters for

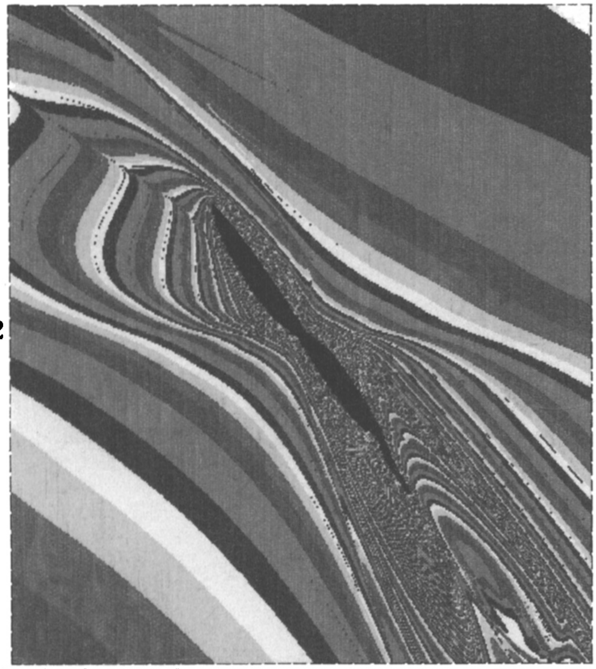

Fig. 1. Pattern of divergence.

researching lower period ratios is rather coarse, more careful survey may be helpful for understanding period ratios of compact star's pulsation.

5. 5:7 Period Ratio by a Cepheid Model Without Convection

More realistic coupling constants of double-mode stellar pulsation are obtained by Takeuti and Aikawa (1981) and Takeuti (1984) as mentioned above. Following the mode coupling models, they use more realistic cepheid model ($M/M_{\odot} = 6.7$, $L/L_{\odot} = 2280$, $T_{\rm eff} = 5850$ K, $X, Y, Z = 0.70$, 0.28, 0.02) and obtain the coupling constants. Using the constants, Seya et al. (1990) demonstrate the behavior of the solutions of Eq. (7)–(10). It should be noted that Moskalik and Buchler (1989) show period doubling bifurcation of coupled oscillators with van der Pol's term, but C_{1ij} equal to zero.

Seya et al. (1990) obtain time developments of mode 1 (the fundamental mode) and 2 (the first overtone) and the orbits in the phase planes of $(x_1, dx_1/dt)$ etc. for various sets of coefficients of the damping terms. They find the phase locking of several ratios. As pointed out by Ishida and Takeuti

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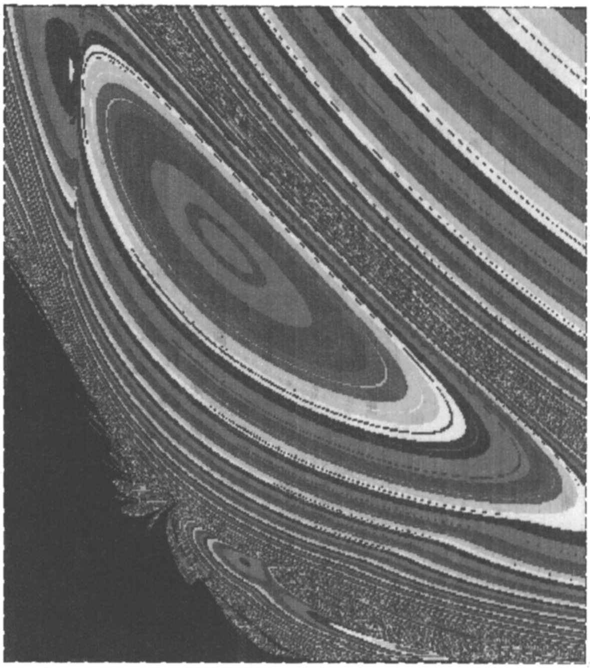

Fig. 2. Enlargement of Fig. 1.

(1991), the ratio of 3/4 is dominant for the given coupling constants and the linear frequency ratio adopted by them. Seya et at. (1990) also show the phase locked orbits such as the period ratio of 3/4 will bifurcate in period doubling as the change of α_2^2. They use the first return map for showing the characteristics of the beat or the quasi-periodicity of the solution. The phenomena of quasi-periodicity may be important for studying the beat of variable stars.

Seya et al. (1991) demonstrate the behavior of solutions of Eqs. (7)–(10) in the Poincare section. On the section, the complex features can be understood as the chaotic states of solutions. Thus they conclude that the chaos of coupled oscillator are produced through folding the surface of torus. Ishida and Takeuti (1991) give the condition of synchronization of coupled oscillator with non-zero C_{ijk} of Takeuti (1985). They also find the phase locking of other ratios than 2/1.

The period ratio of 5/7 as phase locking is reported by Seya et al. (1990), Tanaka et al. (1990) and Ishida and Takeuti (1991). Ishida and Takeuti (1991) obtain it for fixed ratio of linear periods and fixed coupling constants,

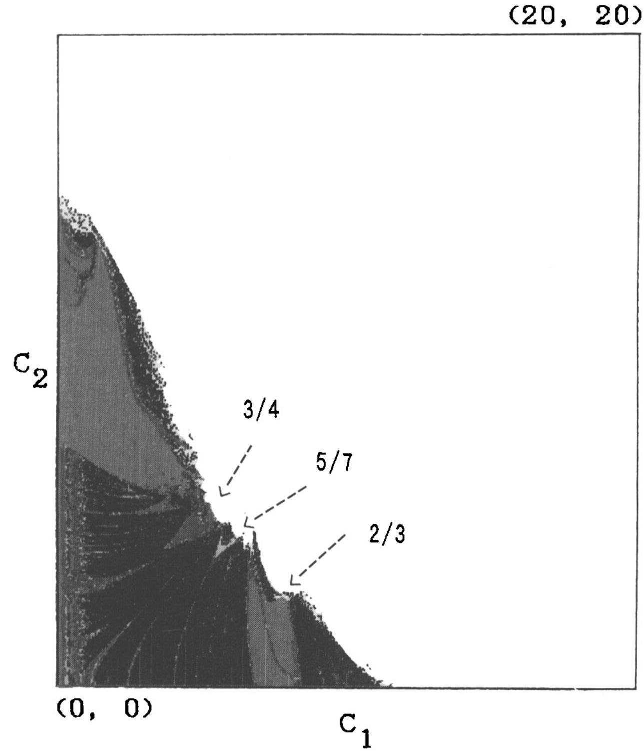

Fig. 3. Structure of non-divergent region. The phase locking can be seen as the Arnold's tongues.

while the others change the linear period ratios for showing the ratio of 5/7. The period ratio of 5/7 is important, because the double-mode cepheids seem have rather narrow period ratios 0.7 to 0.71. The standard stellar evolutionary theory derives the period ratios higher than those of observation. It is known that the mass of cepheids should be considerably reduced for fitting them. The review on this problem by Balona (1985) should be referred.

Tanaka et al. (1992c, d) try to search the condition of coupling constants for the period ratio of 5/7 with the realistic linear periods and the coefficients of the damping terms. Using the relation of $C_{112}/C_{211} = C_{122}/C_{212}$ and fixed C_{2jk}, they find out the relation of C_{111} and A where A is a multiplier of the original coupling constants of C_{1j2}.It is shown that the 5/7 phase locking is realized by the increase of self- and mutual-coupling constants of mode 1.

Takeuti et al. (1992) and Yamakawa et al. (1992) calculate the coupling constants, taking the nonadiabatic effect in radiative models. They obtain rather large values of the constants in order of 10–100. Such increases may make the oscillation to be synchronized or sometime diverge. Zalewski

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(1992), however, show that the nonadiabatic constants become much smaller for the cepheid models if convection is taken into account. This may be the real model of double-mode cepheids which have the period ratio of 5/7.

It is noted that the present coupled oscillator model is still not very realistic. As the further correction, we should mention that Ishida et al. (1992) try to examine the effect of third order nonlinear terms. Their results will give us more information on the direction for studying double-mode cepheids and its evolutionary masses.

6. Summary

We have the constants of modal coupling for the standard model and some cepheid models. With these constants, the behaviors of the coupled oscillator model of stellar pulsation are analyzed by many authors. They find out the phase locking, quasi-periodicity, chaos and divergent states which depend on the constant and the coefficients of the van der Pol's damping term. The coupling constants for nonadiabatic models are now examined with and without convection. As the result, the period ratio of 5/7 and the mystery of mass reduction of cepheids are expected to be understood. Thus we may expect to resolve soon one of mysteries of multi-period variables, but no one can say the step might not be the devil's staircase. It is desired to work with further knowledge from observation, hydrodynamical simulations, and nonlinear dynamics.

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