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Asymptotic freedom and dimensional transmutation

In this chapter we return to the weak coupling limit of non-Abelian gauge theories. At the level of tree Feynman diagrams, relativistic field theory has no divergences and thus needs no renormalization. The bare coupling acquires cutoff dependence only after divergent one-loop diagrams are encountered. This implies that in the perturbative limit of our gauge theory of quarks and gluons

$$\gamma(g_0) = O(g_0^3). \tag{13.1}$$

At the outset we know that one zero of the renormalization group function occurs at vanishing coupling. For this root to be ultraviolet attractive and therefore useable for a continuum limit requires a positive sign for the first non-vanishing term in this perturbative expansion. Politzer (1973) and Gross and Wilczek (1973*a*, *b*) first calculated the relevant term for non-Abelian gauge theories. Defining the coefficients γ_0 and γ_1 from the asymptotic series

$$\gamma(g_0) = \gamma_0 g_0^3 + \gamma_1 g_0^5 + O(g_0^7), \tag{13.2}$$

we have the result for $SU(n)$ gauge theory with n_f fermionic species

$$\gamma_0 = (1/16\pi^2)(11n/3 - 2n_f/3). \tag{13.3}$$

Thus as long as

$$n_f < 11n/2, \tag{13.4}$$

the fixed point at the origin can potentially give a continuum limit. The two-loop contribution (Caswell, 1974; Jones, 1974) is

$$\gamma_1 = (1/16\pi^2)^2(34n^2/3 - 10nn_f/3 - n_f(n^2-1)/n). \tag{13.5}$$

Although in general $\gamma(g_0)$ is scheme dependent, these first two terms in its perturbative expansion are not. Consider two different schemes both defining a bare coupling as a function of cutoff: $g_0(a)$ and $g_0'(a)$. In the weak coupling limit each formulation should reduce to the classical Yang–Mills theory, and thus to lowest order they must agree

$$\left.\begin{aligned} g_0' &= g_0 + cg_0^3 + O(g_0^5), \\ g_0 &= g_0' - cg_0'^3 + O(g_0'^5). \end{aligned}\right\} \tag{13.6}$$

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We now calculate the new renormalization group function

$$\begin{aligned}\gamma'(g_0') &= a(\mathrm{d}/\mathrm{d}a)\, g_0' = (\partial g_0'/\partial g_0)\,\gamma(g_0)\\ &= (1+3cg_0^2)(\gamma_0 g_0^3+\gamma_1 g_0^5)+O(g_0^7)\\ &= \gamma_0 g_0'^3+\gamma_1 g_0'^5+O(g_0'^7).\end{aligned} \tag{13.7}$$

To order $g_0'^5$ all dependence on the parameter c cancels.

Thus far our discussion of the renormalization group has been in terms of the bare charge in the theory with a cutoff in place. This is a natural procedure in the lattice theory; however, the renormalization group is still useful in the continuum theory if we define a finite renormalized coupling constant. Like the generic physical function H of the last chapter, a renormalized coupling is first of all an observable which remains finite in the continuum limit

$$\lim_{a\to 0} g_R(r, a, g_0(a)) = g_R(r). \tag{13.8}$$

In general, the renormalized coupling g_R retains a dependence on the scale r of its definition. The masses and radii of the physical particles determine the typical dimensions for this dependence.

Secondly, to be properly called a renormalization of the classical coupling, g_R should be normalized such that it reduces to the bare coupling in lowest order perturbation theory when the cutoff is still in place.

$$g_R(r, a, g_0) = g_0+O(g_0^3). \tag{13.9}$$

Beyond this, the definition of g_R is totally arbitrary. In particular, given any physical observable H satisfying the perturbative expansion

$$H(r, a, g_0) = h_0+h_1 g_0^2+O(g_0^4), \tag{13.10}$$

we can define a renormalized coupling

$$g_H^2(r) = (H-h_0)/h_1. \tag{13.11}$$

For perturbative purposes one often uses a renormalized three-gluon vertex with all legs at a given scale of momentum, representing the inverse of the scale r, and with a gauge fixing imposed.

In the continuum limit it should be possible to re-express physical observables such as H in terms of renormalized quantities. The renormalized perturbation expansion then takes the form

$$H(r, r', g_R(r')) = h_0+h_1 g_R^2+O(g_R^4). \tag{13.12}$$

Here r represents the scale on which H is defined and r' is the scale used to define the renormalized coupling. In general the coefficients in this series will differ from those in eq. (13.10); however to second order they agree. As r' is selected for convenience, changing its value should not alter real physical observables. This gives rise to the usual continuum renormalization-group equation

$$r'(\mathrm{d}/\mathrm{d}r')\,H(r,r',g_R(r')) = 0 = r'(\partial/\partial r')\,H+\gamma_R(g_R)\,H. \quad (13.13)$$

Here we have introduced the renormalized renormalization-group function

$$\gamma_R(g_R) = r(\mathrm{d}/\mathrm{d}r)\,g_R(r). \quad (13.14)$$

We can now draw a remarkable connection between this renormalization-group function and the one defined earlier for the bare coupling. When the cutoff is still in place, g_R is a function of the scale r, the cutoff a, and the bare coupling g_0

$$g_R = g_R(r,a,g_0).$$

However, since we are working with a dimensionless coupling, g_R can depend directly on r and a only through their ratio. This simple application of dimensional analysis implies

$$r(\partial/\partial r)\,g_R = -a(\partial/\partial a)\,g_R. \quad (13.15)$$

Now, as we renormalize the theory, g_R should become a function of r alone as a goes to zero, and we have

$$a(\partial g_R/\partial a)+(\partial g_R/\partial g_0)\,a(\partial g_0/\partial a) = 0. \quad (13.16)$$

Using the above equations and an analysis similar to that in eq. (13.7), we find

$$\gamma_R(g_R) = \gamma_0\, g_R^3+\gamma_1\, g_R^5+O(g_R^7). \quad (13.17)$$

The renormalized and bare γ functions have the same first two terms in their perturbative expansions. Indeed, it was through consideration of the renormalized coupling that γ_0 and γ_1 were first calculated.

Far from the weak coupling region, there is no simple relationship between the bare and renormalized γ functions. Perverse definitions (or not so perverse; see problem 1) of the renormalized coupling can lead to zeros in γ_R which have no counterpart in the bare quantities.

The perturbative expansion of γ_R has important experimental consequences. If we consider the continuum limit to be taken at $g_0 = 0$, and if g_R is ever small enough that the first terms dominate in eq. (13.17), then the renormalized coupling itself will be driven to zero as r becomes small. Not only does the bare coupling vanish, but any effective coupling becomes arbitrarily weak when the scale of measurement decreases. This is the physical implication of asymptotic freedom; phenomena involving only short-distance effects may be accurately described with the perturbative expansion. Indeed, asymptotically free gauge theories were first invoked for the strong interactions as an explanation of the apparently free parton behavior manifested in the structure functions of deeply inelastic scattering of leptons from hadrons.

Returning to the bare renormalization-group function, we wish to investigate how rapidly g_0 decreases with cutoff. Separating the variables in the form

$$\frac{dg_0}{\gamma_0 g_0^3 + \gamma_1 g_0^5 + O(g_0^7)} = d(\log a), \tag{13.18}$$

we can integrate to obtain the result

$$g_0^{-2} = \gamma_0 \log(a^{-2}\Lambda_0^{-2}) + (\gamma_1/\gamma_0)\log(\log(a^{-2}\Lambda_0^{-2}) + O(g_0^2)). \tag{13.19}$$

Here the parameter Λ_0 represents a constant of integration. This equation indicates the well-known logarithmic decrease of the coupling with scale. The subscript on Λ_0 is to remind us that it has been defined from the bare charge and with the Wilson lattice cutoff. For the renormalized coupling, this equation should be rewritten for g_R with the cutoff a replaced by r and with a possibly different integration constant Λ_R.

The constant appearing upon integration of the renormalization-group equation represents a yardstick for measurement of the scales of the strong interactions. Its value is scheme dependent as can be seen by considering two different bare couplings related as in eq. (13.6). From the analog of eq. (13.19) for g_0' with its own Λ_0', we see

$$\log(\Lambda_0'^2/\Lambda_0^2) = c/\gamma_0, \tag{13.20}$$

where c is the parameter appearing in eq. (13.6). Thus, perturbation theory relates the values of Λ_0 in two different schemes. Furthermore, this requires only a one-loop calculation even though two loops were needed to define Λ_0 through eq. (19).

Hasenfratz and Hasenfratz (1980) were the first to perform the necessary one-loop calculations to relate Λ_0 and Λ_R. Defining the renormalized coupling from the three-gluon vertex in the Feynman gauge and with all legs carrying momentum $\mu^2 = r^{-2}$, they found

$$\Lambda_R/\Lambda_0 = \begin{cases} 57.5, SU(2) \\ 83.5, SU(3), \end{cases} \tag{13.21}$$

for the pure gauge theory. Note that not only is Λ scheme dependent, but that different definitions can vary by rather large factors. The original calculation of these numbers was rather tedious, involving intermediate definitions of the coupling and evaluation of one-loop diagrams with the lattice regulator. These numbers have been verified with calculationally more efficient techniques based on a study of the quantum fluctuations around a slowly varying classical background field (Dashen and Gross, 1981). These calculations have been extended to other lattice actions and to theories with fermions (Weisz, 1981; Kawai, Nakayama and Seo, 1981).

We have been discussing the bare coupling as a function of the lattice spacing. A useful alternative considers the coupling as a parameter which determines the cutoff. Inverting eq. (13.19), we have

$$a = \Lambda_0^{-1}(g_0^2\gamma_0)^{-\gamma_1/(2\gamma_0^2)} \exp(-1/(2\gamma_0 g_0^2))(1+O(g_0^2)). \qquad (13.22)$$

Note the essential singularity at vanishing bare coupling. The perturbative renormalization group is about to give us non-perturbative information. Multiplying by the corresponding mass, we can obtain the weak coupling dependence of a correlation length on the lattice

$$ma = \xi^{-1} = (m/\Lambda_0)(g_0^2\gamma_0)^{-\gamma_1/(2\gamma_0^2)} \exp(-1/(2\gamma_0 g_0^2))(1+O(g_0^2)). \quad (13.23)$$

If m is the mass of a physical particle and remains finite in the continuum limit, then its value in units of Λ_0 is given by the coefficient of the weak coupling dependence indicated in eq. (13.23).

For the above discussion we could elect to work with the correlation length between operators which select any desired set of quantum numbers, such as spin, parity, etc. Thus the mass of any particle in units of Λ_0 is the coefficient of the weak coupling dependence of some correlation function, as in eq. (13.23). Furthermore, Λ_0 is universal, determined solely by the initial cutoff scheme. It will drop out of any dimensionless ratio of masses, which is then determined uniquely by the theory. This brings us to the remarkable conclusion that for pure gauge fields the strong interactions have no free parameters. The cutoff is absorbed into $g_0(a)$, which in turn is absorbed into the renormalization-group dependence of eq. (13.23). The only remaining dimensional parameter is Λ_0, which merely sets the scale for all other masses. In a theory considered in isolation, one may define Λ_0 to be unity. Coleman and Weinberg (1973) have given this process, wherein a dimensionless parameter g_0 and a dimensionful one a manage to 'eat' each other, the marvelous name 'dimensional transmutation'.

In the theory including quarks, their masses represent new parameters. Indeed these are the only parameters in the theory of the strong interactions. In the limit where the bare quark masses vanish, referred to as the chiral limit, we return to a zero parameter theory. In this approximation to the physical world, the pion mass is expected to vanish and all dimensionless observables should be uniquely determined by the theory. This applies not only to mass ratios, such as of the rho mass to the proton, but as well to quantities such as the pion–nucleon coupling constant, once regarded as a parameter for a perturbative expansion. As the chiral approximation has been rather successful in the predictions of current algebra, we hope that eventually we may develop the techniques to calculate these quantities. If they seriously disagree with experiment, the theory is wrong because there are no parameters to adjust. Given a qualitative agreement, a fine tuning of the small quark masses should give the pion its mass and complete the theory.

The exciting idea of a parameter-free theory is sadly lacking from treatments of the other interactions such as electromagnetism or the weak force. There the coupling $\alpha = 1/137$ is treated as a parameter. One might optimistically hope for inclusion of the appropriate non-perturbative ideas into a grand unified scheme ultimately rendering α and the quark and lepton masses calculable.

The renormalization group is indeed a rich subject. We have only touched on a few uses which we will find valuable in later chapters. Perhaps the most remarkable result of this chapter is that a perturbative analysis of the renormalization-group function can give important non-perturbative conclusions, such as eq. (13.23).

Problems

1. Define $g_R^2(r)$ to be proportional to r^2 times the force between two quarks separated by a distance r. Argue that the corresponding renormalization-group function in the full theory of strong interactions including quark loops must exhibit a zero at non-vanishing g_R.

2. Show that the γ_1 term in eq. (13.19) is needed to properly define Λ_0.

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