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On the Riemann zeta-function I

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Dedicated to Professor Bernhard H. Neumann

We prove an approximation formula for the Riemann zeta function. We show that a classical theorem:

$$\zeta(s) = O\left(t^{(1-\sigma)/2}\right) \text{ as } t \to \infty \quad (s = \sigma+it)$$

uniformly in the domain $\frac{1}{2} \le \sigma < 1$, is an immediate consequence of our approximation formula. Our method is real and free from complex analysis.

1. Introduction and theorems

1.1. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s \quad (s = \sigma+it) ,$$

which converges in the half plane $\sigma > 1$ and represents a regular function. Riemann proved that it is analytically continued to the whole plane and regular there except at the point $s = 1$, which is its simple pole. Riemann supposed that $\zeta(s) \ne 0$ in the strip $\frac{1}{2} < \sigma < 1$. This is called the Riemann hypothesis. Lindelöf conjectured that

$$\zeta(s) = O\left(t^{\varepsilon}\right) \text{ uniformly for } \tfrac{1}{2} \le \sigma < 1$$

for any $\varepsilon > 0$, which is equivalent to $\zeta(\frac{1}{2}+it) = O\left(t^{\varepsilon}\right)$.

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We shall prove an approximation formula for the zeta function.

THEOREM 1.

$$\zeta(s) = \sum_{j=1}^{t/3\pi} \int_{3/2}^{t/\pi j} \frac{e^{-i(t\log k - 2\pi jk)}}{k^{\sigma}}\, dk + O(\log t) \quad as \quad t \to \infty ,$$

uniformly in the interval $\frac{1}{2} \le \sigma < 1$.

As an immediate consequence, we get

THEOREM 2.

$$\zeta(s) = O\left(t^{(1-\sigma)/2}\right) \quad as \quad t \to \infty$$

uniformly in $\frac{1}{2} \le \sigma < 1$.

1.2. We use the notation $\sum_{n=a}^{b} = \sum_{n=[a]}^{[b]}$ and $k, l, m, n, \ldots$ are used to represent continuous variables as well as discrete variables.

2. Proof of Theorem 1

It is known that

$$\zeta(s) = s \int_{1}^{\infty} \frac{J(u)}{u^{s+1}}\, du + \frac{1}{s-1} + \frac{1}{2} \quad (\mathcal{R}s > 0)$$

(see [2], p. 14), where

$$J(u) = [u] - u + \tfrac{1}{2} \sim \sum_{n=1}^{\infty} \frac{\sin 2\pi nu}{\pi u} .$$

We suppose that $s = \sigma + it$, $\frac{1}{2} < \sigma < 1$, and t is a large positive number. Then

$$\begin{aligned} \zeta(s) &= (\sigma+it) \int_{1}^{\infty} \frac{J(u)}{u^{(1+\sigma)+it}}\, du + O(1) \\ &= it \int_{1}^{\infty} \frac{J(u)}{u^{1+\sigma}}\, e^{-it\log u} du + O(1) \\ &= t \int_{1}^{\infty} \frac{J(u)}{u^{1+\sigma}} \left(i \cos(t \log u) + \sin(t \log u)\right) du + O(1) \\ &= iP + P^{*} + O(1) . \end{aligned}$$

We shall estimate P only since P^* can be done quite similarly.

Using the Fourier expansion of $J(u)$,

$$P = \sum_{m=1}^{\infty} \frac{t}{\pi m} \int_1^{\infty} \cos(t \log u) \sin 2\pi m u \frac{du}{u^{\sigma+1}}$$

$$= \int_{\frac{1}{2}}^{\infty} \frac{t}{\pi m} \left(dm+dJ(m)\right) \int_1^{\infty} \cos(t \log u) \sin 2\pi m u \frac{du}{u^{\sigma+1}}$$

$$= \left(P_1+P_2\right)/\pi .$$

By the transformation $mu = v$,

$$P_1 = t \int_{\frac{1}{2}}^{\infty} \frac{dm}{m^{1-\sigma}} \int_m^{\infty} \cos\left(t \log \frac{v}{m}\right) \sin 2\pi v \frac{dv}{v^{\sigma+1}}$$

$$= t \int_{\frac{1}{2}}^{\infty} \frac{\sin 2\pi v}{v^{\sigma+1}} dv \int_{\frac{1}{2}}^{v} \cos\left(t \log \frac{v}{m}\right) \frac{dm}{m^{1-\sigma}} .$$

Further we write $t \log(v/m) = n$; then

$$m = ve^{-n/t} , \quad dm = -(v/t)e^{-n/t}dn ,$$

and then

$$P_1 = \int_{\frac{1}{2}}^{\infty} \frac{\sin 2\pi v}{v} dv \int_0^{t\log 2v} \frac{\cos n}{e^{\sigma n/t}} dn$$

$$= \int_{\frac{1}{2}}^{\infty} dv \left(\int_0^{\infty} dn - \int_{t\log 2v}^{\infty} dn\right) = O(1) .$$

By integration by parts,

$$P_2 = t \int_{\frac{1}{2}}^{\infty} \frac{J(m)}{m^2} dm \int_1^{\infty} \cos(t \log u) \sin 2\pi m u \frac{du}{u^{\sigma+1}}$$

$$- 2\pi t \int_{\frac{1}{2}}^{\infty} \frac{J(m)}{m} dm \int_1^{\infty} \cos(t \log u) \cos 2\pi m u \frac{du}{u^{\sigma}}$$

$$= Q_1 - 2\pi Q_2 .$$

3. Estimation of Q_1

By changing the order of integration,

$$Q_1 = t\int_1^\infty \frac{\cos(t\log u)}{u^{\sigma+1}}\,du \int_{\frac12}^\infty J(m)\sin 2\pi mu\,\frac{dm}{m^2},$$

where the inner integral of the right side is

$$\begin{aligned} Q_1(u) &= \sum_{k=1}^\infty \frac{1}{\pi k}\int_{\frac12}^\infty \frac{\sin 2\pi km \sin 2\pi mu}{m^2}\,dm \\ &= \sum_{k=1}^\infty \frac{1}{2\pi k}\int_{\frac12}^\infty \frac{\cos 2\pi(k-u)m - \cos 2\pi(k+u)m}{m^2}\,dm \\ &= \frac{1}{2\pi}\left(Q_{11}(u) - Q_{12}(u)\right). \end{aligned}$$

We write

$$Q_{11} = t\int_1^\infty \frac{\cos(t\log u)}{u^{\sigma+1}}\,Q_{11}(u)\,du,$$

$$Q_{12} = t\int_1^\infty \frac{\cos(t\log u)}{u^{\sigma+1}}\,Q_{12}(u)\,du.$$

3.1. Estimation of Q_{12}.

$$\begin{aligned} Q_{12}(u) &= \sum_{k=1}^\infty \frac{1}{k}\int_{\frac12}^\infty \frac{\cos 2\pi(k+u)m}{m^2}\,dm \\ &= \sum_{k=1}^\infty \left(1+\frac{u}{k}\right)\int_{(k+u)/2}^\infty \frac{\cos 2\pi m}{m^2}\,dm \end{aligned}$$

and then

$$\begin{aligned} Q_{12} &= t\sum_{k=1}^\infty \int_1^\infty \frac{\cos(t\log u)}{u^{\sigma+1}}\,du \int_{(k+u)/2}^\infty \frac{\cos 2\pi m}{m^2}\,dm \\ &\qquad + \sum_{k=1}^\infty \frac{t}{k}\int_1^\infty \frac{\cos(t\log u)}{u^{\sigma}}\,du \int_{(k+u)/2}^\infty \frac{\cos 2\pi m}{m^2}\,dm \\ &= Q_{121} + Q_{122}, \end{aligned}$$

where

$$Q_{121} = \sum_{k=1}^{\infty} \int_0^{\infty} \frac{\cos v}{e^{\sigma v/t}}\, dv \int_{(k+e^{v/t})/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm$$

$$= \sum_{k=1}^{t} \int_{(k+1)/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm \int_0^{t\log(2m-k)} \frac{\cos v}{e^{\sigma v/t}}\, dv$$

$$+ \sum_{k=m+1}^{\infty} \int_0^{\infty} \frac{\cos v}{e^{\sigma v/t}}\, dv \int_{(k+e^{v/t})/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm$$

$$= O\left(\sum_{k=1}^{t} \int_{(k+1)/2}^{\infty} \frac{dm}{m^2}\right) + O\left(\sum_{k=t+1}^{\infty} \frac{1}{k^2} \int_0^{\infty} \frac{dv}{e^{\sigma v/t}}\right)$$

$$= O(\log t) + O(1) = O(\log t) ,$$

and

$$Q_{122} = \sum_{k=1}^{\infty} \frac{t}{k} \int_{(k+1)/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm \int_1^{2m-k} \frac{\cos(t\log u)}{u^{\sigma}}\, du$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm \int_0^{t\log(2m-k)} e^{(1-\sigma)v/t} \cos v dv$$

$$= O\left\{\sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)/2}^{\infty} \frac{(2m-k)^{1-\sigma}}{m^2}\, dm\right\} = O(1) .$$

Therefore $Q_{12} = O(\log t)$ as $t \to \infty$.

3.2. Estimation of Q_{11} .

$$Q_{11} = t \int_1^{\infty} \frac{\cos(t\log u)}{u^{\sigma+1}} \left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{\frac{1}{2}}^{\infty} \frac{\cos 2\pi(k-u)m}{m^2}\, dm\right) du .$$

Since

$$\int_{\frac{1}{2}}^{\infty} \frac{\cos 2\pi(k-u)m}{m^2}\, dm = |k-u| \int_{|k-u|/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm$$

is bounded for $|k-u| < 1$ and $O(1/|k-u|)$ for $|k-u| > 1$, we can interchange the order of summation and integration on the right side; that is

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$$Q_{11} = \sum_{k=1}^{\infty} \frac{t}{k} \left[\int_1^k \frac{\cos(t\log u)}{u^{\sigma+1}}\, du \int_{\frac{1}{2}}^{\infty} \frac{\cos 2\pi(k-u)m}{m^2}\, dm + \int_k^{\infty} \frac{\cos(t\log u)}{u^{\sigma+1}}\, du \int_{\frac{1}{2}}^{\infty} \frac{\cos 2\pi(u-k)m}{m^2}\, dm \right]$$

$$= Q_{111} + Q_{112} ,$$

where

$$Q_{111} = \sum_{k=2}^{\infty} \frac{t}{k} \left[\int_0^{(k-1)/2} \frac{\cos 2\pi m}{m^2}\, dm \int_{k-2m}^{k} \frac{(k-u)\cos(t\log u)}{u^{\sigma+1}}\, du + \int_{(k-1)/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm \int_1^k \frac{(k-u)\cos(t\log u)}{u^{\sigma+1}}\, du \right]$$

$$= \sum_{k=2}^{\infty} \frac{1}{k} \int_0^{(k-1)/2} \frac{\cos 2\pi m}{m^2}\, dm \int_{t\log(k-2m)}^{t\log k} \frac{k-e^{v/t}}{e^{\sigma v/t}} \cos v dv + O\left[\sum_{k=2}^{\infty} \left| \int_{(k-1)/2}^{\infty} \frac{\cos 2\pi m}{m^2}\, dm \right| \right]$$

$$= O\left[\sum_{k=2}^{\infty} \frac{1}{k} \left\{ \int_0^{k/2t} \frac{1}{m^2} \frac{m}{(k-2m)^{\sigma}} \frac{tm}{k}\, dm + \left(\int_{k/2}^{k/4} + \int_{k/4}^{(k-1)/2} \right) \frac{1}{m^2} \frac{m}{(k-2m)^{\sigma}}\, dm \right\} \right] + O(1)$$

$$= O\left[\sum_{k=2}^{\infty} \frac{1}{k^{1+\sigma}} + \sum_{k=2}^{\infty} \frac{1}{k^{1+\sigma}} \int_{k/2t}^{k/4} \frac{dm}{m} + \sum_{k=2}^{\infty} \frac{1}{k^2} \int_{k/4}^{(k-1)/2} \frac{dm}{(k-2m)^{\sigma}} \right] + O(1)$$

$$= O(\log t) ,$$

and

$$Q_{112} = \sum_{k=1}^{\infty} \frac{t}{k} \int_k^{\infty} \frac{\cos(t\log u)}{u^{\sigma+1}}\, du \int_{(u-k)/2}^{\infty} \frac{(u-k)\cos 2\pi m}{m^2}\, dm$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} \frac{\cos 2\pi m}{m^2}\, dm \int_{t\log k}^{t\log(k+2m)} \frac{(e^{v/t}-k)\cos v}{e^{\sigma v/t}}\, dv$$

$$= O\left[\sum_{k=1}^{\infty} \frac{1}{k} \left\{ \int_0^{k/2t} \frac{1}{m^2} \frac{tm}{k} \frac{m}{(m+2k)^{\sigma}}\, dm + \left(\int_{k/2t}^{k} + \int_k^{\infty} \right) \frac{1}{m^2} \frac{m}{(m+2k)^{\sigma}}\, dm \right\} \right]$$

$$= O(\log t) .$$

Thus we have proved that $Q_{11} = O(\log t)$ and then

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$$Q_1 = Q_{11} - Q_{12} = O(\log t) \ .$$

It remains to estimate Q_2 .

4. Estimation of Q_2

$$Q_2 = t\int_{\frac{1}{2}}^{\infty} \frac{J(m)}{m}\, dm \int_1^{\infty} \cos(t\log u)\cos 2\pi mu \frac{du}{u^{\sigma}}$$

$$= t\int_1^{\infty} \frac{\cos(t\log u)}{u^{\sigma}}\, du \int_{\frac{1}{2}}^{\infty} \frac{J(m)}{m}\cos 2\pi mu\, dm \ ,$$

which will be proved in the Appendix. The inner integral of the right side is

$$Q_2(u) = \sum_{k=1}^{\infty} \frac{1}{\pi k}\int_{\frac{1}{2}}^{\infty} \sin 2\pi km \cos 2\pi mu \frac{dm}{m}$$

$$= \sum_{k=1}^{\infty} \frac{1}{2\pi k}\int_{\frac{1}{2}}^{\infty} \{\sin 2\pi(k+u)m + \sin 2\pi(k-u)m\} \frac{dm}{m}$$

$$= Q_{21}(u) + Q_{22}(u) \ .$$

We write

$$Q_2 = t\int_1^{\infty} \frac{\cos(t\log u)}{u^{\sigma}} \big(Q_{21}(u)+Q_{22}(u)\big)\, du$$

$$= Q_{21} + Q_{22} \ .$$

5. Estimation of Q_{21}

We get

$$Q_{21} = \sum_{k=1}^{\infty} \frac{t}{2\pi k}\int_1^{\infty} \frac{\cos(t\log u)}{u^{\sigma}}\, du \int_{(k+u)/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$= \sum_{k=1}^{\infty} \frac{1}{2\pi k}\int_{(k+1)/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_0^{t\log(2m-k)} e^{(1-\sigma)v/t} \cos v\, dv$$

$$= \sum_{k=1}^{\infty} \frac{1}{4\pi k}\int_{(k+1)/2}^{\infty} \frac{dm}{m}\int_0^{t\log(2m-k)} \{\sin(2\pi m+v) + \sin(2\pi m-v)\} e^{(1-\sigma)v/t}\, dv$$

$$= \frac{1}{4\pi}\big(Q_{211}+Q_{212}\big) \ .$$

5.1. Estimation of Q_{211}. By the transformation $2\pi m + v = w$,

$$Q_{211} = \sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)/2}^{\infty} \frac{dm}{m} \int_{2\pi m}^{t\log(2m-k)+2\pi m} e^{(1-\sigma)(w-2\pi m)/t} \sin w dw .$$

The function $w = t \log(2m-k) + 2\pi m$ is an increasing function of m on the interval $[(k+1)/2, \infty)$ and then its inverse function $m = M(w)$ is also increasing on the interval $[(k+1)\pi, \infty)$; therefore

$$Q_{211} = \sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)}^{\infty} \sin w dw \left[e^{(1-\sigma)w/t} \int_{M(w)}^{w/2\pi} \frac{dm}{me^{2\pi(1-\sigma)m/t}} \right] .$$

If we denote by $y(w)$ the function in the bracket on the right side, that is,

$$y(w) = e^{(1-\sigma)w/t} \int_{M(w)}^{w/2\pi} \frac{dm}{me^{2\pi(1-\sigma)m/t}} ,$$

then $y(w)$ is non-negative on the interval $[(k+1)\pi, \infty)$ and vanishes at both ends of the interval. For, by the second mean-value theorem,

$$\begin{aligned} y(w) &= \frac{e^{(1-\sigma)w/t}}{M(w)} \int_{M(w)}^{\theta} \frac{dm}{e^{2\pi m(1-\sigma)/t}} \quad (M(w) < \theta < w/2\pi) \\ &\le \frac{t}{2\pi(1-\sigma)M(w)} e^{(1-\sigma)\left(w-2\pi M(w)\right)/t} \\ &= \frac{t\left(2M(w)-k\right)^{1-\sigma}}{2\pi(1-\sigma)M(w)} = o(1) \quad \text{as} \quad w \to \infty , \end{aligned}$$

since

(1) $$w = t \log\left(2M(w)-k\right) + 2\pi M(w) .$$

Now

$$\begin{aligned} y'(w) &= \frac{1-\sigma}{t} y(w) + \frac{1}{w} - \frac{M'(w)e^{(1-\sigma)w/t}}{M(w)e^{2\pi(1-\sigma)M(w)/t}} \\ &= \frac{1-\sigma}{t} y(w) + \frac{1}{w} - \frac{M'(w)\left(2M(w)-k\right)^{1-\sigma}}{M(w)} \\ &= \frac{\left(2M(w)-k\right)^{1-\sigma}}{2\pi M(w)} - \frac{M'(w)\left(2M(w)-k\right)^{1-\sigma}}{M(w)} - \frac{1}{2\pi} e^{(1-\sigma)w/t} \int_{M(w)}^{w/2\pi} \frac{dm}{m^2 e^{2\pi m(1-\sigma)/t}} \\ &= \frac{tM'(w)}{\pi M(w)\left(2M(w)-k\right)} - \frac{1}{2\pi} e^{(1-\sigma)w/t} \int_{M(w)}^{w/2\pi} \frac{dm}{m^2 e^{2\pi(1-\sigma)m/t}} , \end{aligned}$$

using (1) at the first step, integration by parts at the second step, and using the relation

$$M'(w) = 1/\left(2\pi + \frac{2t}{2M(w)-k}\right) \tag{2}$$

at the last step. Therefore

$$\begin{aligned} Q_{211} &= \sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)\pi}^{\infty} y'(w) \cos w\,dw \\ &= \sum_{k=1}^{\infty} \frac{t}{k\pi} \int_{(k+1)\pi}^{\infty} \frac{M'(w)\cos w}{M(w)\left(2M(w)-k\right)^{\sigma}}\,dw \\ &\qquad - \sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_{(k+1)\pi}^{\infty} e^{(1-\sigma)w/t} \cos\,dw \int_{M(w)}^{w/2\pi} \frac{dm}{m^2 e^{2\pi(1-\sigma)m/t}} \\ &= \frac{1}{\pi} R_1 - \frac{1}{2\pi} R_2\ . \end{aligned}$$

We shall first estimate R_1 . By (2),

$$\begin{aligned} M(w)\left(2M(w)-k\right)^{\sigma}/M'(w) &= M(w)\left(2M(w)-k\right)^{\sigma}\left(2\pi+2t/\left(2M(w)-k\right)\right) \\ &= M(w)\left(2\pi\left(2M(w)-k\right)+2t\right)/\left(2M(w)-k\right)^{1-\sigma}\ . \end{aligned}$$

Consider the function of x :

$$z(x) = x\left(2\pi(2x-k)+2t\right)/(2x-k)^{1-\sigma}\ ;$$

then its logarithmic differential coefficient is

$$\begin{aligned} \frac{z'(x)}{z(x)} &= \frac{1}{x} + \frac{4\pi}{2\pi(2x-k)+2t} - \frac{2(1-\sigma)}{2x-k} \\ &= z_0(x)/x\left(2\pi(2x-k)+2t\right)(2x-k)\ , \end{aligned}$$

where

$$\begin{aligned} z_0(x) &= (2x-k)\left(2\pi(2x-k)+2t\right) + 4\pi x(2x-k) - 2(1-\sigma)x\left(2\pi(2x-k)+2t\right) \\ &= 8\pi(1+\sigma)x^2 - 4\left((2+\sigma)\pi k-\sigma t\right)x + 2k(\pi k-t)\ , \end{aligned}$$

which vanishes at

$$\begin{aligned} x_0 &= \frac{1}{4\pi(1+\sigma)}\left\{(2+\sigma)\pi k - \sigma t + \sqrt{\left((2+\sigma)\pi k-\sigma t\right)^2-4\pi(1+\sigma)k(\pi k-t)}\right\} \\ &\le k\ . \end{aligned}$$

Therefore, $z(x)$ takes a minimum between $(k+1)/2$ and k or is monotone increasing in the interval of x : $[(k+1)/2, \infty)$. For each case

$$R_1 = O\left(\sum_{k=1}^{\infty} \frac{1}{k^{1+\sigma}}\right) = O(1) .$$

The integral in the kth term of R_2 is

$$R_2(k) = \int_{(k+1)\pi}^{\infty} e^{(1-\sigma)w/t} \cos w\,dw \int_{M(w)}^{w/2} \frac{dm}{m^2 e^{2\pi(1-\sigma)m/t}}$$

$$= \int_{(k+1)/2}^{\infty} \frac{dm}{m^2 e^{2\pi(1-\sigma)m/t}} \int_{2\pi m}^{2\pi m+t\log(2m-k)} e^{(1-\sigma)w/t} \cos w\,dw$$

$$= O\left(\int_{(k+1)/2}^{\infty} \frac{(2m-k)^{1-\sigma}}{m^2}\,dm\right) = O\left(\frac{1}{k^{\sigma}}\right)$$

and then

$$R_2 = \sum_{k=1}^{\infty} \frac{1}{k} R_2(k) = O\left(\sum_{k=1}^{\infty} \frac{1}{k^{1+\sigma}}\right) = O(1) .$$

Therefore,

$$Q_{211} = O(|R_1|+|R_2|) = O(1) .$$

5.2. Estimation of Q_{212}.

$$Q_{212} = \sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)/2}^{\infty} \frac{dm}{m} \int_0^{t\log(2m-k)} e^{(1-\sigma)v/t} \sin(2\pi m-v)\,dv$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} e^{(1-\sigma)v/t}\,dv \int_{(e^{v/t}+k)2}^{\infty} \frac{\sin(2\pi m-v)}{m}\,dm$$

$$= \sum_{k=1}^{\infty} \frac{1}{\pi k} \int_0^{\infty} e^{(1-\sigma)v/t} \frac{\cos((e^{v/t}+k)\pi-v)}{e^{v/t}+k}$$

$$- \sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_0^{\infty} e^{(1-\sigma)v/t}\,dv \int_{(e^{v/t}+k)/2}^{\infty} \frac{\cos(2\pi m-v)}{m^2}\,dm$$

$$= \frac{1}{\pi} S_1 - \frac{1}{2\pi} S_2 ,$$

where

$$S_2 = \sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)/2}^{\infty} \frac{dm}{m^2} \int_0^{t\log(2m-k)} e^{(1-\sigma)v/t} \cos(2\pi m-v)dv$$

$$= O\left(\sum_{k=1}^{\infty} \frac{1}{k} \int_{(k+1)/2}^{\infty} \frac{(2m-k)^{1-\sigma}}{m^2} dm\right) = O(1) .$$

Now

$$S_1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_0^{\infty} \frac{e^{(1-\sigma)v/t}}{e^{v/t}+k} \cos\left(\pi e^{v/t}-v\right)dv$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k} \int_0^{\infty} \frac{e^{(1-\sigma)v/t}\cos\left(\pi e^{v/t}-v\right)}{\left(e^{v/t}+2k\right)\left(e^{v/t}+2k+1\right)} dv$$

$$+ \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \int_0^{\infty} \frac{e^{(1-\sigma)v/t}\cos\left(\pi e^{v/t}-v\right)}{e^{v/t}+2k+1} dv - \int_0^{\infty} \frac{e^{(1-\sigma)v/t}}{e^{v/t}+1} \cos\left(\pi e^{v/t}-v\right)dv$$

$$= S_{11} + S_{12} - S_{13} .$$

We consider the function $x(v) = \pi e^{v/t} - v$, which has two roots v_1 and v_3 such that $\pi < v_1 = \pi + O(1/t)$ and v_3 satisfies the relations

$$v_3 = t \log\left(v_3/\pi\right) , \quad v_3 > t \log(t/\pi)$$

and

$$t \log \frac{t}{\pi} < t \log\left(\frac{t}{\pi} \log \frac{t}{\pi}\right) < t \log\left(\frac{t}{\pi} \log\left(\frac{t}{\pi} \log \frac{t}{\pi}\right)\right) < \ldots$$

$$< v_3 = t\left\{\log \frac{t}{\pi} + \log \log \frac{t}{\pi}\left(1 + \frac{1+o(1)}{\log(t/\pi)}\right)\right\} .$$

We write

$$S_{11} = \sum_{k=1}^{\infty} \frac{1}{2k} \left(\int_0^{v_1} + \int_{v_1}^{v_2} + \int_{v_2}^{v_3} + \int_{v_3}^{\infty}\right)dv$$

$$= S_{111} + S_{112} + S_{113} + S_{114} ,$$

where $v_2 = \frac{4}{5} t \log \frac{t}{\pi}$ and

$$S_{111} = O\left(\sum_{k=1}^{\infty} \frac{1}{k^3}\right) = O(1) .$$

We shall use the transformation $w = v - \pi e^{v/t}$ for the estimation of S_{112}. We denote it by $w = w(v)$ and its inverse function by $v = v(w)$. Then

$$S_{112} = \sum_{k=1}^{\infty} \frac{1}{2k} \int_{w(v_1)}^{w(v_2)} \frac{e^{(1-\sigma)v(w)/t} \cos w dw}{\left(e^{v(w)/t}+2k\right)\left(e^{v(w)/t}+2k+1\right)\left(1-\pi e^{v(w)/t}/t\right)} .$$

For large t and for $w(v_1) < w < w(v_2)$,

$$1 - \frac{\pi}{t} e^{v(w)/t} > 1 - (\pi/t)e^{v_2/t} = 1 - \frac{\pi}{t}\left(\frac{t}{\pi}\right)^{4/5} > \tfrac{1}{2} ,$$

and the function of x : $y(x) = x^{1-\sigma}/(x+2k)(x+2k+1)$ $(x > 0)$ takes its maximum at the point

$$x_0 = \frac{1}{2(1+\sigma)} \left\{\sqrt{(4k+1)^2-\left(1-\sigma^2\right)} + \sigma(4k+1)\right\} \cong 2k + \tfrac{1}{2} .$$

Therefore,

$$\begin{aligned} S_{112} &= O\left(\sum_{k=1}^{(t/\pi)^{4/5}} \frac{y(x_0)}{k}\right) + O\left(\sum_{k=(t/\pi)^{4/5}+1}^{\infty} \frac{y\left(e^{v_2/t}\right)}{k}\right) \\ &= O\left(\sum_{k=1}^{(t/\pi)^{4/5}} \frac{1}{k} \cdot \frac{1}{k^{1+\sigma}}\right) + O\left(\sum_{k=(t/\pi)^{4/5}}^{\infty} \frac{t^{(1-\sigma)4/5}}{k^3}\right) \\ &= O(1) , \end{aligned}$$

and

$$\begin{aligned} S_{113} &= O\left(\sum_{k=1}^{\infty} \frac{1}{k} \frac{t^{(1-\sigma)4/5}}{\left(t^{4/5}+k\right)^2} v_3\right) \\ &= O\left(\frac{\log t}{t^{(4\sigma-1)/5}} \sum_{k=1}^{t^{4/5}} \frac{1}{k}\right) + O\left(t^{(9-4\sigma)/5} \log t \sum_{k=t^{4/5}+1}^{\infty} \frac{1}{k^3}\right) \\ &= O(1) . \end{aligned}$$

Using the transformation

$$w = \pi e^{v/t} - v , \quad dw = \left((\pi/t)e^{v/t}-1\right)dv$$

for the integral of S_{114}, we get

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$$S_{114} = O\left(\sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{(1-\sigma)v_3/t}}{\left(e^{v_3/t}+k\right)^2} \frac{1}{\log(t/\pi)}\right) = O(1) .$$

Thus we have proved that $S_{11} = O(1)$. We shall now estimate

$$S_{12} = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \int_0^{\infty} \frac{e^{(1-\sigma)v/t}\cos\left(\pi e^{v/t}-v\right)}{e^{v/t}+2k+1}\, dv$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)} \left(\int_0^{v_1} + \int_{v_1}^{v_2} + \int_{v_2}^{t\log(t/\pi)-\sqrt{t}} + \int_{t\log(t/\pi)-\sqrt{t}}^{t\log(t/\pi)} + \int_{t\log(t/\pi)}^{t\log(t/\pi)+\sqrt{t}} + \int_{t\log(t/\pi)+\sqrt{t}}^{v_3} + \int_{v_3}^{\infty}\right) dv$$

$$= \sum_{i=1}^{7} S_{12i} .$$

We have

$$S_{121} = O\left(\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{k}\right) = O(1) .$$

Since the function $y(x) = x^{1-\sigma}/(x+2k+1)$ takes its maximum at the point $x = (1-\sigma)/(2k+1)/\sigma$,

$$S_{122} = O\left(\sum_{k=1}^{L} \frac{1}{k^{2+\sigma}}\right) + O\left(\sum_{k=L+1}^{\infty} \frac{t^{(1-\sigma)4/5}}{k^3}\right) = O(1) ,$$

where $L = \frac{\sigma}{2(1-\sigma)} \left(\frac{t}{\pi}\right)^{4/5} - \frac{1}{2}$, using the transformation $w = v - \pi e^{v/t}$

For the estimation of S_{123} , we use the transformation

$$w = w(v) = v - \pi e^{v/t} ,$$

$$dw = \left(1-(\pi/t)e^{v/t}\right)dv .$$

By $v = v(w)$ we denote the inverse function of $w = w(v)$ in the interval $\left(v_2,\ t\log(t/\pi)-\sqrt{t}\right)$. Then the integral of the kth term of S_{123} becomes

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$$S_{123} = \int_{v_2}^{v_2'} \frac{e^{(1-\sigma)v/t}\cos\left(v-\pi e^{v/t}\right)}{e^{v/t}+2k+1}\,dv$$

$$= \int_{w(v_2)}^{w(v_2')} \frac{e^{(1-\sigma)v(w)/t}\cos w}{\left(e^{v(w)/t}+2k+1\right)\left(1-(\pi/t)e^{v(w)/t}\right)}\,dw ,$$

where $v_2' = t\log(t/\pi) - \sqrt{t}$. We consider the function of x :

$$y = y(x) = x^{1-\sigma}/(x+2k+1)(1-\pi x/t) ,$$

whose logarithmic differential coefficient is

(3) $$\frac{y'}{y} = \frac{1-\sigma}{x} - \frac{1}{x+2k+1} + \frac{1}{(t/\pi)-x}$$

$$= \frac{(1+\sigma)x^2+\sigma(2k+1-t/\pi)x+(1-\sigma)(2k+1)t/\pi}{x(x+2k+1)\left((t/\pi)-x\right)} .$$

If $2k+1 > t/\pi$, then $y' > 0$. In the case $t/\pi \le 2k+1$, the discriminant of the number on the right side is

$$\sigma^2(2k+1-t/\pi)^2 - 4(1-\sigma^2)(2k+1)t/\pi$$

$$= \sigma^2\left\{\frac{t}{\pi} - \left(\frac{1-\sqrt{1-\sigma^2}}{\sigma}\right)^2(2k+1)\right\}\left\{\frac{t}{\pi} - \left(\frac{1+\sqrt{1-\sigma^2}}{\sigma}\right)^2(2k+1)\right\} ,$$

which is negative for

$$\frac{t}{\pi} \ge 2k+1 > \left(\frac{1+\sqrt{1-\sigma^2}}{\sigma}\right)^{-2}\frac{t}{\pi} = \left(\frac{1-\sqrt{1-\sigma^2}}{\sigma}\right)^2\frac{t}{\pi} .$$

Therefore, if

$$k > \tfrac{1}{2}\left(\frac{1-\sqrt{1-\sigma^2}}{\sigma}\right)^2\frac{t}{\pi} - \tfrac{1}{2} = k_0 ,$$

then the function $y\left(e^{v(w)/t}\right)$ increases in the interval $\left(w(v_2), w(v_2')\right)$ and

$$S_{123}(k) = O\left(\frac{e^{(1-\sigma)v_2'/t}}{\left(e^{v_2'/t}+2k+1\right)\left(1-(\pi/t)e^{v_2'/t}\right)}\right)$$

$$= O\left(\frac{t^{1-\sigma}e^{-(1-\sigma)/\sqrt{t}}}{\left(te^{-1/\sqrt{t}}+2k+1\right)\left(1-e^{-1/\sqrt{t}}\right)}\right)$$

$$= O\left(\frac{t^{1-\sigma}\sqrt{t}}{t}\right) = O(1) ,$$

and then

$$\sum_{k=k_0+1}^{\infty} \frac{1}{2k(2k+1)} S_{123}(k) = O(1) .$$

In the case $k < k_0$, the numerator of the right side of (3) has two roots x' and x'' $(x' < x'')$ and

$$x'' = \frac{1}{2(1+\sigma)} \left\{\sigma\left(\frac{t}{\pi} - (2k+1)\right) + \sqrt{\sigma^2\left(\frac{t}{\pi} - (2k+1)\right)^2 - 4\left(1-\sigma^2\right)(2k+1)\frac{t}{\pi}}\right\}$$

$$\leq \frac{\sigma}{1+\sigma}\left(\frac{t}{\pi} - (2k+1)\right) .$$

Writing $x'' = e^{v''/t}$,

$$v'' \leq t \log\left(\frac{t}{\pi} - (2k+1)\right) - t \log \frac{1+\sigma}{\sigma} \leq v_2' ,$$

$$x' = \frac{1}{2(1+\sigma)} \left\{\sigma\left(\frac{t}{\pi} - (2k+1)\right) - \sqrt{\sigma^2\left(\frac{t}{\pi} - (2k+1)\right)^2 - 4\left(1-\sigma^2\right)(2k+1)\frac{t}{\pi}}\right\}$$

$$\cong \frac{(1-\sigma)(2k+1)t/\pi}{\sigma\left(t/\pi-(2k+1)\right)} .$$

If $k > (2\sigma-1)/2(1-\sigma)$, then $x' > 1$, and then, writing $x' = e^{v'/t}$,

$$S_{123}(k) = O\left(\frac{e^{(1-\sigma)v'/t}}{\left(e^{v'/t}+2k+1\right)\left(1-(\pi/t)e^{v'/t}\right)}\right) + O\left(\frac{e^{(1-\sigma)v_2'/t}}{\left(e^{v_2'/t}+2k+1\right)\left(1-(\pi/t)e^{v_2'/t}\right)}\right)$$

$$= O(1) ,$$

and then

$$\sum_{k=(2\sigma-1)/2(1-\sigma)+1}^{k_0} \frac{1}{2k(2k+1)} S_{123}(k) = O(1) .$$

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If $1 \le k \le (2\sigma-1)/2(1-\sigma)$, then $v' \le 0$. Similarly as above $S_{123}(k) = O(1)$ for such k . Thus we have proved that S_{123} is bounded.

The function $y(v) = e^{(1-\sigma)v/t}/(e^{v/t}+2k+1)$ takes a maximum at $v = t\log((1-\sigma)(2k+1)/\sigma)$ and its maximum value is $O(1/k^{\sigma})$. Writing

$$k_2 = \frac{\sigma t}{2(1-\sigma)\pi} - \tfrac{1}{2}$$

and

$$k_1 = \frac{t}{2(1-\sigma)\pi e^{1/\sqrt{t}}} - \tfrac{1}{2} \cong k_2 - \frac{\sigma\sqrt{t}}{2(1-\sigma)\pi} ,$$

we get

$$\begin{aligned} S_{124} &= O\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\sqrt{t} \max_{v_2'\le v\le t\log(t/\pi)} y(v)\right) \\ &= O\left(\sum_{k=1}^{k_1} \frac{1}{k^2}\frac{\sqrt{t}t^{1-\sigma}}{t} + \sum_{k=k_1+1}^{k_2} \frac{1}{k^2}\frac{\sqrt{t}}{k^{\sigma}} + \sum_{k=k_2+1}^{\infty} \frac{1}{k^2}\frac{\sqrt{t}t^{1-\sigma}}{t}\right) \\ &= O(1) . \end{aligned}$$

S_{125} and S_{126} become bounded by similar estimations. Putting $w = w(v) = \pi e^{v/t} - v$ and denoting by $v = v(w)$ the solution of the equation in the interval (v_3, ∞) , the integral of the kth term of S_{127} is

$$\begin{aligned} \int_{v_3}^{\infty} \frac{e^{(1-\sigma)v/t}}{e^{v/t}+2k+1} \cos\left(\pi e^{v/t}-v\right) dv &= \pi^{\sigma} \int_0^{\infty} \frac{(w+v(w))^{1-\sigma}\cos w\,dw}{(w+v(w)+(2k+1)\pi)\left((\pi/t)e^{v(w)/t}-1\right)} \\ &= O\left(\int_0^{2\pi} \frac{dw}{(w+v(w))^{\sigma}\log(t/\pi)}\right) = O(1) , \end{aligned}$$

which shows that S_{127} is also bounded.

Summing up the above estimate, we see that S_{12} is bounded. Since S_{13} is also bounded by an estimation similar to S_{12} , S_1 is bounded. Combining with the estimation of S_2 , we get

$$Q_{212} = O(1) .$$

Combining the estimation of §5.1 and §5.2, we get

$$Q_{21} = O(1) .$$

Therefore it remains to estimate Q_{22} .

6. Estimation of Q_{22}

$$Q_{22} = \sum_{k=1}^{\infty} \frac{t}{2\pi k} \left\{ \int_1^k \frac{\cos(t\log u)}{u^\sigma} du \int_{(k-u)/2}^{\infty} \frac{\sin 2\pi m}{m} dm - \int_k^{\infty} \frac{\cos(t\log u)}{u^\sigma} du \int_{(u-k)/2}^{\infty} \frac{\sin 2\pi m}{m} dm \right\}$$

$$= \sum_{k=2}^{\infty} \frac{t}{2\pi k} \left\{ \int_0^{k-1} \frac{\cos\{t\log(k-u)\}}{(k-u)^\sigma} du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m} dm - \int_0^{\infty} \frac{\cos\{t\log(k+u)\}}{(k+u)^\sigma} du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m} dm \right\}$$

$$- \frac{t}{2\pi} \int_1^{\infty} \frac{\cos(t\log u)}{u^\sigma} du \int_{(u-1)/2}^{\infty} \frac{\sin 2\pi m}{m} dm$$

$$= \sum_{k=2}^{\infty} \frac{t}{2\pi k} \int_0^K \left\{ \frac{\cos\{t\log(k-u)\}}{(k-u)^\sigma} - \frac{\cos\{t\log(k+u)\}}{(k+u)^\sigma} \right\} du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m} dm$$

$$+ \sum_{k=2}^{\infty} \frac{t}{2\pi k} \int_K^{k-1} \frac{\cos\{t\log(k-u)\}}{(k-u)^\sigma} du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m} dm$$

$$- \sum_{k=2}^{\infty} \frac{t}{2\pi k} \int_K^{\infty} \frac{\cos\{t\log(k+u)\}}{(k+u)^\sigma} du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m} dm$$

$$- \frac{t}{2\pi} \int_1^{\infty} \frac{\cos(t\log u)}{u^\sigma} du \int_{(u-1)/2}^{\infty} \frac{\sin 2\pi m}{m} dm$$

$$= \frac{1}{2\pi} (Q_{221} + Q_{222} - Q_{223} - Q_{224}) ,$$

where K ($< k-1$) will be determined later.

By the transformations $t \log(k-u) = v$ and $t \log(k+u) = v$, we get

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$$Q_{221} =$$

$$= \sum_{k=2}^{\infty} \frac{t}{k} \int_0^K \left\{ \frac{\cos\left(t\log(k-u)\right)}{(k-u)^{\sigma}} - \frac{\cos\left(t\log(k+u)\right)}{(k+u)^{\sigma}} \right\} du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$= \sum_{k=2}^{\infty} \frac{1}{k} \left\{ \int_{t\log(k-u)}^{t\log k} e^{(1-\sigma)v/t} \cos v dv \int_{(k-e^{v/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right.$$

$$\left. - \int_{t\log k}^{t\log(k+K)} e^{(1-\sigma)v/t} \cos v dv \int_{(e^{v/t}-k)/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right\}$$

$$= \sum_{k=2}^{\infty} \frac{1}{k} \left\{ \int_0^{-t\log(1-K/k)} e^{-(1-\sigma)w/t} \cos(t \log k-w) dw \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right.$$

$$\left. - \int_0^{t\log(1+K/k)} e^{(1-\sigma)w/t} \cos(t \log k+w) dw \int_{k(e^{w/t}-1)/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right\}$$

$$= \sum_{k=2}^{\infty} \frac{1}{k^{\sigma}} \left\{ \int_0^{t\log(1+K/k)} \left[e^{-(1-\sigma)w/t} \cos(t \log k-w) \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right. \right.$$

$$\left. - e^{(1-\sigma)w/t} \cos(t \log k+w) \int_{k(e^{w/t}-1)/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right] dw$$

$$\left. + \int_{t\log(1+K/k)}^{-t\log(1-K/k)} e^{-(1-\sigma)w/t} \cos(t \log k-w) dw \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right\}$$

$$= \sum_{k=2}^{\infty} \frac{1}{k^{\sigma}} \left\{ \int_0^{t\log(1+K/k)} \left(\cos(t \log k-w) - \cos(t \log k+w)\right) e^{-(1-\sigma)w/t} dw \right.$$

$$\cdot \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm + \int_0^{t\log(1+K/k)} \cos(t \log k+w)$$

$$\cdot \left[e^{-(1-\sigma)w/t} \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm - e^{(1-\sigma)w/t} \int_{k(e^{w/t}-1)/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right] dw$$

$$\left. + \int_{t\log(1+K/k)}^{-t\log(1-K/k)} \cos(t \log k-w) e^{-(1-\sigma)w/t} dw \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm \right\}$$

$$= 2T_1 + T_2 + T_3 .$$

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7. Estimation of T_1

$$T_1 = \sum_{k=2}^{\infty} \frac{\sin(t\log k)}{k^\sigma} \int_0^{t\log(1+K/k)} e^{-(1-\sigma)w/t} \sin w\,dw \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\,dm$$

$$= \sum_{k=2}^{\infty} \frac{\sin(t\log k)}{k^\sigma}$$

$$\cdot \left\{\int_0^{K/2(1+K/k)} \frac{\sin 2\pi m}{m}\,dm \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t} \sin w\,dw\right.$$

$$\left. + \int_{K/2(1+K/k)}^{\infty} \frac{\sin 2\pi m}{m}\,dm \int_0^{t\log(1+K/k)} e^{-(1-\sigma)w/t} \sin w\,dw\right\}$$

$$= T_{11} + T_{12} ,$$

$$T_{11} = \int_{3/2}^{\infty} \frac{\sin(t\log k)}{k^\sigma} \left(dk + dJ(k)\right)$$

$$\cdot \int_0^{K/2(1+K/k)} \frac{\sin 2\pi m}{m}\,dm \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t} \sin w\,dw$$

$$= T_{111} + T_{112} .$$

7.1. Estimation of T_{111} . We take

$$K = k^{\sigma-\varepsilon} \quad (0 < \varepsilon < \sigma) ;$$

then

$$T_{111} = \int_0^{c_0} \frac{\sin 2\pi m}{m}\,dm \int_{3/2}^{\infty} \frac{\sin(t\log k)}{k^\sigma}\,dk \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t} \sin w\,dw$$

$$+ \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\,dm \int_{M(m)}^{\infty} \frac{\sin(t\log k)}{k^\sigma}\,dk \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t} \sin w\,dw$$

$$= U_1 + U_2 ,$$

where $c_0 = (3/2)^{\sigma-\varepsilon}/2\left(1+(3/2)^{\sigma-\varepsilon-1}\right)$ and $M = M(m)$ is the solution of the equation $m = k^{\sigma-\varepsilon}/2\left(1+k^{\sigma-\varepsilon-1}\right)$; then $M \cong (2m)^{1/(\sigma-\varepsilon)}$. We shall first estimate U_1 . By the formula

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$$(1+a^2)\int_0^b e^{aw}\sin w\,dw = 1 - e^{ab}\cos b + ae^{ab}\sin b$$

with $a = -(1-\sigma)/t$ and $b = -t\log(1-2m/k)$, the inner double integral of U_1 becomes

$$\int_{3/2}^{\infty}\frac{\sin(t\log k)}{k^\sigma}dk\int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t}\sin w\,dw$$

$$= \left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1}\int_{3/2}^{\infty}\frac{\sin(t\log k)}{k^\sigma}\left\{\left(1-\left(1-\frac{2m}{k}\right)^{1-\sigma}\cos\left(t\log\left(1-\frac{2m}{k}\right)\right)\right)\right.$$

$$\left. - \frac{1-\sigma}{t}\left(1-\frac{2m}{k}\right)^{1-\sigma}\sin\left(t\log\left(1-\frac{2m}{k}\right)\right)\right\}dk$$

$$= \left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1}\left(U_{11}-(1-\sigma)U_{12}\right) .$$

Now

$$U_{11} = \int_{3/2}^{\infty}\frac{\sin(t\log k)}{k^\sigma}\left(1-\cos\left(t\log(1-2m/k)\right)\right)dk + O(1)$$

$$= \int_{3/2}^{t^{2/(1+\sigma)}}\frac{\sin(t\log k)}{k^\sigma}\left(1-\cos\left(t\log(1-2m/k)\right)\right)dk + O(1)$$

$$= \tfrac{1}{2}\int_{3/2}^{t^{2/(1+\sigma)}}\{2\sin(t\log k) - \sin\left(t\log(k-2m)\right)$$

$$- \sin\left(t\log\left(t\log\left(k^2/(k-2m)\right)\right)\right)\}\frac{dk}{k^\sigma} + O(1)$$

$$= U_{111} - U_{112} - U_{113} ,$$

where

$$U_{111} = \frac{1}{t}\int_{t\log(3/2)}^{(2/(1+\sigma))t\log t} e^{(1-\sigma)k/t}\sin k\,dk$$

$$= O\left(\frac{1}{t}\,t^{2(1-\sigma)/(1+\sigma)}\right) = O(1) ,$$

and similarly U_{112} and U_{113} are bounded. Therefore $U_{11} = O(1)$. Since U_{12} is also bounded, we have proved that U_1 is bounded.

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Now, the inner double integral of U_2 is

$$U_2(m) =$$

$$= \int_{M(m)}^{\infty} \frac{\sin(t\log k)}{k^\sigma}\,dk \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t}\sin w\,dw$$

$$= \left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1} \int_{M(m)}^{\infty} \frac{\sin(t\log k)}{k^\sigma}\left\{\left(1-\left(1-\frac{2m}{k}\right)^{1-\sigma}\cos\left(t\log\left(1-\frac{2m}{k}\right)\right)\right)\right.$$

$$\left. - \frac{1-\sigma}{t}\left(1-\frac{2m}{k}\right)^{1-\sigma}\sin\left(t\log\left(1-\frac{2m}{k}\right)\right)\right\}dk$$

$$= \left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1}\left(U_{21}(m)-(1-\sigma)U_{22}(m)\right) ,$$

where

$$U_{21}(m) = \int_{M(m)}^{\infty}\frac{\sin(t\log k)}{k^\sigma}\left\{1-\cos\left(t\log\left(1-\frac{2m}{k}\right)\right)\right\}dk + O\left(\frac{1}{m^{\varepsilon/(\sigma-\varepsilon)}}\right)$$

$$= O\left(\frac{t^2m^2}{m^{(1+\sigma)/(\sigma-\varepsilon)}}\right) + O\left(\frac{1}{m^{\varepsilon/(\sigma-\varepsilon)}}\right)$$

and

$$U_{22}(m) = O\left(\frac{m}{m^{\sigma/(\sigma-\varepsilon)}}\right) = O\left(\frac{1}{m^{\varepsilon/(\sigma-\varepsilon)}}\right) .$$

Therefore, writing $a = 2(\sigma-\varepsilon)/(1-\sigma+2\varepsilon)$,

$$U_2 = \left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1}\int_{c_0}^{t^a}\frac{\sin 2\pi m}{m}\,dm$$

$$\cdot \int_{M(m)}^{\infty}\frac{\sin(t\log k)}{k^\sigma}\left\{1-\cos\left(t\log\left(1-\frac{2m}{k}\right)\right)\right\}dk + O(1)$$

$$= \tfrac{1}{2}\left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1}\int_{c_0}^{t^a}\frac{\sin 2\pi m}{m}\,dm\int_{M(m)}^{(mt)^{1/\sigma}}\left\{\frac{\sin(t\log k)-\sin\left(t\log(k-2m)\right)}{k^\sigma}\right.$$

$$\left. + \frac{\sin(t\log k)-\sin\left(t\log\left(k^2/(k-2m)\right)\right)}{k^\sigma}\right\}dk + O(\log t)$$

$$= \tfrac{1}{2}\left(1+\frac{(1-\sigma)^2}{t^2}\right)^{-1}\left(U_{21}+U_{22}\right) + O(\log t) ;$$

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$$U_{21} = \frac{1}{t}\int_{c_0}^{t^a} \frac{\sin 2\pi m}{m}\,dm \left\{\int_{t\log M(m)}^{(1/\sigma)t\log(mt)} e^{(1-\sigma)k/t}\sin k\,dk - \int_{t\log(M(m)-2m)}^{t\log((mt)^{1/\sigma}-2m)} \frac{e^{k/t}}{(e^{k/t}+2m)^{\sigma}}\sin k\,dk\right\}$$

$$= \frac{1}{t}\int_{c_0}^{t^a} \frac{\sin 2\pi m}{m}\,dm \left\{\int_{t\log M(m)}^{(1/\sigma)t\log(mt)} \left(\frac{1}{e^{\sigma k/t}} - \frac{1}{(e^{i/t}+2m)^{\sigma}}\right)e^{i/t}\sin k\,dk + \int_{t\log((mt)^{1/\sigma}-2m)}^{(1/\sigma)t\log(mt)} \frac{e^{k/t}}{(e^{k/t}+2m)^{\sigma}}\sin k\,dk - \int_{t\log(M(m)-2m)}^{t\log M(m)} \frac{e^{k/t}}{(e^{k/t}+2m)^{\sigma}}\sin k\,dk\right\}$$

$$= U_{211} + U_{212} - U_{213}\ .$$

Since

$$x^{1-\sigma} - \frac{x}{(x+m)^{\sigma}} = -\sum_{j=1}^{\infty}\binom{-\sigma}{j}\frac{m^j}{x^{j-1+\sigma}}\ ,$$

we get, writing $x = e^{k/t}$,

$$U_{211} = O\left(\frac{1}{t}\int_{c_0}^{t^a}\left(\sum_{j=1}^{\infty}\left|\binom{-\sigma}{j}\right|\frac{m^j}{M(m)^{j-1+\sigma}}\right)\frac{dm}{m}\right)$$

$$= O\left(\frac{1}{t}\int_{c_0}^{t^a}\frac{dm}{m^{\sigma/(\sigma-\varepsilon)}} + \frac{1}{t}\sum_{j=2}^{\infty}\left|\binom{-\sigma}{j}\right|\int_{c_0}^{t^a}\frac{dm}{m^{(j-1)(1-\sigma+\varepsilon)/(\sigma-\varepsilon)+\sigma/(\sigma-\varepsilon)}}\right)$$

$$= O(1)\ .$$

Since the function $y = x/(x+2m)^{\sigma}$ is increasing,

$$U_{212} = O\left(\frac{1}{t}\int_{c_0}^{t^a}\frac{1}{m}(mt)^{(1-\sigma)/\sigma}\frac{1}{(mt)^{(1-\sigma)/\sigma}}\,dm\right) = O(1)$$

and

$$U_{213} = O\left(\frac{1}{t}\int_{c_0}^{t^a}\frac{1}{m}M(m)^{1-\sigma}\frac{dm}{M(m)}\,dm\right) = O\left(\int_{c_0}^{t^a}\frac{dm}{m^{\sigma/(\sigma-\varepsilon)}}\right) = O(1)\ .$$

Thus we have proved that U_{21} is bounded. Similarly U_{22} is bounded, and then

$$U_2 = O(\log t) .$$

Therefore, T_{111} is also of order $O(\log t)$.

7.2. Estimation of T_{112} .

$$T_{112} = \int_{3/2}^{\infty} \frac{\sin(t\log k)}{k^{\sigma}}\, dJ(k) \cdot \int_0^{K/2(1+K/k)} \frac{\sin 2\pi m}{m}\, dm \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t} \sin w\, dw .$$

We use the formula: if f is absolutely continuous and f' is of bounded variation in the range of integration, then

$$\int f(k)dJ(k) = -\int f'(k)J(k)dk$$

$$= -\sum_{j=1}^{\infty} \frac{1}{\pi j} \int f'(k) \sin 2\pi jk dk = 2 \sum_{j=1}^{\infty} \int f(k) \cos 2\pi jk dk ,$$

where both limits of integration are half of odd integers. Then

$$T_{112} = 2\sum_{j=1}^{\infty} \int_{3/2}^{\infty} \frac{\sin(t\log k)\cos 2\pi jk}{k^{\sigma}}\, dk \cdot \int_0^{K/2(1+K/k)} \frac{\sin 2\pi m}{m}\, dm \int_0^{-t\log(1-2m/k)} e^{-(1-\sigma)w/t} \sin w\, dw$$

$$= 2\left[1 - \frac{(1-\sigma)^2}{t^2}\right]^{-1} \sum_{j=1}^{\infty} \int_{3/2}^{\infty} \frac{\sin(t\log k)\cos 2\pi jk}{k^{\sigma}}\, dk \cdot \int_0^{K/2(1+K/k)} \frac{\sin 2\pi m}{m} \left\{\left[1 - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \cos\left(t \log\left(1 - \frac{2m}{k}\right)\right)\right] - \frac{1-\sigma}{t}\left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log\left(1 - \frac{2m}{k}\right)\right)\right\} dm$$

$$= 2\left[1 - \frac{(1-\sigma)^2}{t^2}\right]^{-1} \left(V_1 - (1-\sigma)V_2\right) ;$$

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$$V_1 = \sum_{j=1}^{\infty} \int_0^{c_0} \frac{\sin 2\pi m}{m}\, dm$$

$$\cdot \int_{3/2}^{\infty} \frac{\sin(t\log k)\cos 2\pi jk}{k^{\sigma}} \left\{1 - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \cos\left(t \log\left(1 - \frac{2m}{k}\right)\right)\right\} dk$$

$$+ \sum_{j=1}^{\infty} \int_{c_0}^{\infty} dm \int_{M(m)}^{\infty} dk$$

$$= V_{11} + V_{12} \,.$$

7.2.1. Estimation of V_{11} .

$$V_{11} = \tfrac{1}{4} \sum_{j=1}^{\infty} \int_0^{c_0} \frac{\sin 2\pi m}{m}\, dm$$

$$\cdot \int_{3/2}^{\infty} \left\{2 \sin(t \log k + 2\pi jk) - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log(k-2m) + 2\pi jk\right)\right.$$

$$- \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log\left(k^2/(k-2m)\right) + 2\pi jk\right)$$

$$+ 2 \sin(t \log k - 2\pi jk) - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log(k-2m) - 2\pi jk\right)$$

$$\left.- \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log\left(k^2/(k-2m)\right) - 2\pi jk\right)\right\} \frac{dk}{k^{\sigma}}$$

$$= \tfrac{1}{4} \int_0^{c_0} \frac{\sin 2\pi m}{m} \left(2V_{111} - V_{112} - V_{113} + 2V_{114} - V_{115} - V_{116}\right) dm \,.$$

We use the transformation $p = t \log k + 2\pi jk$ and denote by $k(p)$ the solution of k of this equation for fixed p . Then

$$V_{111} = \sum_{j=1}^{\infty} \int_{3\pi j + t\log 3/2}^{\infty} \frac{\sin p\, dp}{k(p)^{\sigma}\left(2\pi j + t/k(p)\right)}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}\cos(t\log 3/2)}{(3/2)^{\sigma}(2\pi j + 2t/3)} + \sum_{j=1}^{\infty} \int_{3\pi j + t\log 3/2}^{\infty} \frac{(1-\sigma)k\ (p)\cos p}{k(p)^{\sigma}\left(2\pi jk(p)+t\right)}\, dp$$

$$+ \sum_{j=1}^{\infty} \int_{3\pi j + t\log 3/2}^{\infty} \frac{2\pi jk\ (p)k(p)^{1-\sigma}\cos p}{\left(2\pi jk(p)+t\right)^2}\, dp$$

$$= O(1) \;,$$

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since $k'(p) = 1/\frac{dp}{dk} = 1/\left(2\pi j + t/k(p)\right)$. Similarly V_{112} and V_{113} are also bounded.

For the estimation of V_{114} , we use the transformation

(4) $$p = |t \log k - 2\pi jk| .$$

The curves

(5) $$y = t \log k \quad \text{and} \quad y = 2\pi jk$$

touch each other at the point $k = e$ for $j = t/2\pi e$. The lines $y = 2\pi jk$ $(j > t/2\pi e)$ do not intersect the curve $y = t \log k$, but the lines $y = 2\pi jk$ $(j < t/2\pi e)$ intersect the curve $y = t \log k$. If $j < \frac{t}{3\pi} \log 3/2$, then they intersect at only one point k_j in the range $(3/2, \infty)$ such that

$$\frac{k_j}{\log k_j} = \frac{t}{2\pi j} ;$$

that is,

$$k_j = \frac{t}{2\pi j} \log\left(\frac{t}{2\pi j} \log \frac{t}{2\pi j}\right) + O\left(\frac{\log\log(t/2\pi j)}{\left(\log(t/2\pi j)\right)^2}\right) .$$

Further, the curve (4) takes its maximum at the point $k = t/2\pi j$. If $\frac{t}{3\pi} \log 3/2 \le j < \frac{t}{2\pi e}$, then the curves (5) intersect at two points in the range $(3/2, \infty)$, one being less than e and the other greater than e . If $t/2\pi e \le j < t/3\pi$, then the curve (4) becomes

(6) $$p = 2\pi jk - t \log k$$

and takes a minimum between $3/2$ and e . If $j \ge t/3\pi$, then the curve (6) increases; that is, $2\pi j - t/k > 0$. We write

$$V_{114} = \left\{\sum_{j=1}^{(t/3\pi)\log(3/2)} + \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi} + \sum_{j=t/3\pi+1}^{\infty}\right\} \cdot \int_{3/2}^{\infty} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma} dk$$

$$= W_1 + W_2 + W_3 .$$

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Using the transformation, (6) and denoting by $k(p)$ the solution of the equation (6) for fixed p,

$$W_3 = -\sum_{j=t/3\pi+1}^{\infty}\int_{3/2}^{\infty}\frac{\sin(2\pi jk-t\log k)}{k^{\sigma}}\,dk$$

$$= -\sum_{j=t/3\pi+1}^{\infty}\int_{3\pi j-t\log 3/2}^{\infty}\frac{\sin p\,dp}{k(p)^{\sigma}\left(2\pi j-t/k(p)\right)}$$

$$= \sum_{j=t/3\pi+1}^{\infty}\frac{(-1)^{j}\cos(t\log 3/2)}{(3/2)^{\sigma}(2\pi j-2t/3)} - \sum_{j=t/3\pi+1}^{\infty}\int_{3\pi j-t\log 3/2}^{\infty}\frac{(1-\sigma)k'(p)\cos p}{k(p)^{\sigma}\left(2\pi jk(p)-t\right)}\,dp$$

$$- \sum_{j=t/3\pi+1}^{\infty}\int_{3\pi j-t\log 3/2}^{\infty}\frac{2\pi jk'(p)k(p)^{1-\sigma}\cos p}{\left(2\pi jk(p)-t\right)^{2}}\,dp$$

$$= O(1),$$

since $k'(p) = 1/\left(2\pi j-t/k(p)\right)$.

$$W_1 = \sum_{j=1}^{(t/3\pi)\log(3/2)}\left[\int_{3/2}^{k_j} + \int_{k_j}^{\infty}\right]\frac{\sin(t\log k-2\pi jk)}{k^{\sigma}}\,dk$$

$$= W_{11} + W_{12}.$$

By the transformation (6),

$$W_{12} = \sum_{j=1}^{(t/3\pi)\log(3/2)}\int_{2\pi jk_j-t\log k_j}^{\infty}\frac{\sin p\,dp}{k(p)^{\sigma}\left(2\pi j-t/k(p)\right)}$$

$$= O\left(\sum_{j=1}^{(t/3\pi)\log(3/2)}\frac{1}{(t/j)^{\sigma}j}\right) = O(1).$$

Now we shall estimate

$$W_2 = \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi}\int_{3/2}^{\infty}\frac{\sin(t\log k-2\pi jk)}{k^{\sigma}}\,dk$$

$$= \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi}\left[\int_{3/2}^{10} + \int_{10}^{\infty}\right]dk$$

$$= W_{21} + W_{22}.$$

For $(t/3\pi) < j < (t/3\pi)\log(3/2)$, we have

$$(3/2)/\log(3/2) < (t/2\pi j) < (3/2)$$

and then $k_j < 10$. By the transformation $p = 2\pi jk - t\log k$,

$$W_{22} = \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi} \int_{20j\pi - t\log 10}^{\infty} \frac{\sin p\,dp}{k(p)^{\sigma}\left(2\pi j - t/k(p)\right)}$$

$$= O\left(\sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi} \frac{1}{j}\right) = O(1)$$

and

$$W_{21} = \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \left(dj + dJ(j)\right) \int_{3/2}^{10} \frac{\sin(t\log k - 2jk)}{k^{\sigma}}\,dk$$

$$= W_{211} + W_{212} ,$$

where

$$W_{211} = \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} dj \int_{3/2}^{10} \frac{\sin(t\log k)\cos 2\pi jk - \cos(t\log k)\sin 2\pi jk}{k^{\sigma}}\,dk$$

$$= \int_{3/2}^{10} \frac{\sin(t\log k)}{k^{\sigma}}\,dk \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \cos 2\pi jk\,dj$$

$$- \int_{3/2}^{10} \frac{\cos(t\log k)}{k^{\sigma}}\,dk \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \sin 2\pi jk\,dj$$

$$= O(1) ,$$

and similarly $W_{212} = O(\log t)$. Thus we have proved that $W_2 = O(\log t)$.

Now we shall estimate V_{115} :

$$V_{115} = \sum_{j=1}^{\infty} \int_{3/2}^{\infty} \left(1 - \frac{2m}{k}\right)^{1-\sigma} \frac{\sin\left(t\log(k-2m) - 2\pi jk\right)}{k^{\sigma}}\,dk$$

$$= \left(\sum_{j=1}^{t/3\pi} + \sum_{j=t/3\pi+1}^{\infty}\right) \int_{3/2}^{\infty} dk$$

$$= W_4 + W_5 ,$$

$$W_5 = \sum_{j=t/3\pi+1}^{t/(3-4c_0)\pi} \left(\int_{3/2}^{3/2+2m} + \int_{3/2+2m}^{\infty}\right) dk + \sum_{j=t/(3-4c_0)\pi+1}^{\infty} \int_{3/2}^{\infty} dk$$

$$= W_{51} + W_{52} + W_{53} .$$

For W_{51} , change the order of summation and integration and use the order

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of the integral of Dirichlet and conjugate Dirichlet kernels; then we can easily see that it is $O(\log t)$. For W_{52} and W_{53} , we use the transformation $p = 2\pi jk - t\log(k-2m)$, then $dp/dk > 0$ and the method used for V_{111} can be applied to them. Thus we get $W_5 = O(\log t)$. Now

$$
\begin{aligned}
W_4 &= \sum_{j=1}^{t/3\pi} \int_{3/2}^{\infty} \left(1 - \frac{2m}{k}\right)^{1-\sigma} \frac{\sin(t\log(k-2m)-2\pi jk)}{k^{\sigma}}\, dk \\
&= \sum_{j=1}^{t/3\pi} \left(\int_{3/2}^{t/2\pi j+2m} + \int_{t/2\pi j+2m}^{k_j'} + \int_{k_j'}^{\infty}\right) dk \\
&= W_{41} + W_{42} + W_{43} ,
\end{aligned}
$$

where k_j' is the solution of the equation $k = \frac{t}{2\pi j}\log(k-2m)$, greater than $t/2\pi_j + 2m$; that is,

$$
k_j' = \frac{t}{2\pi j}\log\left(\frac{t}{2\pi j}\log\frac{t}{2\pi j}\right) - \frac{2m}{\log(t/2\pi j)} + O\left(\frac{1}{(\log(t/2\pi j))^2}\right) ;
$$

$$
W_{41} = \sum_{j=1}^{t/3\pi} \int_{3/2-2m}^{t/2\pi j} \left(1 - \frac{2m}{k+2m}\right)^{1-\sigma} \frac{\sin(t\log k - 2\pi j(k+2m))}{(k+2m)^{\sigma}}\, dk ;
$$

and then

$$
\begin{aligned}
&\int_0^{c_0} \frac{\sin 2\pi m}{m} W_{41}\, dm \\
&\quad = \sum_{j=1}^{t/3\pi} \left\{ \int_0^{c_0} \frac{\sin 2\pi m}{m} \cos 4\pi jm\,dm \int_{3/2-m}^{t/2\pi j} \left(1 - \frac{2m}{k+2m}\right)^{1-\sigma} \frac{\sin(t\log k-2\pi jk)}{(k+2m)^{\sigma}}\, dk \right. \\
&\qquad \left. - \int_0^{c_0} \frac{\sin 2\pi m}{m} \sin 4\pi jm\,dm \int_{3/2-2m}^{t/2\pi j} \left(1 - \frac{2m}{k+2m}\right)^{1-\sigma} \frac{\cos(t\log k-2\pi jk)}{(k+2m)^{\sigma}}\, dk \right\} \\
&\quad = W_{411} - W_{412} ,
\end{aligned}
$$

$$
\begin{aligned}
W_{411} = \sum_{j=1}^{t/3\pi} &\left\{ \left[\frac{\sin 2\pi m}{m} \frac{\sin 4\pi jm}{4\pi j} \int_{3/2-2m}^{t/2\pi j} \left(1 - \frac{2m}{k+2m}\right)^{1-\sigma} \frac{\sin(t\log k-2\pi jk)}{(k+2m)^{\sigma}}\, dk \right]_{m=0}^{c_0} \right. \\
&- \int_0^{c_0} \left(\frac{\sin 2\pi m}{m}\right)' \frac{\sin 4\pi jm}{4\pi j}\, dm \int_{3/2-2m}^{t/2\pi j} \left(1 - \frac{2m}{k+2m}\right)^{1-\sigma} \frac{\sin(t\log k-2\pi jk)}{(k+2m)^{\sigma}}\, dk
\end{aligned}
$$

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$$- 2(-1)^j \int_0^{c_0} \frac{\sin 2\pi m}{m} \frac{\sin 4\pi jm}{4\pi j} (1-4m/3)^{1-\sigma} \frac{\sin\left(t\log(3/2-2m)+4\pi jm\right)}{(3/2)^\sigma} dm$$

$$+ 2 \int_0^{c_0} \frac{\sin 2\pi m}{m} \frac{\sin 4\pi jm}{4\pi j} dm \int_{3/2-2m}^{t/2\pi j} \frac{k^{1-\sigma}}{(k+2m)^2} \sin(t \log k - 2\pi jk) dk \Bigg\} .$$

Now we put $p = t \log k - 2\pi jk$; then p increases as k increases in the interval $(3/2-2m, t/2\pi j)$, and the function

$$y(k) = k^{2-\sigma}/(k+2m)(t/k-2\pi j)$$

increases as k increases. If $k = t/2\pi j$, then $p = t \log(t/2\pi je)$. Suppose $k = t/2\pi j - \theta$ $(\theta > 0)$; then

$$\begin{aligned} p &= t \log(t/2\pi j-\theta) - t + 2\pi j\theta \\ &= t \log(t/2\pi je) + t \log(1-2\pi j\theta/t) + 2\pi j\theta . \end{aligned}$$

If we take $\theta = \frac{1}{j} \sqrt{t/\pi}$, we see that p increases more than 2π when k increases from $t/2\pi j - \sqrt{t/\pi}/j$ to $t/2\pi j$. Therefore

$$\begin{aligned} \int_{3/2-2m}^{t/2\pi j} \frac{k^{1-\sigma}}{k+2m} \sin(t \log k-2\pi jk)dk &= O\left(\frac{1}{j} \sqrt{t/\pi} \frac{1}{(t/2\pi j)^\sigma}\right) \\ &= O\left(1/t^{\sigma-\frac{1}{2}} j^{1-\sigma}\right) , \end{aligned}$$

and then $W_{411} = O(\log t)$. Similarly $W_{412} = O(\log t)$. Therefore W_{41} is of order $\log t$. Similarly W_{42} is also of order $\log t$. Since W_{43} is bounded, W_4 is of order $\log t$. Thus we have proved that

$$\int_0^{c_0} \frac{\sin 2\pi m}{m} V_{115} = O(\log t) .$$

A similar estimate holds for V_{116} . Collecting the above estimations, we get

$$\begin{aligned} V_{11} &= \frac{2}{\pi} \int_0^{c_0} \frac{\sin 2\pi m}{m} W_{11} dm + O(\log t) \\ &= \frac{2}{\pi} \sum_{j=1}^{(t/3\pi)\log(3/2)} \int_0^{c_0} \frac{\sin 2\pi m}{m} dm \int_{3/2}^{k_j} \frac{\sin(t\log k-2\pi jk)}{k^\sigma} dk + O(\log t) . \end{aligned}$$

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7.2.2. Estimation of V_{12} . We shall estimate

$$V_{12} = \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$\cdot \int_{M(m)}^{\infty} \frac{\sin(t\log k)\cos 2\pi jk}{k^{\sigma}} \left\{1 - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \cos\left(t \log\left(1 - \frac{2m}{k}\right)\right)\right\} dk$$

$$= \tfrac{1}{4} \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$\cdot \int_{M(m)}^{\infty} \left\{2 \sin(t \log k + 2\pi jk) - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log(k-2m) + 2\pi jk\right)\right.$$

$$- \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log\left(k^2/(k-2m)\right) + 2\pi jk\right)$$

$$+ 2 \sin(t \log k - 2\pi jk) - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log(k-2m) - 2\pi jk\right)$$

$$\left. - \left(1 - \frac{2m}{k}\right)^{1-\sigma} \sin\left(t \log\left(k^2/(k-2m)\right) - 2\pi jk\right)\right\} \frac{dk}{k^{\sigma}}$$

$$= \tfrac{1}{4}\left(2V_{121} - V_{122} - V_{123} + 2V_{124} - V_{125} - V_{126}\right) .$$

By the transformation $p = t \log k + 2 jk$, used for V_{111} , we get

$$V_{121} = \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{M(m)}^{\infty} \frac{\sin(t\log k + 2\pi jk)}{k^{\sigma}}\, dk$$

$$= \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{t\log M(m) + 2\pi jM(m)}^{\infty} \frac{\sin p\, dp}{k(p)^{\sigma}\left(2\pi j + t/k(p)\right)}$$

$$= \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m \cos\left(t\log M(m) + 2\pi jM(m)\right)}{mM(m)^{\sigma}\left(2\pi j + t/M(m)\right)}\, dm$$

$$- (1-\sigma) \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{t\log M(m) + 2\pi jM(m)}^{\infty} \frac{k'(p)\cos p\, dp}{k(p)^{\sigma}\left(2\pi mk(p) + t\right)}$$

$$- \sum_{j=1}^{\infty} 2\pi j \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{t\log M(m) + 2\pi jM(m)}^{\infty} \frac{k(p)^{1-\sigma} k'(p)\cos p}{\left(2\pi jk(p) + t\right)^2}\, dp$$

$$= X_{11} - X_{12} - X_{13} ,$$

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$$X_{11} = \tfrac{1}{2} \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \{\sin(t \log M(m)+2\pi jM(m)+2\pi m) - \sin(t \log M(m)+2\pi jM(m)-2\pi m)\} \frac{dm}{mM(m)^{\sigma}(2\pi j+t/M(m))}$$

$$= \tfrac{1}{2}(X_{111}-X_{112}) .$$

By the transformation $q = t \log M(m) + 2\pi jM(m) + 2\pi m$,

$$dq = (tM'(M)/M(m) + 2\pi jM'(m) + 2\pi)dm$$
$$\cong \left(\frac{t}{(\sigma-\varepsilon)m} + \frac{2^{1/(\sigma-\varepsilon)}2\pi jm^{(1-\sigma+\varepsilon)/(\sigma-\varepsilon)}}{\sigma-\varepsilon} + 2\pi\right)dm ,$$

since

$$M'(m) - 1 \Big/ \frac{d}{dk}\left(\frac{k^{\sigma-\varepsilon}}{2(1+k^{\sigma-\varepsilon-1})}\right) \cong \frac{2k^{1-\sigma+\varepsilon}}{\sigma-\varepsilon} \cong \frac{2(2m)^{(1-\sigma+\varepsilon)/(\sigma-\varepsilon)}}{\sigma-\varepsilon} ,$$

$$M'(m)/M(m) \cong 1/m(\sigma-\varepsilon) ,$$

and then

$$X_{111} = \sum_{j=1}^{\infty} \int_{t\log M(c_0)+2\pi jM(c_0)+2\pi c_0}^{\infty} \frac{1}{m(q)M(m(q))^{\sigma}(2\pi j+t/M(m(q)))} \cdot \frac{1}{tM'(m(q))/M(m(q))+2\pi jM'(m(q))+2\pi} \sin q\,dq$$

$$= O\left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right) = O(1) ,$$

since $m/M(m)^{1-\sigma}$ increases. We can see that X_{112} is also bounded by a similar estimate, and then X_{11} is bounded. For the estimation of X_{12} , use the relation $k'(p) = 1/(2\pi j+t/k(p))$ and that the function of x : $y = x^{1-\sigma}/(2\pi jx+t)^2$ takes its maximum at the point $x = (1-\sigma)t/2(1+\sigma)\pi j$; then we can see that X_{12} is bounded. X_{13} is also similarly bounded. Therefore V_{121} is bounded. Similarly V_{122} and V_{123} are bounded. We shall estimate V_{124} .

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$$V_{124} = \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{M(m)}^{\infty} \frac{\sin(t\log k - 2\pi jk)}{k^{\sigma}}\, dk$$

$$= \sum_{j=1}^{(t/3\pi)\log(3/2)} + \sum_{j=(t/3\pi)\log(3/2)}^{t/3\pi} + \sum_{j=t/3\pi+1}^{\infty}$$

$$= X_{41} + X_{42} + X_{43}\,.$$

By the transformation $p = 2\pi jk - t\log k$ and integration by parts, we get

$$X_{43} = -\sum_{j=t/3\pi+1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{2\pi jM(m)-t\log M(m)}^{\infty} \frac{\sin p\, dp}{k(p)^{\sigma}\big(2\pi j - t/k(p)\big)}$$

$$= \sum_{j=t/3\pi+1}^{\infty} \Bigg\{ \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m} \frac{\cos\big(2\pi jM(m) - t\log M(m)\big)}{M(m)^{\sigma}\big(2\pi j - t/M(m)\big)}$$

$$- \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{2\pi jM(m)-t\log M(m)}^{\infty} \frac{(1-\sigma)k'(p)\cos p}{k(p)^{\sigma}\big(2\pi jk(p) - t\big)}\, dp$$

$$- \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{2\pi jM(m)-t\log M(m)}^{\infty} \frac{2\pi jk(p)^{1-\sigma}k'(p)\cos p}{\big(2\pi jk(p) - t\big)^2}\, dp$$

$$= X_{431} - X_{432} - X_{433}\,,$$

$$X_{431} = \sum_{j=t/3\pi+1}^{\infty} \int_{c_0}^{\infty} \frac{dm}{2mM(m)^{\sigma}\big(2\pi j - t/M(m)\big)} \Big\{\sin\big(2\pi jM(m) - t\log M(m) + 2\pi m\big) - \sin\big(2\pi jM(m) - t\log M(m) - 2\pi m\big)\Big\}$$

$$= X_{4311} - X_{4312}\,.$$

By the transformation $q = 2\pi jM(m) - t\log M(m) + 2\pi m$, we get

$$X_{4311} = \sum_{j=t/3\pi+1}^{\infty} \int_{3\pi j - t\log 3/2 + 2\pi c_0}^{\infty} \frac{1}{2m(q)M\big(m(q)\big)^{\sigma}\big(2\pi j - t/M(m(q))\big)} \cdot \frac{\sin q\, dq}{2\pi jM'\big(m(q)\big) - tM'\big(m(q)\big)/M\big(m(q)\big) + 2\pi}$$

$$= O\left(\sum_{j=t/3\pi}^{\infty} 1/(j - t/3\pi + 1)^2\right) = O(1)\,,$$

since $M'(m) > 0$ and increases and $M'(m)/M(m)$ decreases. Similarly $X_{4312} = O(1)$. Therefore $X_{431} = O(1)$. Since $k'(p) = 1/\big(2\pi j - t/k(p)\big)$, we have $X_{432} = O(1)$, $X_{433} = O(1)$. Thus we have proved $X_{43} = O(1)$.

$$X_{42} =$$

$$= \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\,dm \int_{M(m)}^{\infty} \frac{\sin(t\log k - 2\pi jk)}{k^{\sigma}}\,dk$$

$$= \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3} \int_{3/2}^{\infty} \frac{\sin(t\log k - 2\pi jk)}{k^{\sigma}}\,dk \left(\int_{c_0}^{\infty} - \int_{K/2(1+K/k)}^{\infty}\right) \frac{\sin 2\pi m}{m}\,dm$$

$$= X_{421} - X_{422} ,$$

where X_{421} is bounded.

$$X_{422} = \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi} \left(\int_{3/2}^{9} + \int_{9}^{\infty}\right) dk \int_{K/2(1+K/k)}^{\infty} dm$$

$$= X_{4221} + X_{4222} ,$$

$$X_{4222} = \sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi} \int_{18\pi j - t\log 9}^{\infty} \frac{\sin p\,dp}{k(p)^{\sigma}\left(2\pi j - t/k(p)\right)} \cdot \int_{k(p)^{\sigma-\varepsilon}/2\left(1+k(p)^{\sigma-\varepsilon-1}\right)}^{\infty} \frac{\sin 2\pi m}{m}\,dm$$

$$= O\left(\sum_{j=(t/3\pi)\log(3/2)+1}^{t/3\pi} \frac{1}{j}\right) = O(1) ,$$

and

$$X_{4221} = \int_{(t/3\pi)\log(3/2)+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \left(dj + dJ(j)\right) \int_{3/2}^{9} \frac{\sin(t\log k - 2\pi jk)}{k^{\sigma}}\,dk \cdot \int_{K/2(1+K/k)}^{\infty} \frac{\sin 2\pi m}{m}\,dm$$

$$= 2\int_{(t/3\pi)\log(3/2)+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} J(j)\,dj \int_{3/2}^{9} k^{1-\sigma}\cos(t\log k - 2\pi jk)\,dk \cdot \int_{K/2(1+K/k)}^{\infty} \frac{\sin 2\pi m}{m}\,dm + O(1) .$$

Writing

$$I(k) = \int_{K/2(1+K/k)}^{\infty} \frac{\sin 2\pi m}{m}\, dm ,$$

we get

$$X_{4221} = 2 \sum_{n=1}^{\infty} \frac{1}{n} \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \sin 2\pi nj\,dj$$

$$\cdot \int_{3/2}^{9} I(k)k^{1-\sigma} \cos(t \log k - 2\pi jk)dk + O(1)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \int_{3/2}^{9} I(k)k^{1-\sigma}dk$$

$$\cdot \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \big(\sin\big(t \log k - 2\pi j(k-n)\big) - \sin\big(t \log k - 2\pi j(k+n)\big)\big)dj$$

$$+ O(1)$$

$$= Y_{11} - Y_{12} + O(1) ,$$

$$Y_{11} = \sum_{n=1}^{\infty} \frac{1}{n} \Big\{\int_{3/2}^{9} I(k)k^{1-\sigma} \sin(t \log k)dk$$

$$\cdot \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \cos\big(2\pi j(k-n)\big)dj + \int_{3/2}^{9} I(k)k^{1-\sigma} \cos(t \log k)dk$$

$$\cdot \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} \sin\big(2\pi j(k-n)\big)dj\Big\}$$

$$= Y_{111} + Y_{112} .$$

We shall write $\theta_1 = 1/\big([(t/3\pi)\log(3/2)]+\frac{1}{2}\big)$, $\theta_2 = 1/([t/3\pi]+\frac{1}{2})$, and

$$Y_{111} = \sum_{n=2}^{8} \frac{1}{n} \Big(\int_{3/2}^{n} dk + \int_{n}^{9} dk\Big) + \sum_{n=1,9} + O(1)$$

$$= Z_1 + Z_2 + \sum_{n=1,9} + O(1) ,$$

$$Z_1 = \sum_{n=2}^{8} \frac{1}{n} \Big\{\int_{3/2}^{n-\theta_1} dk \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} dj$$

$$+ \int_{n-\theta_1}^{n-\theta_2} dk \Big(\int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{1/(n-k)} dj + \int_{1/(n-k)}^{[t/3\pi]+\frac{1}{2}} dk\Big)$$

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$$+ \int_{n-\theta_2}^{n} dk \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} dj \Bigg\}$$

$$= Z_{11} + Z_{12} + Z_{13} + Z_{14} ,$$

$$Z_{11} = \sum_{n=2}^{8} \frac{1}{n} \int_{3/2}^{n-\theta_1} \frac{I(k)k^{1-\sigma}\sin(t\log k)}{n-k}$$

$$\left\{\sin\left(2\pi([t/3\pi]+\tfrac{1}{2})(n-k)\right)-\sin\left(2\pi\left([(t/3\pi)\log(3/2)]+\tfrac{1}{2}\right)(n-k)\right)\right\}dk$$

$$= Z_{111} + Z_{112} .$$

In order to estimate Z_{111}, divide the range of integration into $(3/2, 7/4)$ and $(7/4, n-\theta_1)$; then the integrand of the first integral is bounded, but the second is $O(1/\theta_1 t) = O(1)$, using the transformation $p = 2\pi([t/3\pi]+\frac{1}{2})k - t \log k - 2\pi([t/3\pi]+\frac{1}{2})n$. About Z_{112}, in the cases $n \le 3$, use the transformation $p = t \log k + 2\pi\left([(t/3\pi)\log(3/2)]+\frac{1}{2}\right)(n-k)$ and, in the cases $n > 4$, divide the integration range into $(3/2, n-1)$ and $(n-1, n-\theta_1)$; then the integrand of the first integral is bounded and the second integral is $O(1/\theta_1 t) = O(1)$. Thus we have proved that $Z_{11} = O(1)$. Z_{13} is similarly estimated.

$$Z_{12} = \sum_{n=2}^{8} \frac{1}{n} \int_{n-\theta_1}^{n-\theta_2} I(k)k^{1-\sigma} \sin(t \log k)dk$$

$$\cdot \int_{[(t/3\pi)\log(3/2)]+\frac{1}{2}}^{1/(n-k)} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(2\pi j(n-k)\right)^{2m} dj$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{(2\pi)^{2m}}{2m+1} \sum_{n=2}^{8} \frac{1}{n} \int_{n-\theta_1}^{n-\theta_2} I(k)k^{1-\sigma} \sin(t \log k)$$

$$\cdot \left(\frac{1}{(n-k)^{2m+1}} - \left([(t/3\pi)\log(3/2)]+\tfrac{1}{2}\right)^{2m+1}\right)(n-k)^{2m}dk$$

$$= O\left(\sum_{m=0}^{\infty} \frac{(2\pi)^{2m}}{(2m+1)!} \sum_{n=2}^{8} \frac{1}{n\theta_2} (\theta_1-\theta_2)\right) = O(1) .$$

Z_{14} is also similarly estimated to become bounded. Thus we have proved that Z_1 is bounded. Z_2 is also, and then Y_{111}. Y_{112} is derived from

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Y_{111}, interchanging sine and cosine, and hence Y_{112} is bounded, and then Y_{11} is so. Y_{12} is easily seen to be bounded. This proves that X_{4221} is bounded. Thus we get that X_{42} is bounded.

Now we shall estimate X_{41}.

$$X_{41} = \sum_{j=1}^{(t/3\pi)\log(3/2)} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\,dm \int_{M(m)}^{\infty} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma}\,dk$$

$$= \sum_{j=1}^{(t/3\pi)\log(3/2)} \int_{3/2}^{\infty} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma}\,dk \int_{c_0}^{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})} \frac{\sin 2\pi m}{m}\,dm$$

$$= \sum_{j=1}^{(t/3\pi)\log(3/2)} \left[\int_{3/2}^{k_j} dk + \int_{k_j}^{\infty} dk\right] \int_{c_0}^{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})} dm$$

$$= X_{411} + X_{412},$$

$$X_{412} = -\sum_{j=1}^{(t/3\pi)\log(3/2)} \int_{2\pi jk_j - t\log k_j}^{\infty} \frac{\sin p\,dp}{k(p)^\sigma(2\pi j - t/k(p))} \cdot \int_{c_0}^{k(p)^{\sigma-\varepsilon}/2(1+k(p)^{\sigma-\varepsilon-1})} \frac{\sin 2\pi m}{m}\,dm$$

$$= O\left(\sum_{j=1}^{(t/3\pi)\log(3/2)} \frac{1}{j}\right) = O(\log t).$$

Therefore

$$V_{124} = X_{411} + O(\log t)$$

$$= \sum_{j=1}^{(t/3\pi)\log(3/2)} \int_{3/2}^{k_j} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma}\,dk \int_{c_0}^{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})} \frac{\sin 2\pi m}{m}\,dm + O(\log t).$$

Now,

$$V_{125} = \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\,dm \int_{M(m)}^{\infty} \left(1 - \frac{2m}{k}\right)^{1-\sigma} \frac{\sin\left(t\log(k-2m) - 2\pi jk\right)}{k^\sigma}\,dk$$

$$= \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{M(m)-2m}^{\infty} \left(\frac{k}{k+2m}\right)^{1-\sigma} \frac{\sin(t\log k-2\pi jk-4\pi jm)}{(k+2m)^{\sigma}}\, dk$$

$$= \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi m}{m}\, dm \int_{M(m)-2m}^{\infty} \{\sin(t \log k-2\pi jk)\cos 4\pi jm$$

$$- \cos(t \log k-2\pi jk) \sin 4\pi jm\} \frac{k^{1-\sigma}}{k+2m}\, dk$$

$$= X_{51} - X_{52} ,$$

$$X_{51} = \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\sin 2\pi(2j+1)m-\sin 2\pi(2j-1)m}{2m}\, dm \int_{M(m)-2m}^{\infty} \frac{k^{1-\sigma}\sin(t\log k-2\pi jk)}{k+2m}\, dk$$

$$= X_{511} - X_{512} ,$$

$$X_{511} = \sum_{j=1}^{\infty} \frac{\cos 2\pi(2j+1)c_0}{2\pi.2c_0(2j+1)} \int_{3/2-2c_0}^{\infty} \frac{k^{1-\sigma}\sin(t\log k-2\pi jk)}{k+2c_0}\, dk$$

$$- \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\cos 2\pi(2j+1)m}{4\pi m^2(2j+1)}\, dm \int_{M(m)-2m}^{\infty} \frac{k^{1-\sigma}\sin(t\log k-2\pi jk)}{k+2m}\, dk$$

$$- \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\cos 2\pi(2j+1)m}{4\pi m(2j+1)} \frac{\big(M(m)-2m\big)^{1-\sigma}}{M(m)}$$

$$. \big(M'(m)-2\big)\sin\big(t \log\big(M(m)-2m\big) - 2\pi j\big(M(m)-2m\big)\big)\, dm$$

$$- \sum_{j=1}^{\infty} \int_{c_0}^{\infty} \frac{\cos 2\pi(2j+1)m}{4\pi m(2j+1)}\, dm \int_{M(m)-2m}^{\infty} \frac{k^{1-\sigma}\sin(t\log k-2\pi jk)}{(k+2m)^2}\, dk$$

$$= \frac{1}{4\pi c_0} X_{5111} - \frac{1}{4\pi} \left(X_{5112}+X_{5113}+X_{5114}\right) .$$

Let $t' = t/2\pi\big(3/2-2c_0\big)$ and k_j be the solution of the equation $2\pi jk = t \log k$. We use the transformation

(7) $$p = |2\pi jk - t \log k| ,$$

where dp vanishes for $k = t/2\pi j$. We write

$$X_{5111} = \sum_{j=1}^{t'} \frac{\cos 2\pi(2j+1)c_0}{2j+1} \left[\int_{3/2-2c_0}^{t/2\pi j} + \int_{t/2\pi j}^{k_j} + \int_{k_j}^{\infty} \right] dk$$

$$+ \sum_{j=t'+1}^{\infty} \frac{\cos 2\pi(2j+1)c_0}{2j+1} \int_{3/2-c_0}^{\infty} dk$$

$$= Y_{21} + Y_{22} + Y_{23} + Y_{24} .$$

By the transformation (7), the integral in Y_{21} becomes

$$\int_{3/2-c_0}^{t/2\pi j} \frac{k^{1-\sigma}}{k+2c_0} \sin(t \log k - 2\pi jk) dk$$

$$= \int_{t\log(3/2-2c_0)-2\pi j(3/2-2c_0)}^{t\log(t/2\pi je)} \frac{k(p)^{1-\sigma} \sin p dp}{[k(p)+2c_0][t/k(p)-2\pi j]}$$

$$= O\left[\int_{t\log(t/2\pi je)-2\pi}^{t\log(t/2\pi je)} \frac{k(p)^{1-\sigma} dp}{[k(p)+2c_0][t/k(p)-2\pi j]} \right] .$$

For $k = t/2\pi j - \theta$ $(0 < \theta < t/2\pi j)$, in the transformation (7) p is given by

$$p = t \log\left(\frac{t}{2\pi j} - \theta\right) - 2\pi j\left(\frac{t}{2\pi j} - \theta\right)$$

$$= t \log \frac{t}{2\pi je} + t \log\left(1 - \frac{2\pi j\theta}{t}\right) + 2\pi j\theta$$

$$= t \log \frac{t}{2\pi je} - \frac{(2\pi j\theta)^2}{2t} - \frac{(2\pi j\theta)^3}{3t^2} - \cdots \quad \left(0 < \frac{2\pi j\theta}{t} < 1\right) .$$

If we take $\theta = \frac{1}{j}\sqrt{t/\pi}$, then

$$p = t \log \frac{t}{2\pi je} - 2\theta - o(1) .$$

Therefore, the range $(t/2\ j,\ t/2\pi j - \sqrt{t/\pi j^2})$ on the k line can be transformed to the interval on the p line, which covers the range of integration of the last integral with respect to p . Thus we get

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$$Y_{21} = O\left(\sum_{j=1}^{t'} \frac{1}{j} \int_{t/2\pi j - \sqrt{t/\pi j^2}}^{t/2\pi j} \frac{dk}{k^\sigma}\right)$$

$$= O\left(\sum_{j=1}^{t'} \frac{1}{j} \frac{\sqrt{t/\pi j^2}}{(t/2\pi j)^\sigma}\right) = O\left(\sum_{j=1}^{t'} \frac{1}{t^{\sigma-\frac{1}{2}} j^{2-\sigma}}\right)$$

$$= O(1) .$$

Similarly Y_{22}, Y_{23}, and Y_{24} are bounded, and then $X_{5111} = O(1)$.

$$X_{5112} = \sum_{j=1}^{t/2\pi e} \left\{\int_{c_0}^{m_0} dm \left(\int_{M(m)-2m}^{t/2\pi j} + \int_{t/2\pi j}^{k_j} + \int_{k_j}^{\infty}\right) dk\right.$$

$$\left. + \int_{m_0}^{m_1} dm \left(\int_{M(m)-2m}^{k_j} + \int_{k_j}^{\infty}\right) dk + \int_{m_1}^{\infty} dm \int_{M(m)-2m}^{\infty} dk\right\}$$

$$+ \sum_{j=t/2\pi e}^{\infty} \int_{c_0}^{\infty} dm \left(\int_{M(m)-2m}^{\max(M(m)-2m,e)} + \int_{\max(M(m)-2m,e)}^{\infty}\right) dk$$

$$= \sum_{j=1}^{t/2\pi e} O\left(\frac{1}{j^{2-\sigma} t^{\sigma-\frac{1}{2}}}\right) + \sum_{j=1}^{t/2\pi e} O\left(\frac{1}{j}\right) + \sum_{j=t/2\pi e}^{\infty} O\left(\frac{1}{j(j-t/2\pi e)}\right)$$

$$= O(\log t) ,$$

using an estimate similar to Y_{21} where m_0 and m_1 are the solutions of the equation of m : $M(m) - 2m = t/2\pi j$ and $M(m) - 2m = k_j$, respectively.

Since $k = M(m)$ is the solution of the equation

$$m = k^{\sigma-\varepsilon}/2\left(1+k^{\sigma-\varepsilon-1}\right) ,$$

we have

$$M'(m) = \frac{dk}{dm} = 1 \Big/ \frac{dm}{dk}$$

$$= \left(\frac{(\sigma-\varepsilon)k^{\sigma-\varepsilon-1}}{2\left(1+k^{\sigma-\varepsilon-1}\right)} - \frac{(\sigma-\varepsilon-1)k^{2(\sigma-\varepsilon-1)}}{2\left(1+k^{\sigma-\varepsilon-1}\right)^2}\right)^{-1}$$

$$\cong \frac{2}{\sigma-\varepsilon} \frac{1+k^{\sigma-\varepsilon-1}}{k^{\sigma-\varepsilon-1}} \cong \frac{2}{\sigma-\varepsilon} M(m)^{1-\sigma+\varepsilon}$$

and then

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$X_{5113} =$

$$= \sum_{j=1}^{\infty} \frac{1}{4\pi(2j+1)} \int_{c_0}^{\infty} \frac{\{M(m)-2m\}^{1-\sigma}\{M(m)-2\}}{mM(m)}$$

$$\cdot \{\sin t(\log\{M(m)-2m\}-2\pi m-2\pi jM(m))+\sin(t \log\{M(m)-2m\}+2\pi m-2\pi j\{M(m)-4m\})\}dm$$

$$= \sum_{j=1}^{t/2\pi c_0^{1/(\sigma-\varepsilon)}} \left[\int_{c_0}^{(t/2\pi j)^{\sigma-\varepsilon}} + \int_{(t/2\pi j)^{\sigma-\varepsilon}}^{\infty}\right] dm + \sum_{j=t/2\pi c_0^{1/(\sigma-\varepsilon)}}^{\infty} \int_{c_0}^{\infty} dm$$

$= O(1)$.

X_{5114} is also of the same order as X_{5113} , and then

$$X_{511} = O(\log t) .$$

X_{512} is similarly estimated, and then X_{51} is also of order $O(\log t)$, and X_{52} is also. Therefore

$$V_{125} = O(\log t) .$$

V_{126} is quite similar to V_{125} , so that

$V_{12} = \frac{1}{2}V_{124} + O(\log t)$

$$= \frac{1}{2} \sum_{j=1}^{(t/3\pi)\log(3/2)} \int_{3/2}^{k_j} \frac{\sin(t\log k-2\pi jk)}{k^{\sigma}}\, dk \int_{c_0}^{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})} \frac{\sin 2\pi m}{m}\, dm$$

$$+ O(\log t) .$$

Combining with the estimation of §7.2.1 we get

$V_1 = V_{11} + V_{12}$

$$= \frac{1}{2} \sum_{j=1}^{t/3\pi} \int_{3/2}^{k_j} \frac{\sin(t\log k-2\pi jk)}{k^{\sigma}}\, dk \int_{0}^{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})} \frac{\sin 2\pi m}{m}\, dm$$

$$+ O(\log t) .$$

Denoting the inner integral of V_2 by $g(k)$, we have

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$$V_2 = \frac{1}{t}\sum_{j=1}^{\infty}\left\{\left[\frac{\sin(t\log k)\sin 2\pi jk}{2\pi jk^{\sigma}}\,g(k)\right]_{k=3/2}^{\infty} - \int_{3/2}^{\infty}\frac{t\cos(t\log k)\sin 2\pi jk}{2\pi jk^{1+\sigma}}\,g(k)dk + \int_{3/2}^{\infty}\frac{\sin(t\log k)\sin 2\pi jk}{2\pi j}\left(\frac{g(k)}{k^{\sigma}}\right)' dk\right\}$$

$$= O(1)\ .$$

We easily see that k_j , the upper limit of the outer integral in V_1 , can be replaced by $t/j\pi$. Therefore, combining with the estimation of T_{111} and $T_{12} = O(1)$, we get

$$T_1 = \sum_{j=1}^{t/3\pi}\int_{3/2}^{t/j\pi}\frac{\sin(t\log k-2\pi jk)}{k^{\sigma}}\,dk\int_0^{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})}\frac{\sin 2\pi m}{m}\,dm + O(\log t)$$

$$= \sum_{j=1}^{t/3\pi}\left(\int_{3/2}^{t/2\pi j}+\int_{t/2\pi j}^{t/\pi j}\right)dk\left(\int_0^{\infty}-\int_{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})}^{\infty}\right)dm + O(\log t)$$

$$= T'_{11} + T'_{12} - \left(T'_{13}+T'_{14}\right) + O(\log t)\ .$$

Now

$$T'_{13} = \sum_{j=1}^{t/3\pi}\int_{3/2}^{t/2\pi j}\frac{\sin(t\log k)\cos 2\pi jk-\cos(t\log k)\sin 2\pi jk}{k^{\sigma}}\,h(k)dk$$

$$= T'_{131} - T'_{132}\ ,$$

where

$$h(k) = \int_{k^{\sigma-\varepsilon}/2(1+k^{\sigma-\varepsilon-1})}^{\infty}\frac{\sin 2\pi m}{m}\,dm\ .$$

We write

$$T'_{131} = \int_{\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}}\left(dj+dJ(j)\right)\int_{3/2}^{t/2\pi j}\frac{\sin(t\log k)\cos 2\pi jk}{k^{\sigma}}\,h(k)dk$$

$$= U'_1 + U'_2\ ,$$

where

$$U_1' = \int_{3/2}^{t/\pi} \frac{\sin(t\log k)}{k^\sigma} h(k)dk \int_{\frac{1}{2}}^{t/2\pi k} \cos 2\pi jkdj + O(1)$$

$$= O\left(\int_{3/2}^{t/\pi} \frac{dk}{k^{2\sigma-\varepsilon_k}}\right) + O(1) = O(1) ,$$

and using the transformation $t \log(t/2\pi j) = t \log(t/2\pi j') + \pi$,

$$U_2' = \int_{\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} J(j) \frac{t}{2\pi j^2} \frac{\sin\left(t\log(t/2\pi j)\right)\cos t}{(t/2\pi j)^\sigma} h\left(\frac{t}{2\pi j}\right) dj$$

$$+ 2\pi \int_{\frac{1}{2}}^{[t/3\pi]+\frac{1}{2}} J(j)dj \int_{3/2}^{t/2\pi j} k^{1-\sigma} \sin(t \log k) \sin 2\pi jk \cdot h(k)dk$$

$$= O(1) + 2\pi \int_{3/2}^{t/\pi} k^{1-\sigma} h(k) \sin(t \log k)dk \int_{\frac{1}{2}}^{t/2\pi k} \sin 2\pi jkJ(j)dj + O(1)$$

$$= O(1) + \int_{3/2}^{t/\pi} \frac{h(k)\sin(t\log k)}{k^\sigma} dk\left\{- \cos tJ\left(\frac{t}{2\pi k}\right) + \int_{\frac{1}{2}}^{t/2\pi k} \cos 2\pi jkdJ(j)\right\}$$

$$= O(1) + O\left(\int_{3/2}^{t/\pi} \frac{\log(t/2\pi k)}{k^{2\sigma-\varepsilon}} dk\right) = O(\log t) .$$

Thus we have proved that $T'_{131} = O(1)$. Similarly T'_{132} is bounded, and then T'_{13} is also. Since T'_{14} is also bounded by a similar estimate, we get

$$T_1 = \frac{\pi}{2} \sum_{j=1}^{t/3\pi} \int_{3/2}^{t/j\pi} \frac{\sin(t\log k-2\pi jk)}{k^\sigma} dk + O(\log t) .$$

8. Estimation of the remaining terms

8.1. Estimation of T_2 .

$T_2 =$

$$= \sum_{k=2}^{\infty} \frac{1}{k^\sigma} \int_0^{t\log(1+K/k)} \cos(t \log k+\omega)$$

$$\cdot \left[e^{-(1-\sigma)\omega/t} \int_{k(1-e^{-\omega/t})/2}^{\infty} \frac{\sin 2\pi m}{m} dm - e^{(1-\sigma)\omega/t} \int_{k(e^{\omega/t}-1)/2}^{\infty} \frac{\sin 2\pi m}{m} dm\right] d\omega$$

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$$= \sum_{k=2}^{\infty} \frac{1}{k^{\sigma}} \left\{ \int_0^{t\log(1+K/k)} \cos(t \log k+w)\left(e^{-(1-\sigma)w/t} - e^{(1-\sigma)w/t}\right) dw \right.$$

$$\cdot \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$\left. - \int_0^{t\log(1+K/k)} \cos(t \log k+w) e^{(1-\sigma)w/t} dw \int_{k(1-e^{-w/t})/2}^{k(e^{w/t}-1)/2} \frac{\sin 2\pi m}{m}\, dm \right\}$$

$$= -T_{21} - T_{22} ,$$

$$T_{21} = \sum_{n=1}^{\infty} \frac{2(1-\sigma)^{2n-1}}{(2n-1)} \sum_{k=2}^{\infty} \frac{1}{k^{\sigma} t^{2n-1}} \int_0^{t\log(1+K/k)} \cos(t \log k+w) w^{2n-1} dw$$

$$\cdot \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

by using the expansion formula for e^x , and denoting by $I(k, w)$ the inner integral in the last formula, we get

$$\int_0^{t\log(1+K/k)} \cos(t \log k+w) w^{2n-1} I(k, w) dw$$

$$= \left[\sin(t \log k+w) w^{2n-1} I(k, w)\right]_{w=0}^{t\log(1+K/k)}$$

$$- \int_0^{t\log(1+K/k)} \sin(t \log k+w)$$

$$\left\{(2n-1)w^{2n-2} I(k, w) + w^{2n-1} \frac{k \sin \pi k\left(1-e^{-w/t}\right)}{kt\left(1-e^{-w/t}\right)} e^{-w/t}\right\} dw ,$$

and then $T_{21} = O(1)$. T_{22} is also bounded by the same method. Thus we have proved that $T_2 = O(1)$.

8.3. Estimation of T_3 . Similarly as the estimation of T_2 , we get

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$$T_3 = \sum_{k=2}^{\infty} \frac{1}{k^\sigma} \int_{t\log(1+K/k)}^{-t\log(1-K/k)} \cos(t \log k - w) e^{-(1-\sigma)w/t} dw \cdot \int_{k(1-e^{-w/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$= O(1) .$$

8.3. Estimation of Q_{222}, Q_{223}, and Q_{224}.

$$Q_{222} = \sum_{k=2}^{\infty} \frac{t}{2\pi k} \int_{k}^{k-1} \frac{\cos(t\log(k-u))}{(k-u)^\sigma}\, du \int_{u/2}^{\infty} \frac{\sin 2\pi m}{m}\, du$$

$$= \sum_{k=2}^{\infty} \frac{1}{2\pi k} \int_0^{t\log(k-K)} e^{(1-\sigma)v/t} \cos v dv \int_{(k-e^{v/t})/2}^{\infty} \frac{\sin 2\pi m}{m}\, dm$$

$$= \sum_{k=2}^{\infty} \frac{1}{2\pi k} \int_0^{t\log(k-K)} e^{(1-\sigma)v/t} \cos v dv \cdot \left\{ \frac{\cos(k-e^{v/t})}{(k-e^{v/t})} - \int_{(k-e^{v/t})/2}^{\infty} \frac{\cos 2\pi m}{2\pi m^2}\, dm \right\}$$

$$= \frac{1}{2\pi^2} R_1' - \frac{1}{4\pi^2} R_2' ,$$

where

$$R_1' = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \int_0^{t\log(k-K)} \frac{e^{(1-\sigma)v/t}}{k-e^{v/t}} \cos(\pi e^{v/t} - v)\, dv + O(1)$$

$$= \sum_{k=2}^{t/\pi} + \sum_{k=t/\pi+1}^{\infty} + O(1)$$

$$= R_{11}' + R_{12}' + O(1) .$$

We use the transformation $p = v - \pi e^{v/t}$, where $dp = \left(1 - \frac{\pi}{t} e^{v/t}\right) dv$, vanishing at $v = t \log \frac{t}{\pi}$. For $v = t \log(t/\pi)$, we have $p = t \log(t/\pi e)$ and for $v = t \log(t/\pi) - \theta$,

$$p = t \log(t/\pi e) - \theta - t(1 - e^{-\theta/t})$$

$$\cong p_0 - \tfrac{1}{2} \theta^2/t .$$

If we take $\theta = \sqrt{4\pi t}$, then we see that p changes over an interval of

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length greater than 2π when v changes from $t\log(t/\pi)-\sqrt{4\pi t}$ to $t\log(t/\pi)$. Therefore

$$R'_{12} = O\left\{\sum_{k=t/\pi+1}^{\infty}\frac{1}{k}\int_{t\log(t/\pi)-\sqrt{4\pi t}}^{t\log(t/\pi)}\frac{e^{(1-\sigma)v/t}}{k-e^{v/t}}\,dv\right\}$$

$$= O\left\{t^{1-\sigma+\frac{1}{2}}\sum_{k=t/\pi+1}^{\infty}\frac{1}{k(k-t/\pi)}\right\} = O(1)\;.$$

Similarly,

$$R'_{11} = O\left\{\sum_{k=2}^{t/\pi}\frac{1}{k}\,\frac{k^{1-\sigma}}{k^{\sigma-\varepsilon}}\,\frac{t}{t-k\pi+K\pi}\right\} = O(1)\;.$$

Thus $R'_1 = O(1)$.

$$R'_2 = \sum_{k=2}^{\infty}\frac{1}{k}\int_0^{t\log(k-K)} e^{(1-\sigma)v/t}\cos v\,dv\int_{\left(k-e^{v/t}\right)/2}^{\infty}\frac{\cos 2\pi m}{m^2}\,dm$$

$$= \sum_{k=2}^{\infty}\frac{1}{k}\left\{\int_K^{k/2-\frac{1}{2}}dm\int_{t\log(k-2m)}^{t\log(k-K)}dv + \int_{k/2-\frac{1}{2}}^{\infty}dm\int_0^{t\log(k-K)}dv\right\}$$

$$= O(1)\;.$$

Thus we have proved that Q_{222} is bounded. Similarly Q_{223} and Q_{224} are bounded.

Collecting the above estimates we get

$$I\zeta(s) = -\sum_{j=1}^{t/3\pi}\int_{3/2}^{t/\pi j}\frac{\sin(t\log k-2\pi jk)}{k^{\sigma}}\,dk + O(\log t)\;.$$

For the real part of $\zeta(s)$, we get a corresponding formula where sine is replaced by cosine. Therefore

$$\zeta(s) = -\sum_{j=1}^{t/3\pi}\int_{3/2}^{t/\pi j}e^{i(t\log k-2\pi jk)}\,\frac{dk}{k^{\sigma}} + O(\log t)\;.$$

Thus Theorem 1 is proved.

9. Proof of Theorem 2

$$G = \sum_{j=1}^{t/3\pi} \int_{3/2}^{t/\pi j} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma}\, dk$$

$$= \int_{\frac12}^{[t/3\pi]+\frac12} \left(dj + dJ(j)\right) \int_{3/2}^{t/\pi j} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma}\, dk$$

$$= G_1 + G_2 .$$

By change of order of integration, we can easily see that $G_1 = O(1)$.

$$G_2 = \int_{\frac12}^{[t/3\pi]+\frac12} dJ(j) \int_{3/2}^{t/2\pi j} \frac{\sin(t\log k - 2\pi jk)}{k^\sigma}\, dk + \int_{\frac12}^{[t/3\pi]+\frac12} dJ(j) \int_{t/2\pi j}^{t/\pi j} dk$$

$$= G_{21} + G_{22} .$$

By integration by parts,

$$G_{21} = \int_{\frac12}^{[t/3\pi]+\frac12} J(j)\, \frac{t}{2\pi j^2}\, \frac{\sin\left(t\log(t/2\pi je)\right)}{(t/2\pi j)^\sigma}\, dj$$

$$+ 2\pi \int_{\frac12}^{[t/3\pi]+\frac12} J(j)dj \int_{3/2}^{t/2\pi j} k^{1-\sigma} \cos(t \log k - 2\pi jk) dk$$

$$= \frac{1}{(2\pi)^{1-\sigma}}\, G_{211} + 2\pi G_{212} ,$$

where

$$G_{211} = t^{1-\sigma} \int_{\frac12}^{[t/3\pi]+\frac12} \frac{J(j)}{j^{2-\sigma}} \sin\left(t \log(t/2\pi je)\right) dj$$

$$= t^{1-\sigma} \sum_{n=1}^{t/3\pi} \int_n^{n+1} dj + O(1)$$

$$= t^{1-\sigma} \sum_{n=1}^{t/3\pi} \int_{t\log\left(t/2\pi(n+1)e\right)}^{t\log(t/2\pi ne)} \frac{e^{(1-\sigma)p/t} \sin p}{(t/2\pi e)^{2-\sigma}}\, dp$$

$$= O\left(\frac{1}{t^\sigma} \sum_{n=1}^{t/3\pi} \frac{1}{n^{1-\sigma}}\right) = O(1) ,$$

$$G_{212} = \int_{3/2}^{t/\pi} k^{1-\sigma} \cos(t \log k)dk \int_{\frac{1}{2}}^{t/2\pi k} \cos(2\pi jk)J(j)dj$$

$$+ \text{(the term where the cosine is replaced by sine)} + O(1)$$

$$= H_1 + H_2 + O(1) .$$

Using integration by parts,

$$H_1 = \int_{3/2}^{t/\pi} k^{1-\sigma} \cos(t \log k)\left\{\left[\frac{\sin 2\pi jk}{2\pi k} J(j)\right]_{j=\frac{1}{2}}^{t/2\pi k} - \frac{1}{2\pi k}\int_{\frac{1}{2}}^{t/2\pi k} \sin 2\pi jk dJ(j)\right\}$$

$$= \frac{1}{2\pi} H_{11} - \frac{1}{2\pi} H_{12} ,$$

$$H_{11} = \sin t \int_{3/2}^{t/\pi} \frac{\cos(t\log k)}{k^{\sigma}} J\left(\frac{t}{2\pi k}\right)dk$$

$$= \sin t \sum_{n=1}^{t/\pi} \int_{t/2\pi(n+1)}^{t/2\pi n} + O(1)$$

$$= \sin t \sum_{n=1}^{t/\pi} \left(\int_{t/2\pi(n+1)}^{\theta} - \int_{\theta'}^{t/2\pi n}\right) \frac{\cos(t\log k)}{k^{\sigma}} dk ,$$

where $t/2\pi(n+1) < \theta < t/2\pi(n+\frac{1}{2}) < \theta' < t/2\pi n$. Using the transformation $p = t \log k$, $dp = (t/k)dk$, we get

$$H_{11} = O\left(\sum_{n=1}^{t/\pi} \frac{1}{t} \left(\frac{t}{n}\right)^{1-\sigma}\right) = O(1) .$$

Further

$$H_{12} = \int_{3/2}^{t/\pi} \frac{\cos(t\log k)}{k^{\sigma}} \bar{D}_{[t/2\pi k]}(k)dk$$

where $\bar{D}_n(k)$ denotes the nth conjugate Dirichlet kernel at the point k , and then

$$H_{12} = \sum_{n=1}^{\sqrt{t/2\pi}} \int_{t/2\pi(n+1)}^{t/2\pi n} + \int_{t/2\pi}^{t/\pi} + \sum_{m=1}^{\sqrt{t}-1} \int_{m}^{m+1} + \int_{[\sqrt{t}]}^{t/2\pi([\sqrt{t/2\pi}]+1)}$$

$$= H_{121} + H_{122} + H_{123} + H_{124} ,$$

where

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$$H_{121} = \sum_{n=1}^{\sqrt{t/2\pi}} \int_{t/2\pi(n+1)}^{t/2\pi n} \frac{\cos(t\log k)}{k^\sigma} \bar{D}_n(2\pi k)dk$$
$$= \sum_{n=1}^{\sqrt{t/2\pi}} \left\{ \sum_{l=1}^{n} \int_{t/2\pi(n+1)}^{t/2\pi n} \frac{\sin(t\log k - 2\pi l k)}{k^\sigma} dk \right\} + O(1)$$
$$= O\left(\sum_{n=1}^{\sqrt{t/2\pi}} \left(\sum_{l=1}^{n-1} \left(\frac{t}{n}\right)^{1-\sigma} \frac{1}{t - lt/n} + \left(\frac{n}{t}\right)^\sigma \frac{\sqrt{t}}{n} \right) \right) + O(1)$$
$$= O\left(t^{(1-\sigma)/2} \log t\right) ,$$
$$H_{123} = O\left(\sum_{m=1}^{\sqrt{t}-1} \frac{1}{m^\sigma} \log \frac{t}{2\pi m} \right) = O\left(t^{(1-\sigma)/2} \log t\right) ,$$

and H_{122} and H_{124} are bounded. Thus we have proved that H_1 is of order $O\left(t^{(1-\sigma)/2} \log t\right)$. H_2 is also and then G_{21} is of the same order. G_{22} can be estimated similarly and more easily. Therefore we have proved that $G = O\left(t^{(1-\sigma)/2} \log t\right)$, which proves Theorem 2.

Appendix

It is sufficient to prove that

$$I_1 = \int_{\frac{1}{2}}^{M} \frac{J(m)}{m} dm \int_L^\infty \frac{\cos(t\log u)\cos 2\pi m u}{u^\sigma} du \to 0 \quad \text{as} \quad L \to \infty , \text{ for all } M > 0 ,$$

and

$$I_2 = \int_1^\infty \frac{\cos(t\log u)}{u^\sigma} du \int_M^\infty J(m) \cos 2\pi m u \frac{dm}{m} \to 0 \quad \text{as} \quad M \to \infty .$$

Now, suppose $L > t \log t$,

$$I_1 = \tfrac{1}{2} \int_{\frac{1}{2}}^{M} \frac{J(m)}{m} dm \int_L^\infty \frac{\cos(2\pi m u - t\log u) + \cos(2\pi m u + t\log u)}{u^\sigma} du$$
$$= \tfrac{1}{2}\left(I_{11} + I_{12}\right) .$$

By the transformation $v = 2\pi m u - t \log u$ and denoting by $u(v)$ the solution for u of the above equation for fixed v ,

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$$I_{11} = \int_{\frac{1}{2}}^{M} \frac{J(m)}{m}\, dm \int_{2\pi mL - t\log L}^{\infty} \frac{\cos v\, dv}{u(v)^{\sigma}\left(2\pi m - t/u(v)\right)}$$

$$= O\left(\frac{1}{L^{\sigma}} \int_{\frac{1}{2}}^{\infty} \frac{dm}{m^2}\right) = o(1) \quad \text{as} \quad L \to \infty .$$

By the transformation $v = 2\pi mu + t \log u$, we can prove similarly that $I_{12} = o(1)$.

$$\int_M^{\infty} J(m) \cos 2\pi mu \frac{dm}{m} = \left[\frac{1}{m} \int_M^m J(n) \cos 2\pi nu dn\right]_{m=\infty} - \int_M^{\infty} \frac{dm}{m^2} \int_M^m J(n) \cos 2\pi nu dn$$

where

$$\int_M^m J(n) \cos 2\pi nu dn$$

$$= \sum_{k=1}^{\infty} \frac{1}{\pi k} \int_M^m \sin 2\pi kn \cos 2\pi nu dn$$

$$= \sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_M^m \{\sin 2\pi(k+u)n + \sin 2\pi(k-u)n\} dn$$

$$= \sum_{k=1}^{\infty} \frac{1}{2\pi k} \left\{\frac{\cos 2\pi(k+u)M - \cos 2\pi(k+u)m}{2\pi(k+u)} + \frac{\cos 2\pi(k-u)M - \cos 2\pi(k-u)m}{2\pi(k-u)}\right\} ,$$

and then

$$\lim_{m\to\infty} \frac{1}{m} \int_M^m J(n) \cos 2\pi nu dn = o(1) ,$$

$$\int_M^{\infty} J(m) \cos 2\pi mu \frac{dm}{m} = - \sum_{k=1}^{\infty} \frac{1}{(2\pi)^2 Mk} \frac{\cos 2\pi(k+u)M}{k+u}$$

$$+ \sum_{k=1}^{\infty} \frac{1}{(2\pi)^2 Mk} \left(\frac{1}{k+u} \int_M^{\infty} \frac{\cos 2\pi(k+u)m}{m^2} dm + \frac{1}{k+u} \int_M^{\infty} \frac{\cos 2\pi(k-u)m - \cos 2\pi(k-u)M}{m^2} dm\right) ,$$

and

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$$I_2 = \int_1^\infty \frac{\cos(t\log u)}{u^\sigma} \left\{ \sum_{k=1}^\infty \frac{1}{(2\pi)^2(k-u)k} \cdot \int_M^\infty \frac{\cos 2\pi(k-u)m - \cos 2\pi(k-u)M}{m^2} dm \right\} du + o(1)$$

$$= \left\{ \int_1^{3/2} + \sum_{j=2}^\infty \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \right\} \frac{\cos(t\log u)}{u^\sigma} du \left\{ \sum_{k=1}^{j-1} + \sum_{k=j} + \sum_{k=j+1}^\infty \right\} + o(1)$$

$$= \frac{1}{(2\pi)^2} \int_1^{3/2} \frac{\cos(t\log u)}{(u-1)u^\sigma} du \int_M^\infty \frac{\cos 2\pi(j-u)m - \cos 2\pi(j-u)M}{m^2} dm$$

$$+ \frac{1}{(2\pi)^2} \sum_{j=2}^\infty \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{\cos(t\log u)}{ju^\sigma(j-u)} du \int_M^\infty \frac{\cos 2\pi(j-u)m - \cos 2\pi(j-u)M}{m^2} dm + o(1)$$

$$= O\left[\int_1^{1+1/M} \frac{du}{d^\sigma} \left\{ \int_{(u-1)M}^1 \frac{dm}{m} + \int_1^\infty \frac{dm}{m^2} \right\} + \frac{1}{M} \int_{1+1/M}^{3/2} \frac{du}{(u-1)u^\sigma} \right]$$

$$+ \frac{1}{(2\pi)^2} \sum_{j=2}^\infty \frac{1}{j} \int_0^{\frac{1}{2}} \left\{ \frac{\cos\{t\log(j+u)\}}{(j+u)^\sigma} - \frac{\cos\{t\log(j-u)\}}{(j-u)^\sigma} \right\} du \cdot \int_{(j-u)M}^\infty \frac{\cos 2\pi m - \cos 2\pi(j-u)M}{m^2} dm + o(1)$$

$$= o(1) .$$

Thus we have proved that the order of integration of Q_2 can be interchanged, that is,

$$Q_2 = t \int_1^\infty \frac{\cos(t\log u)}{u^\sigma} du \int_{\frac{1}{2}}^\infty \frac{J(m)}{m} \cos 2\pi mu dm .$$

References

[1] H.M. Edwards, *Riemann's zeta function* (Academic Press, New York, London, 1974).

[2] G.H. Hardy and J.E. Littlewood, "The approximate functional equation in the theory of the zeta-function, with applications to the divisor-problems of Dirichlet and Piltz", *Proc. London Math. Soc.* (2) 21 (1922), 39-74.

[3] E.C. Titchmarsh, *The theory of the Riemann zeta-function* (Clarendon Press, Oxford, 1951).

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