

ON RINGS WHOSE RIGHT ANNIHILATORS ARE BOUNDED

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Abstract. Jacobson said a right ideal would be called *bounded* if it contained a non-zero ideal, and Faith said a ring would be called *strongly right bounded* if every non-zero right ideal were bounded. In this paper we introduce a condition that is a generalisation of strongly bounded rings and insertion-of-factors-property (IFP) rings, calling a ring *strongly right AB* if every non-zero right annihilator is bounded. We first observe the structure of strongly right AB rings by analysing minimal non-commutative strongly right AB rings up to isomorphism. We study properties of strongly right AB rings, finding conditions for strongly right AB rings to be reduced or strongly right bounded. Relating to Ramamurthi's question (i.e. Are right and left SF rings von Neumann regular?), we show that a ring is strongly regular if and only if it is strongly right AB and right SF, from which we may generalise several known results. We also construct more examples of strongly right AB rings and counterexamples to several naturally raised situations in the process.

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1. Introduction. Throughout this paper all rings are associative with identity unless otherwise stated, and all modules are unitary. Let R be a ring. The prime radical, the set of all nilpotent elements and the Jacobson radical of R are denoted by $P(R)$, $N(R)$ and $J(R)$, respectively. For any non-empty subset X of R , $r_R(X)$ and $\ell_R(X)$ denote the right annihilator and the left annihilator of X in R , respectively. If X is singleton, say $X = \{a\}$, then we write $r_R(a)$ ($\ell_R(a)$) in place of $r_R(\{a\})$ ($\ell_R(\{a\})$). We use $Mat_n(R)$ to denote the $n \times n$ matrix ring over R . An element c of R is called *right regular* if $r_R(c) = 0$, *left regular* if $\ell_R(c) = 0$ and *regular* if $r_R(c) = 0 = \ell_R(c)$. A zero divisor means an element whose left and right annihilators are both non-zero. A ring is called a *domain* if every non-zero element is regular. It is well known that for a division ring D , a matrix in $Mat_n(D)$ is regular if and only if it is invertible.

The notion of bounding a one-sided ideal by a two-sided ideal goes back at least to Jacobson [21]. He said that a right ideal of R is *bounded* if it contains a non-zero ideal of R . This concept has been extended in several ways. From Faith [13], a ring R is called *strongly right* (resp. *left*) *bounded* if every non-zero right (resp. left) ideal of R

contains a non-zero ideal. A ring is called *strongly bounded* if it is both strongly right and strongly left bounded. The class of strongly one-sided bounded rings has been observed by many authors [e.g. 7, 21, 34, 35, 37].

Due to Bell [6], a right (or left) ideal I of a ring R is said to have the *insertion-of-factors-property* (simply, *IFP*) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So we shall call a ring R an *IFP ring* if the zero ideal of R has the IFP. Shin [32] used the term *SI* for the IFP, while Narbonne [11] called IFP rings *semi-commutative*. A ring is called *reduced* if it has no non-zero nilpotent elements. A ring is called *abelian* if each idempotent is central. Simple computations give that reduced rings are IFP, and IFP rings are abelian. Note that a ring R is IFP if and only if any right annihilator is an ideal if and only if any left annihilator is an ideal if and only if $ab = 0$ implies $aRb = 0$ for $a, b \in R$ [32, Lemma 1.2].

Due to Feller [14], a ring is *right* (resp. *left*) *duo* if every right (resp. left) ideal is an ideal. Right (resp. left) duo rings are both strongly right (resp. left) bounded and IFP. A ring is called *duo* if it is both right and left duo. Birkenmeier and Tucci [7, Proposition 6] showed that a ring R is right duo if and only if R/T is strongly right bounded for all ideals T of R . For IFP rings we have a similar result in the following.

REMARK. For a ring R the following conditions are equivalent:

- (1) R is IFP;
- (2) for any right annihilator A and any ideal I in R with $I \subseteq A$, A/I is bounded in R/I whenever $A/I \neq 0$;
- (3) for any left annihilator B and any ideal J in R with $J \subseteq B$, B/J is bounded in R/J whenever $B/J \neq 0$.

Proof. (1) \Rightarrow (2) and (1) \Rightarrow (3) are obvious.

(2) \Rightarrow (1): Assume on the contrary that there is a right annihilator A with $RA \not\subseteq A$. We apply the proving method of [7, Proposition 6]. Let H be the sum of all ideals contained in A . Then $A/H \neq 0$, and then A/H is bounded; hence there is a non-zero ideal K/H of R/H contained in A/H , a contradiction. The proof of (3) \Rightarrow (1) is similar. \square

Based on this remark we introduce the following concepts which generalise strongly one-sided bounded rings and IFP rings. A ring R is called *right* (resp. *left*) *AB* if every essential right (resp. left) annihilator of R is bounded; i.e. it contains a non-zero ideal of R . A ring R is called *strongly right* (resp. *left*) *AB* if every non-zero right (resp. left) annihilator of R is bounded; R is called (*strongly*) *AB* if R is (strongly) right and (strongly) left AB. Obviously strongly right bounded rings and IFP rings are both strongly right AB, but the converses need not be true in either case as we see in the next section. We in this paper concentrate on strongly one-sided AB rings.

We study the properties of strongly one-sided AB rings and the relationship between strongly one-sided AB rings and related rings. As an application of strongly one-sided AB rings, we observe the von Neumann regularity of SF rings. A ring R is called *von Neumann regular* (simply *regular*) if for any $a \in R$, $a = aba$ for some $b \in R$. A ring R is called *strongly regular* if for any $a \in R$, $a = a^2b$ for some $b \in R$. A ring is strongly regular if and only if it is abelian regular if and only if it is reduced regular if and only if it is (one-sided) duo regular [15, Theorems 3.2 and 3.5]. A ring R is called a *right* (resp. *left*) *SF* if every simple right (resp. left) R -module is flat. A ring R is regular if and only if all right (left) R -modules are flat [15, Corollary 1.3].

Ramamurthi [29] initiated the study of the von Neumann regularity of SF rings, asking whether a right and left SF ring is necessarily von Neumann regular. This question has drawn the attention of many authors [e.g. 9, 16, 30, 33, 37, 38]. For example, Zhang and Du [38] proved that a ring is strongly regular if and only if it is strongly right bounded and right SF. Also Ramamurthi [29] proved that a ring is strongly regular if and only if it is IFP and right SF. Some affirmative situations have been found, but the question remains open.

We lastly extend the class of strongly one-sided AB rings, concerning some conditions and some ring extensions.

2. Basic forms of strongly one-sided AB rings. In this section we study basic examples of strongly one-sided AB rings. We first construct a typical example of strongly one-sided AB rings. Use E_{ij} for the matrix with (i, j) -entry 1 and elsewhere 0. Given any ring R we consider a ring extension

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & 0 & a & \cdots & a_{3(n-1)} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

where $n(\geq 2)$ is a positive integer.

LEMMA 2.1. *Let R be a ring and $0 \neq a \in R$. Then a is right (resp. left) regular if and only if*

$$\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n$$

is right (resp. left) regular.

Proof. Suppose that $a \in R$ is right regular, and let

$$A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n.$$

We proceed by induction on n . Put $\begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \begin{pmatrix} b & b_{12} \\ 0 & b \end{pmatrix} = 0$ for some $\begin{pmatrix} b & b_{12} \\ 0 & b \end{pmatrix} \in R_2$. Then $ab = 0$ and $ab_{12} + a_{12}b = 0$. Since a is right regular, $b = 0$, and so $b_{12} = 0$; hence $\begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix}$ is right regular. Next let

$$AB = 0 \text{ for } B = \begin{pmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \in R_n.$$

Then we get

$$\begin{pmatrix} a & a_{12} & \cdots & a_{1(n-1)} \\ 0 & a & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b & b_{12} & \cdots & b_{1(n-1)} \\ 0 & b & \cdots & b_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} = 0.$$

By the induction hypothesis we obtain $b = 0$ and $b_{ij} = 0$ for $1 \leq i, j \leq n - 1$; hence $ab_{1n} + a_{12}b_{2n} + \cdots + a_{1(n-1)}b_{(n-1)n} = 0$, $ab_{2n} + a_{23}b_{3n} + \cdots + a_{2(n-2)}b_{(n-2)n} + a_{2(n-1)}b_{(n-1)n} = 0$, \dots , $ab_{(n-3)n} + a_{(n-3)(n-2)}b_{(n-2)n} + a_{(n-3)(n-1)}b_{(n-1)n} = 0$, $ab_{(n-2)n} + a_{(n-2)(n-1)}b_{(n-1)n} = 0$, $ab_{(n-1)n} = 0$. From $ab_{(n-1)n} = 0$ we get $b_{(n-1)n} = 0$, and then from $ab_{(n-2)n} + a_{(n-2)(n-1)}b_{(n-1)n} = 0$ we also get $b_{(n-2)n} = 0$. Inductively we obtain $b_{in} = 0$ for $i = 1, 2, \dots, n - 1$, concluding that A is right regular. Conversely assume that A is right regular, and let $ab = 0$ for some $b \in R$. Then from $A(bE_{1n}) = 0$ we have $b = 0$. Thus a is regular. The proof of left case is similar. \square

With the help of Lemma 2.1 we construct a typical example of strongly one-sided AB rings in the following.

THEOREM 2.2. *A ring R is strongly right (resp. left) AB if and only if R_n is strongly right (resp. left) AB for any $n \geq 2$.*

Proof. Suppose that R is strongly right AB and $X \subseteq R_n$ with $r_{R_n}(X) \neq 0$. Then any diagonal in matrices in X is not right regular by Lemma 2.1. Let Y be the set of all elements in R , which occur as diagonal entries of elements in X . If $Y = 0$, then $r_{R_n}(X)$ contains a non-zero ideal RE_{1n} of R_n . Next we suppose $Y \neq 0$ and let a be any in Y . Take

$$0 \neq \begin{pmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \text{ in } r_{R_n}(X).$$

We will show $r_R(Y) \neq 0$.

(Case 1) If $b \neq 0$, then $r_R(Y) \neq 0$;

(Case 2) Assume $b = 0$, and proceed by induction on n .

Set $n = 2$. Then clearly $0 \neq b_{12} \in r_R(Y)$.

Set $n = 3$. If $b_{23} \neq 0$, then we are done. If $b_{23} = 0$, then we get $ab_{13} = 0$. Here if $b_{13} \neq 0$, then we are done; otherwise we get $0 \neq b_{12} \in r_R(Y)$. Consequently $r_R(Y) \neq 0$. Set

$$\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \begin{pmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} = 0 \text{ for } \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in X.$$

If $b_{(n-1)n} \neq 0$, then we are done; otherwise we have $ab_{(n-2)n} = 0$. Check whether $b_{(n-2)n} = 0$. Inductively we can check whether there is a non-zero $b_{in} \in r_R(Y)$ for some $i \in \{1, 2, \dots, n - 1\}$; if every b_{in} is zero, then there is a non-zero $b_{ij} \in r_R(Y)$

for $i \in \{1, 2, \dots, n - 2\}$ and $j \in \{1, 2, \dots, n - 1\}$, by the induction hypothesis. Thus $r_R(Y) \neq 0$.

Now since R is strongly right AB, there is a non-zero ideal I of R with $I \subseteq r_R(Y)$. Then $r_{R_n}(X)$ contains a non-zero ideal IE_{1n} of R_n . Thus R_n is strongly right AB.

Conversely suppose that R_n is strongly right AB and $V \subseteq R$ with $r_R(V) \neq 0$. Let $W = \{aI_n \mid a \in V\} \subseteq R_n$, where I_n is the $n \times n$ identity matrix. Then $r_{R_n}(W) \neq 0$ because $WU = 0$ for any non-zero matrix U in R_n with entries in $r_R(V)$. Since R_n is strongly right AB, there exists a non-zero ideal J of R_n such that $r_{R_n}(W) \supseteq J$. Set $K = \{c \in R \mid c \text{ is an entry of a matrix in } J\}$. Then K is a non-zero ideal of R from the computations $(aI_n)(rI_n)C = 0$ and $(aI_n)C(rI_n) = 0$ for $a \in V, r \in R$ and $C \in J$. Moreover $aK = 0$ for all $a \in V$ from $(aI_n)J = 0$, entailing $r_R(X) \supseteq K$. Thus R is strongly right AB. The proof of the left case is similar. \square

REMARK. Let U and L be the $n \times n$ upper and lower triangular matrix rings over a ring, respectively; U and L are isomorphic, via the correspondence $(a_{ij}) \mapsto (b_{st})$ with $b_{st} = a_{(n-s+1)(n-t+1)}$; R_n and R'_n are also isomorphic under the preceding correspondence, where R'_n is the image of R_n . Thus every property obtained for U (resp. R_n) is also true for L (resp. R'_n).

We use $GF(p^n)$ to denote the Galois field of order p^n . Given a ring R we denote the order of R and the characteristic of R by $|R|$ and $Ch(R)$, respectively. Strongly right bounded rings and IFP rings are both strongly right AB, but the converse need not be true as we see in the following.

EXAMPLE 2.3. (1) Let R be any ring and M be the 2×2 matrix ring over R . Each of $r_M(E_{11}) = RE_{21} = RE_{22}$ and $\ell_M(E_{11}) = RE_{12} + RE_{22}$ cannot contain a non-zero ideal of M . So M is neither strongly right nor strongly left AB. Let U be the 2×2 upper triangular matrix ring over R . Each of $r_U(E_{11}) = RE_{22}$ and $\ell_U(E_{22}) = RE_{11}$ cannot contain a non-zero ideal of U . So U is neither strongly right nor strongly left AB. By similar computations, any $n \times n$ (upper or lower triangular) matrix ring is neither strongly right nor strongly left AB.

(2) There exists a strongly right AB ring that is not strongly right bounded. Let R be a strongly right AB ring and consider R_3 over R ; R_3 is strongly right AB by Theorem 2.2, but it is not strongly right bounded by the following: $E_{23}R_3 = RE_{23}$ and $R_3E_{12} = RE_{12}$ cannot contain any non-zero ideal of R_3 . By this computation, R_n ($n \geq 3$), over any ring R , is neither strongly right nor strongly left bounded.

(3) There exists a strongly right AB ring that is not IFP. Let R be a strongly right AB ring and consider R_4 over R ; R_4 is strongly right AB by Theorem 2.2, but it is not IFP by [23, Example 1.3].

(4) There are no containing relations between the classes of strongly right bounded rings and IFP rings. Let R be a reduced ring, and consider R_3 over R ; R_3 is IFP by [23, Proposition 1.2], but it is not strongly right bounded by the computation in (2).

Next let $R = \begin{pmatrix} GF(2) & GF(2) \\ 0 & \mathbb{Z} \end{pmatrix}$, where \mathbb{Z} is the ring of integers. Note that each non-zero right ideal of R is one of the following: $\begin{pmatrix} GF(2) & GF(2) \\ 0 & n\mathbb{Z} \end{pmatrix}, \begin{pmatrix} GF(2) & GF(2) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & GF(2) \\ 0 & n\mathbb{Z} \end{pmatrix}, \begin{pmatrix} 0 & GF(2) \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & n\mathbb{Z} \end{pmatrix}$ for $n = 1, 2, \dots$. Since $n\mathbb{Z}E_{22}$ contains an ideal $2n\mathbb{Z}E_{22}$ of R and remainders are two-sided, R is strongly right bounded. However $\mathbb{Z}E_{22} = r_R(E_{11})$ is not two-sided, and thus R is not IFP.

(5) There is a domain that is neither strongly right nor strongly left bounded ring. Let R be the first Weyl algebra over a field F of characteristic zero. Then R is a simple right Noetherian domain which is not a division ring. Thus R is reduced, and so R

is strongly AB. There is $0 \neq a \in R$ such that $aR \neq R$, since R is not a division ring. Assume that there exists a non-zero ideal I of R such that $I \subseteq bR$. Then we have $R = I \subseteq bR$, a contradiction; hence R is not strongly right bounded. Similarly R is also not strongly left bounded.

We next observe the basic structure of strongly one-sided AB rings in the following several results.

The following results are obtained from Eldridge [12].

LEMMA 2.4. (1) [12, Proposition] *Let R be a finite non-commutative ring. If the order of R is p^3 , with p being a prime, then R is isomorphic to the 2×2 upper triangular matrix ring over $GF(p)$.*

(2) [12, Theorem] *Let R be a finite ring of order m . If m has a cube-free factorisation, then R is a commutative ring.*

EXAMPLE 2.5. (1) By Lemma 2.4(1, 2) every minimal non-commutative ring is isomorphic to the 2×2 upper triangular matrix ring over $GF(2)$. So by Theorem 2.2 and Example 2.3(1, 2), R_3 over $GF(2)$ is a minimal non-commutative strongly AB ring that is neither strongly right nor strongly left bounded.

(2) There is a strongly right bounded ring but not right duo. We refer to the ring in [34, Remark 4]. Let S and T be the 2×2 upper and lower triangular matrix rings over $GF(p)$ respectively. Set R be the subring of $S \oplus T$ consisting of all the elements of the form $\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ d & b \end{pmatrix}\right)$. Then R is strongly right bounded but not right duo as can be seen by the right ideal of R generated by (E_{11}, E_{11}) . Note that R is a minimal non-commutative strongly right (hence left by [34, Proposition 2]) bounded ring of order 16 when $p = 2$, with the help of Example 2.3(1) and Lemma 2.4.

(3) We refer to the three rings in [35, Example 2] as follows: Let $S\{x, y\}$ be the free algebra generated by x, y over a ring S .

Set $B_1 = GF(2)\{x, y\}/(x^3, y^3, yx, x^2 - xy, y^2 - xy)$, where $(x^3, y^3, yx, x^2 - xy, y^2 - xy)$ is the ideal of $GF(2)\{x, y\}$ generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$.

Let $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be the ring of integers modulo 4 and

$$B_2 = \mathbb{Z}_4\{x, y\}/(x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y),$$

where $(x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y)$ is the ideal of $\mathbb{Z}_4\{x, y\}$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y$.

Let B_3 be the ring of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix}$ over $GF(2^2)$.

Then each B_i is a non-commutative duo ring of order 16 by [34, Proposition 3]. Note that $Ch(B_1) = 2 = Ch(B_3)$, $Ch(B_2) = 4$ and $|J(B_1)| = 8$, $|J(B_3)| = 4$. Thus $B_1 \not\cong B_2$, $B_1 \not\cong B_3$ and $B_2 \not\cong B_3$. Any minimal non-commutative duo ring is isomorphic to B_i for some $i = 1, 2, 3$ by [35, Theorem 3].

(4) Consider two subrings of the 4×4 matrix ring over $GF(p^n)$. Define

$$C_1 = \left\{ \begin{pmatrix} a & 0 & 0 & c \\ 0 & a & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & b \end{pmatrix} \mid a, b, c, d \in GF(p^n) \right\}$$

and

$$C_2 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ c & 0 & d & b \end{pmatrix} \mid a, b, c, d \in GF(p^n) \right\};$$

$$\begin{pmatrix} 0 & 0 & 0 & GF(p^n) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & GF(p^n) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are non-zero ideals of C_1 ; hence every non-zero right annihilator contains at least one of them, and so C_1 is strongly right AB. However

$$\ell_{C_1} \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a \in GF(p^n) \right\}$$

is a minimal left ideal of C_1 that is not an ideal, and so it is not strongly left AB. Moreover C_1 is not strongly right bounded, since

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} C_1$$

is a minimal right ideal of C_1 that is not an ideal.

By a similar computation we get that C_2 is strongly left AB but neither strongly right AB nor strongly left bounded.

In the next theorem we see all minimal non-commutative strongly right AB rings up to isomorphism, with the help of [35, Theorem 3].

THEOREM 2.6. (1) *If R is a minimal non-commutative strongly right AB ring, then R is isomorphic to A_k for some $k \in \{1, 2, 3, 4, 5, 6\}$, where A_i is the ring B_i in Example 2.5(3) for $i = 1, 2, 3$; A_4 is R_3 over $GF(2)$; A_5 is the ring R in Example 2.5(2); $A_6 = C_1$ with $p^n = 2$ in Example 2.5(4).*

(2) *If R is a minimal non-commutative strongly left AB ring, then R is isomorphic to A_k for some $k \in \{1, 2, 3, 4, 5, 6\}$, where A_i is the ring B_i in Example 2.5(3) for $i = 1, 2, 3$; A_4 is R_3 over $GF(2)$; A_5 is the ring R in Example 2.5(2); $A_6 = C_2$ with $p^n = 2$ in Example 2.5(4).*

Proof. (1) Any minimal non-commutative strongly right AB ring must be of order 16 by Theorem 2.2 (consider R_3 over $GF(2)$), Example 2.3(1) and Lemma 2.4(1, 2). Let R be a minimal non-commutative strongly right AB ring.

(The case of local rings) We have three cases of $|J(R)| = 2, |J(R)| = 4$ or $|J(R)| = 8$. Then $R/J(R)$ is clearly a field, and by the argument in [35, Theorem 3], $J(R)$ is a vector space over $R/J(R)$. If $|J(R)| = 2$, then $\frac{R}{J(R)} \cong GF(2^3)$, and so $|J(R)| \geq 8$, a contradiction. So have two cases of $|J(R)| = 4$ and $|J(R)| = 8$.

Suppose $|J(R)| = 8$. Then $\frac{R}{J(R)} \cong GF(2)$, and by [25, Theorem 2.3.6], $J(R)$ has a basis $\{a, b, c\}$ over $GF(2)$ such that $a^2 = ab = c, ba = 0, b^2 = 0$, or $a^2 = ab = c, ba = 0, b^2 = c$.

Assume that $a^2 = ab = c, ba = 0, b^2 = c$. Then [35, Theorem 3] implies that $R \cong A_1$ if $Ch(R) = 2$ and $R \cong A_2$ if $Ch(R) = 4$.

Assume that $a^2 = ab = c, ba = 0, b^2 = 0$. Then bR is a minimal right ideal but not an ideal, and so R is not strongly right bounded. Consider A_4 . Then

$$J(A_4) = \begin{pmatrix} 0 & GF(2) & GF(2) \\ 0 & 0 & GF(2) \\ 0 & 0 & 0 \end{pmatrix}$$

and $R \cong A_4$, letting

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose $|J(R)| = 4$. Then $\frac{R}{J(R)} \cong GF(2^2)$, and by [35, Theorem 3] $R \cong A_3$.

(The case of non-local rings) We apply in part the proof of [34, Proposition 2]. First assume $R = A \oplus B$; then $|A| = 2^3$ or $|B| = 2^3$ by Lemma 2.4(2), since R is non-commutative. But by Lemma 2.4(1) and Example 2.3(1), any case is impossible; hence R is indecomposable. Since R is finite and non-local, R contains a non-trivial idempotent e (i.e. $e \neq 0$ and $e \neq 1$). Put $f = 1 - e$ and consider the Pierce decomposition $R = \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$.

Suppose that $eRf \neq 0$ and $fRe \neq 0$. Then $|eRe| = |eRf| = |fRe| = |fRf| = 2$. If $eRfRe \neq 0$, then $eRfRe = eRe$ and $r_R\left(\begin{pmatrix} 0 & eRf \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} eRe & eRf \\ 0 & 0 \end{pmatrix}$ cannot contain a non-zero ideal of R , a contradiction to the strong right ABness of R . If $fReRf \neq 0$, then we also have a contradiction by a similar method. Consequently $eRfRe = 0 = fReRf$. Then $\begin{pmatrix} 0 & eRf \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ fRe & 0 \end{pmatrix}$ are ideals of R ; hence R is strongly right bounded. In this case $R \cong A_5$ when $p = 2$ with $e = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$ and $f = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

Suppose that $eRf \neq 0$ and $fRe = 0$, i.e. $R = \begin{pmatrix} eRe & eRf \\ 0 & fRf \end{pmatrix}$. If $|fRf| = 2$, then $r_R\left(\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & fRf \end{pmatrix}$, but this is a minimal right ideal of R that does not contain a non-zero ideal of R , a contradiction to the strong right ABness of R . So we must assume $|fRf| = 2^2$. If fRf is decomposable, say $fRf = D_1 \oplus D_2$, then $eRfD_i \neq 0$ for some $i = 1, 2$ because $eRf \neq 0$, say $eRfD_1 \neq 0$. Hence R has a minimal right ideal $r_R\left(\begin{pmatrix} e & 0 \\ 0 & D_2 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}$ that is not an ideal, and thus R is not strongly right AB. Consequently fRf is a local ring. But $|J(fRf)| = 2$. For, if $J(fRf) = 0$, then fRf is a field and $|eRf| \geq 4$, since eRf is a vector space over fRf , a contradiction. Note $J(fRf)^2 = 0$. For if $J(fRf)^2 \neq 0$, then $J(fRf)^2 = J(fRf)$, and so $J(fRf)$ is non-nilpotent, a contradiction. Since $\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}R$ is a minimal right ideal of R but not ideal, R is not strongly right bounded, where $0 \neq x \in eRf$ and $0 \neq y \in J(fRf)$. In this case $R \cong A_6$ with

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2) The proof is similar to that of (1). □

REMARK. The proof of the case of non-local rings in Theorem 2.6 is applicable for any prime p in place of 2.

COROLLARY 2.7. *If R is a minimal non-commutative strongly right bounded ring, then R is isomorphic to A_i for some $i \in \{1, 2, 3, 5\}$, where A_i denotes the rings in Theorem 2.6.*

Proof. In the proof of Theorem 2.6, A_i is right duo for $i \in \{1, 2, 3\}$; A_5 is strongly right bounded; and the rings A_4, A_6 are not strongly right bounded. □

3. Properties of strongly one-sided AB rings and related rings. In this sections we study the properties of strongly one-sided AB rings and the relationship between strongly one-sided AB rings and related rings. As an application of strongly one-sided AB rings, we observe the von Neumann regularity of right SF rings under some specific conditions (e.g. strong right ABness).

LEMMA 3.1. *Let R_i be a ring for $i \in I$ and D be the direct product of R_i 's.*

(1) *Every R_i is strongly right (resp. left) AB if and only if D is strongly right (resp. left) AB. This result also holds for direct sums (possibly without identity).*

(2) *Every R_i is strongly right (resp. left) bounded if and only if D is strongly right (resp. left) bounded. This result also holds for direct sums (possibly without identity).*

Proof. (1) Suppose that R_i is strongly right AB for each $i \in I$. Let $X \subseteq D$ with $r_D(X) \neq 0$ and Y_i be the set of all the i th components of the sequences in X . From $r_D(X) \neq 0$ there is $j \in I$ such that $r_{R_j}(Y_j) \neq 0$. Since R_j is strongly right AB there is a non-zero ideal K of R_j with $K \subseteq r_{R_j}(Y_j)$. Set H be the subset of D such that $e_j H = K$ and $e_k H = 0$ for $k \neq j$, where e_i is the sequence in D with i th component 1_{R_i} and elsewhere zero. Since H is a non-zero ideal of D with $XH = 0$, D is strongly right AB. Conversely suppose that D is strongly right AB and R_j is not strongly right AB for some $j \in I$. Take $X \subseteq R_j$ such that $r_{R_j}(X)$ is non-zero, and this does not contain any non-zero ideal of R_j . Consider $Y \subseteq D$ such that $e_j Y = X$ and $e_k Y = \{1_{R_k}\}$ for $k \neq j$. Then $e_j r_D(Y) = r_{R_j}(X)$ and $e_k r_D(Y) = 0$ for $k \neq j$; hence $r_D(Y)$ does not contain any non-zero ideal of D , a contradiction. The direct sum case and the left case can be proved similarly. The proof of (2) is similar to (1). □

From Lemma 3.1 it is natural to ask whether the class of strongly one-sided AB rings is closed under subrings and factor rings. However the answer is negative by the following.

EXAMPLE 3.2. Let R be the strongly right AB ring C_1 in Example 2.5(4), and consider the subring

$$S = \left\{ \left(\begin{array}{cccc} a & 0 & 0 & c \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{array} \right) \mid a, b, c \in GF(p^n) \right\}$$

of R . Then S is not strongly right AB because

$$r_S \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \mid b \in GF(p^n) \right\}$$

is a minimal right ideal of S that is not an ideal. Next consider the factor ring R/I by the ideal

$$I = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid d \in GF(p^n) \right\};$$

R/I is isomorphic to the preceding ring S and so is not strongly right AB. The left cases can be shown similarly.

The IFPness is left–right symmetric, but the strong ABness is not left–right symmetric by Example 2.5(4) in spite of being finite rings. The strong boundedness and duoness are also not left–right symmetric by the following.

EXAMPLE 3.3 (1) Let R be the ring in Example 2.3(4). Then R is strongly right bounded. But $\ell_R \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \left(\begin{matrix} GF(2) & 0 \\ 0 & 0 \end{matrix} \right)$ cannot contain a right ideal of R , concluding that R is not strongly left AB (hence not strongly left bounded).

(2) Let F be a field and $F(x)$ be the field of rational functions over F . Due to Chatters and Xue [8], set $R = F(x) \oplus F(x)$ to be an additive group with the multiplication $(f_1(x), g_1(x))(f_2(x), g_2(x)) = (f_1(x)f_2(x), f_1(x^2)g_2(x) + g_1(x)f_2(x))$. Then R is right duo but not left duo by the computation in the example in [8].

In the following lemma we check some conditions, obtained by Chatters, Courter and Xue, under which the strong boundedness and duoness can be left–right symmetric. We denote by $c(M)$ the composition length of a module M over a ring R .

LEMMA 3.4 (1) [34, Proposition 3] *A finite right duo ring is left duo.*

(2) [10, Theorem 2.2] *A right Artinian right duo ring R is left duo if $c(R_R) = c({}_R R)$.*

(3) [10, Corollary 2.3] *A right duo ring is left duo when it is a finite-dimensional algebra over a field.*

(4) [34, Theorem] *If R is a strongly right bounded finite ring and $|R|$ has no factors of the form p^5 , then R is strongly left bounded.*

(5) [34, Proposition 2] *Let R be a strongly right bounded ring with $|R| = p^4$, p a prime. If R is non-local, then R is strongly left bounded.*

(6) *If R is a local ring with nilpotent $J(R)$, then R is strongly AB.*

Proof. (6) Let R be a local ring such that $J(R)^k = 0$ and $J(R)^{k-1} \neq 0$. Let $X \subset R$ with $r_R(X) \neq 0$. Then $X \subseteq J(R)$, and so $XJ(R)^{k-1} = 0$; hence R is strongly right AB. Similarly R is strongly left AB. □

Local rings (especially of order 2^4) with nilpotent Jacobson radical need not be strongly one-sided bounded as can be seen by the ring R_3 over $GF(2)$ in Example 2.3(2). But in the following we see situations for which finite strongly one-sided AB rings can be two-sided.

THEOREM 3.5 (1) *In Theorem 2.6, every A_i is left–right symmetric for $i \in \{1, 2, 3, 4, 5\}$.*

(2) *Let R be a finite non-local strongly right AB ring such that $|R|$ has no factors of the form p^5 . Then R is strongly right bounded if and only if R is strongly left AB.*

Proof. (1) By Lemma 3.4(1) A_i is duo for $i = 1, 2, 3$; A_4 is strongly AB by Lemma 3.4(6); and A_5 is strongly bounded by Lemma 3.4(4) or Lemma 3.4(5).

(2) It is well known that R is a finite direct product of rings each of which has prime power order. Through Lemmas 2.4 and 3.1, we can assume R is of order p^4 with p a prime. If R is strongly right bounded, then it is strongly left bounded by Lemma 3.4(5). Conversely if R is strongly left AB, then R is strongly AB, and so by Theorem 2.6(2) R is isomorphic to the ring R in Example 2.5(2); hence R is strongly (right) bounded. \square

We next find more conditions under which the strong ABness or strong boundedness can be left–right symmetric.

PROPOSITION 3.6. *For a ring R the following conditions are equivalent:*

- (1) *R is reduced;*
- (2) *R is semiprime and strongly right AB;*
- (3) *R is semiprime and strongly left AB;*
- (4) *R is semiprime and IFP.*

Proof. (1) \Rightarrow (4), (4) \Rightarrow (2) and (4) \Rightarrow (3) are clear.

(2) \Rightarrow (1): Assume on the contrary that there exists $0 \neq a \in R$ satisfying $a^2 = 0$. Then $r_R(\ell_R(a))$ is a non-zero right annihilator in R . Since R is strongly right AB, there exists a non-zero ideal I of R such that $I \subseteq r_R(\ell_R(a))$, i.e. $\ell_R(a)I = 0$. We claim that $\ell_R(a) \cap I \neq 0$. If $Ia = 0$, then we are done. If $Ia \neq 0$, then we also have $0 \neq Ia \subseteq \ell_R(a) \cap I$ because $(Ia)a = 0$. However $(\ell_R(a) \cap I)^2 \subseteq \ell_R(a)I = 0$, and so $\ell_R(a) \cap I = 0$, since R is semiprime, a contradiction. Thus R is reduced. The proof of (3) \Rightarrow (1) is similar. \square

A ring is called *PI* if it satisfies a polynomial identity with coefficients in the ring of integers.

PROPOSITION 3.7. *For a PI ring R the following conditions are equivalent:*

- (1) *R is reduced;*
- (2) *R is semiprime and strongly right AB;*
- (3) *R is semiprime and strongly left AB;*
- (4) *R is semiprime and IFP;*
- (5) *R is semiprime and strongly right bounded;*
- (5) *R is semiprime and strongly left bounded.*

Proof. The equivalences of (1)–(4) come from Proposition 3.6; (1) \Rightarrow (5) and (1) \Rightarrow (6) are obtained by [4, Theorem C]; (5) \Rightarrow (2) and (6) \Rightarrow (3) are obvious. \square

The PI condition in Proposition 3.7 is not superfluous. The non-PI ring R in Example 2.3(5) is a domain (hence reduced) which is neither strongly right nor strongly left bounded.

Due to Marks [27], a ring R is called *NI* if $N(R) = N^*(R)$, where $N^*(R)$ is the upper nilradical of R . It is obvious that a ring R is NI if and only if $R/N^*(R)$ is reduced and that IFP rings are NI. A prime ideal P of R is called *completely prime* if R/P is a domain. Hong and Kwak [18, Corollary 13] proved that a ring R is NI if

and only if every minimal strongly prime ideal of R is completely prime. According to Köthe [24], a ring R is called an *I-ring* if every non-nil right ideal of R contains a non-zero idempotent. Algebraic algebras and π -regular rings are I-rings by [22, Proposition 9.4.1]. It is easy to check that Jacobson radicals of I-rings are nil.

LEMMA 3.8 (1) *The definition of I-ring is left–right symmetric.*

(2) *A ring R is an I-ring if and only if for each non-nilpotent element $x \in R$ there exists $0 \neq y \in R$ satisfying $xyx = y$.*

(3) *Let R be an I-ring. If R is strongly one-sided AB, then $N(R) = J(R)$.*

(4) *Strongly one-sided AB I-rings are NI.*

Proof. (1) Let R be an I-ring and J be a non-nil left ideal of R . Let $a \in J$ be non-nilpotent. Then aR is a non-nil right ideal, and so there is $0 \neq ab = abab$ for some $b \in R$, since R is an I-ring. Consider $baba \in Ra \subseteq J$; then $baba \neq 0$ and $(baba)(baba) = b(ababab)a = baba$.

(2) Let R be an I-ring and $x = a, y = bab$, where a, b are the elements in (1). Then $y \neq 0$ and $xyx = (bab)a(bab) = bab = y$. The converse is obvious.

(3) Let R be an I-ring; then $J(R)$ is nil. Assume that there is $a \in N(R)$, say $a^n = 0$ for some positive integer n , with $a \notin J(R)$. Then aR is a non-nil right ideal of R . But R is an I-ring; so $e \in aR$ for some non-zero $e^2 = e \in R$. Since R is strongly right AB, there exists a non-zero ideal I of R such that $I \subseteq r_R(1 - e) = eR \subseteq aR$. Let $ar \in I$ and $e = as$ for some $r, s \in R$. Then $ar = ear$ and $sar = esar$, since I is two-sided. Thus $ar = ear = asar = a^2s^2ar = a^2es^2ar = \dots = a^n s^n ar = 0$, which is a contradiction. Thus $N(R) \subseteq J(R)$, entailing $N(R) = J(R)$. The proof of the left case is similar.

(4) If R is strongly one-sided AB I-ring, then $R/N(R)$ is reduced by (3) with $N(R) = J(R)$. □

Lemma 3.8(3, 4) is similar to [17, Proposition 3]. The converse of Lemma 3.8(3) need not be true as can be seen by the 2×2 upper triangular matrix ring over a division ring. The condition ‘I-ring’ in Lemma 3.8(3, 4) is not superfluous by the following.

EXAMPLE 3.9. There is a strongly AB ring that is neither an I-ring nor NI. Let K be a field and $D_n = K\{x_n\}$ with relation $x_n^{n+2} = 0$, where n is any non-negative integer and $K\{x_n\}$ is the free algebra generated by x_n over K . Note $D_n \cong K[x]/(x^{n+2})$, where (x^{n+2}) is the ideal of $K[x]$ generated by x^{n+2} , with $K[x]$ the polynomial ring with an indeterminate x over K . We use the ring in [18, Example 1.6]. Define $S_n = \begin{pmatrix} D_n & D_n x_n \\ D_n x_n & D_n \end{pmatrix}$. Notice that $J(S_n) = \begin{pmatrix} D_n x_n & D_n x_n \\ D_n x_n & D_n x_n \end{pmatrix}$ and $\frac{S_n}{J(S_n)} \cong \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$; hence $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in S_n$ is invertible when the constant terms of f_1 and f_4 are both non-zero.

We first claim that S_n is strongly AB. Notice that $I_1 = K \begin{pmatrix} x_n^{n+1} & 0 \\ 0 & 0 \end{pmatrix}, I_2 = K \begin{pmatrix} 0 & x_n^{n+1} \\ 0 & 0 \end{pmatrix}, I_3 = K \begin{pmatrix} 0 & 0 \\ x_n^{n+1} & 0 \end{pmatrix}$ and $I_4 = K \begin{pmatrix} 0 & 0 \\ 0 & x_n^{n+1} \end{pmatrix}$ are non-zero ideals in S_n .

Take $X \subseteq S_n$ such that $r_{S_n}(X) \neq 0$. Then X is contained in one of the following three kinds of subsets:

$$Y_1 = \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in S_n \mid f_i \in D_n x_n \text{ for } i = 1, 2, 3, 4 \right\},$$

$$Y_2 = \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in S_n \mid f_1 \in D_n \text{ and } f_i \in D_n x_n \text{ for } i = 2, 3, 4 \right\},$$

$$Y_3 = \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in S_n \mid f_4 \in D_n \text{ and } f_i \in D_n x_n \text{ for } i = 1, 2, 3 \right\}.$$

If $X \subseteq Y_1$, then $r_{S_n}(X)$ contains all I_j 's; if $X \subseteq Y_2$, then $r_{S_n}(X)$ contains I_3 and I_4 ; if $X \subseteq Y_3$, then $r_{S_n}(X)$ contains I_1 and I_2 . Consequently S_n is strongly right AB, and similarly we can conclude that S_n is strongly left AB.

Next let $R = \prod_{i=0}^{\infty} S_n$. Then R is strongly AB by Lemma 3.1(1), and so is every S_n .

Consider two sequences $(a_n), (b_n) \in R$ such that $a_n = \begin{pmatrix} 0 & x_n \\ 0 & 0 \end{pmatrix}$ and $b_n = \begin{pmatrix} 0 & 0 \\ x_n & 0 \end{pmatrix}$ for all n . Then $(a_n), (b_n) \in N(R)$, since $(a_n)^2 = 0 = (b_n)^2$; but each component of $(a_n) + (b_n)$ is $\begin{pmatrix} 0 & x_n \\ x_n & 0 \end{pmatrix}$, and so $(a_n) + (b_n)$ is non-nilpotent. Thus R is not NI.

The following lemma is due to Ramamurthi [29].

LEMMA 3.10. (1) [29, Proposition 3.2] *An IFP ring R is strongly regular if and only if R is right SF.*

(2) *A ring R is strongly regular if and only if R is reduced and right SF if and only if R is reduced and left SF.*

Proof. (2) is proved by the left version of (1), [15, Theorem 3.2], [15, Corollary 1.13] and the fact that reduced rings are IFP. □

Zhang and Du [38, Theorem 3] proved that a ring R is strongly regular if and only if R is strongly right bounded and right SF. In the following theorems we give partial answers to Ramamurthi's question, extending the result of Zhang and Du and Lemma 3.10(1). Strongly regular rings are I-rings by Lemma 3.8(2) or [22, Proposition 9.4.1].

THEOREM 3.11. *For a ring R the following conditions are equivalent:*

- (1) *R is strongly regular;*
- (2) *R is strongly right AB and right SF;*
- (3) *R is a strongly right AB and left SF I-ring.*

Proof. (1) \Rightarrow (2) and (1) \Rightarrow (3) are obtained from [15, Theorem 3.2], [15, Corollary 1.13] and Lemma 3.8(2).

(2) \Rightarrow (1): Let $0 \neq a \in R$ with $a^2 = 0$. Then $r_R(\ell_R(a))$ is non-zero right annihilator, and so there exists a non-zero ideal I of R such that $I \subseteq r_R(\ell_R(a))$. So $\ell_R(a) = \ell_R(r_R(\ell_R(a))) \subseteq \ell_R(I)$. Here we claim that $\ell_R(I) = R$. Suppose not; then there exists a maximal right ideal M containing $\ell_R(I)$. Since R is right SF and $a \in M$, we can get $b \in M$ with $a = ba$ by the right version of [26, Proposition 5.4.3] (i.e. $Ra \cap M = Ma$). This implies that $1 - b \in \ell_R(a) \subseteq \ell_R(I) \subseteq M$, and so $1 \in M$, a contradiction. Thus $\ell_R(I) = R$, and so $I = 0$, which is also a contradiction, concluding that R is reduced. Moreover R is strongly regular by Lemma 3.10(1).

(3) \Rightarrow (1): First we have that $R/J(R)$ is strongly regular by Lemmas 3.8(3) and 3.10(1). Here we claim that $aR + r_R(a) = R$ for any element $a \in R$. Suppose not; then there exists a maximal right ideal M containing $aR + r_R(a)$. But M is also a maximal left ideal of R by [15, Theorem 3.2], since $R/J(R)$ is strongly regular and $M \supseteq J(R)$. Now since R is left SF and $a \in M$, there exists $b \in M$ such that $a = ab$ by [26, Proposition 5.4.3] (i.e. $aR \cap M = aM$). This implies that $1 - b \in r_R(a) \subseteq M$, and so $1 \in M$, a contradiction. Thus R is strongly regular from $aR + r_R(a) = R$. □

In the following we get the same result as Theorem 3.11 with the NIness in place of the strong one-sided ABness. Strongly regular rings are reduced (hence NI) by [15, Theorem 3.2].

THEOREM 3.12. *For a ring R , the following conditions are equivalent:*

- (1) R is strongly regular;
- (2) R is right SF and NI;
- (3) R is left SF and NI.

Proof. It suffices to show (2) \Rightarrow (1) and (3) \Rightarrow (1) by [15, Corollary 1.13]. Let R be a right SF and NI ring. Then $R/N(R)$ is reduced and right SF; hence $R/N(R)$ is strongly regular by Lemma 3.10(1). Since $J(R/N(R)) = 0$, we get $J(R) \subseteq N(R)$, entailing $J(R) = N(R)$. Then every maximal right or left ideal is two-sided because $R/J(R)$ is duo by [15, Theorem 3.2]. Here we will show $N(R) = 0$. Assume on the contrary that there exists $0 \neq a \in N(R)$ such that $a^2 = 0$. Then there exists a maximal left ideal M of R containing $\ell_R(a)$. But M is also a maximal right ideal of R because $R/N(R)$ is duo, and so R/M is a flat right R -module. Since R is right SF, there exists $b \in M$ such that $a = ba$ by the right version of [26, Proposition 5.4.3]. Hence $1 - b \in \ell_R(a) \subseteq M$, obtaining $1 \in M$, a contradiction. Consequently $N(R) = 0$, and hence R is strongly regular. The proof of (3) \Rightarrow (1) is similar. \square

Recall that a ring R is NI if and only if every minimal strongly prime ideal of R is completely prime [18, Corollary 13]. Thus we can obtain the following from Theorem 3.12 and note that the condition ‘ R is fully idempotent’ is superfluous.

COROLLARY 3.13. [37, Theorem 4] *A ring R is strongly regular if and only if R is a fully idempotent (i.e. every ideal is idempotent) right SF ring such that every prime ideal is completely prime.*

Strongly AB rings need not be abelian by Example 2.5(2), and abelian rings are also need not be strongly one-sided AB by the following.

EXAMPLE 3.14. Consider R_n over $GF(2)$ as in the starting position of Section 2. Let $S_n = R_{2^{n-1}}$ for $n = 1, 2, \dots$, and define a map $\sigma : S_n \rightarrow S_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$; then S_n can be embedded into S_{n+1} via σ (i.e. $A = \sigma(A)$ in S_{n+1} for $A \in S_n$). Then $\{S_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \dots\}$. Set R be the direct limit of $\{S_n, \sigma_{nm}\}$. Then $R = \bigcup_{n=1}^{\infty} S_n$. We first claim that R is semiprime. Take $0 \neq A \in R$ such that A is nilpotent. Then $A \in S_n$ for some n such that the diagonal of A is zero, and every non-zero entry of A is 1, say $A = (a_{st})$.

Let i be the smallest such that the i th row of A contains a non-zero entry and j be the smallest such that $a_{ij} \neq 0$ in the i th row. Note that $i < j$ and $(i + 2^k, j + 2^k)$ -entry of A in S_{k+1} is also 1 for $k = n, n + 1, n + 2, \dots$. Use e_{uv} to denote the square matrix in which (u, v) -entry is 1 and zero elsewhere. Let $A_0 = A$ and $A_1 = A_0 B_0 A_0 \in A_0 R A_0$, where A_0 is considered in S_{n+1} and $B_0 = e_{j(i+2^n)} \in S_{n+1}$. Say $A_1 = (b_{st})$. Then i is the smallest such that the i th row of A_1 contains a non-zero entry and $j + 2^n$ is the smallest such that $b_{i(j+2^n)} \neq 0$, actually $b_{i(j+2^n)} = 1$, in the i th row; hence $(i + 2^{n+1}, j + 2^n + 2^{n+1})$ -entry of A_1 in S_{n+2} is also 1. Let $A_2 = A_1 B_1 A_1 \in A_1 R A_1$, where $B_1 = e_{(j+2^n)(i+2^{n+1})} \in R_{n+2}$. Say $A_2 = (c_{st})$. Then i is the smallest such that the i th row of A_2 contains a non-zero entry, and $j + 2^n + 2^{n+1}$ is the smallest such that $b_{i(j+2^n+2^{n+1})} \neq 0$, actually $b_{i(j+2^n+2^{n+1})} = 1$, in the i th row; hence $(i + 2^{n+2}, j + 2^n + 2^{n+1} + 2^{n+2})$ -entry of A_2 in S_{n+3} is also 1. Proceeding in this manner, we can show that the $(i, j + 2^n + 2^{n+1} + \dots + 2^{n+(k-1)})$ -entry of A_k is also 1 for any k ; hence we can obtain inductively a sequence $(A_k)_{k=0}^{\infty}$ each term of which is non-zero with $A_{k+1} \in A_k R A_k$. Thus A is not strongly nilpotent and $A \notin P(R)$, concluding that R is semiprime.

Assume that R is strongly one-sided AB. Then R is reduced by Proposition 3.6, since R is semiprime, a contradiction. Thus R is neither strongly right nor strongly left AB.

While every S_n is abelian and every non-zero idempotent in S_n is such that the diagonal is 1 and zero elsewhere (i.e. the identity is the only non-zero idempotent), by [19, Lemma 2]. Thus R is also abelian.

A ring R is called *directly finite* if $xy = 1$ implies $yx = 1$ for $x, y \in R$. It is trivial to check that abelian rings are directly finite, whence the following proposition shows that directly finite rings unify the classes of strongly one-sided rings and abelian rings.

PROPOSITION 3.15. *Strongly one-sided AB rings are directly finite.*

Proof. Let R be a strongly right AB ring, and assume on the contrary that $xy = 1$ but $yx \neq 1$ for some $x, y \in R$. Then yx is a non-identity idempotent and $yx(1 - yx) = 0$ with $1 - yx \neq 0$. Since R is strongly right AB, we have $yxRa = 0$ for some non-zero $a \in R$; but $xRa = xyxRa = 0$ implies $0 \neq a = xya \in xRa = 0$, a contradiction. Thus R is directly finite. The proof for strongly left AB rings is similar. \square

Due to Anderson and Camillo [2], a ring R satisfies ZC_2 if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Rings satisfying ZC_2 are clearly IFP, but the converse need not hold by [23, Example 1.5].

COROLLARY 3.16. [2, Theorem 2.1] *If a ring R satisfies ZC_2 , then R is directly finite.*

A one-sided Artinian ring R is π -regular (i.e. for any $x \in R$ there are $y \in R$ and a positive integer n such that $x^n = x^nyx^n$) by a simple computation.

LEMMA 3.17. *Let R be a strongly one-sided AB ring. If R is one-sided Artinian, then $R/J(R)$ is a finite direct product of division rings, and especially each maximal one-sided ideal of R is two-sided.*

Proof. Let R be one-sided Artinian. Then R is an I-ring by [22, Proposition 9.4.1], and so $R/J(R)$ is reduced by Lemma 3.8(3). Thus $R/J(R)$ is a finite direct product of division rings, and it is immediate that each maximal one-sided ideal of R is two-sided. \square

A *quasi-Frobenius* ring, introduced by Nakayama in 1939 [28], is defined to be right Artinian and right self-injective. A ring R is quasi-Frobenius if and only if for each left ideal I and right ideal J in R , $\ell_R(r_R(I)) = I$ and $r_R(\ell_R(J)) = J$ [3, Theorem 30.7].

PROPOSITION 3.18. *Let R be a quasi-Frobenius ring. If R is strongly one-sided AB, then R is strongly bounded.*

Proof. Since R is right Artinian, every right ideal of R contains a minimal right ideal. To prove that R is strongly right bounded it suffices to show that every minimal right ideal is two-sided. Let I be a minimal right ideal of R . Then since R is quasi-Frobenius, $J = \ell(I)$ is a maximal left ideal of R with $I = r_R(J)$. But J is two-sided by Lemma 3.17, and then we have $I = r_R(J)$ is two-sided. The proof for the left case is similar. \square

Denote the right (resp. left) socle – i.e. the intersection of all essential right (resp. left) ideals – of a ring R by $Soc(R_R)$ (resp. $Soc({}_R R)$). The simple right Noetherian domain in Example 2.3(5) is neither strongly right nor strongly left bounded. But

reduced rings are strongly bounded when they satisfy descending chain condition (DCC) on essential right ideals as follows.

PROPOSITION 3.19. *Let R be a reduced ring satisfying DCC on essential right ideals. Then R is a strongly bounded ring such that R is a subdirect product of division rings, and every proper one-sided ideal consists of zero divisors.*

Proof. We use freely the fact that reduced rings are non-singular, semiprime and abelian. Note that $\text{Soc}(R_R)$ is itself an essential right ideal of R , since a finite intersection of essential submodules is also essential and R satisfies DCC on essential right ideals.

Let I be a non-zero right ideal of R . Then $\text{Soc}(R_R) \cap I \neq 0$, and since R is semiprime and abelian, $\text{Soc}(R_R) \cap I$ contains a non-zero central idempotent. Thus R is strongly right bounded.

To prove that R is strongly left bounded it is enough to show that $\text{Soc}({}_R R)$ is essential left ideal of R , using a similar manner to the preceding one. First we get $\text{Soc}({}_R R) = \text{Soc}(R_R)$, since R is semiprime and abelian. Assume that $\text{Soc}({}_R R) \cap J = 0$ for a left ideal J of R . Then $\text{Soc}({}_R R)J = 0$ and since R is semiprime, we have $J\text{Soc}({}_R R) = 0$; consequently $0 = J\text{Soc}({}_R R) = J\text{Soc}(R_R)$. But R is right non-singular and $\text{Soc}(R_R)$ is an essential right ideal of R , inducing $J = 0$. Thus R is strongly left bounded.

Every minimal prime ideal of R is completely prime by [32, Proposition 1.11], since R is reduced. So R is a subdirect product of domains each of which satisfies DCC on essential right ideals by [5, Proposition 1.2(a)]. But a domain, satisfying DCC on essential right ideals, is a division ring by [5, Proposition 1.6]. Thus R is a subdirect product of division rings. Every regular element of a semiprime ring is invertible by [5, Proposition 1.6] when it satisfies DCC on essential right ideals, and hence each proper right or left ideal of R consists of zero divisors. \square

Note that the simple right Noetherian domain in Example 2.3(5) does not satisfy DCC on essential right ideals. Due to Yu [36], a ring is called *right* (resp. *left*) *quasi-duo* if every maximal right (resp. left) ideal is two-sided. Right duo rings are clearly right quasi-duo, and it is straightforward that a right quasi-duo right primitive ring is a division ring. Yu [36, Lemma 2.3] showed that $R/J(R)$ is reduced for a right or left quasi-duo ring R . By Lemma 3.17, strongly one-sided AB rings are quasi-duo when they are one-sided Artinian. But quasi-duo rings need not be strongly one-sided AB as can be seen by the 2×2 upper triangular matrix ring over a simple ring.

COROLLARY 3.20. *Let R be a one-sided quasi-duo ring satisfying DCC on essential right ideals. Then $R/J(R)$ has the properties as in Proposition 3.19.*

Proof. Since R is one-sided quasi-duo and satisfies DCC on essential right ideals, $R/J(R)$ is a reduced ring satisfying DCC on essential right ideals by [36, Lemma 2.3] and [5, Proposition 1.2(a)]. Then Proposition 3.19 gives the result. \square

4. More examples of strongly one-sided AB rings. In this section we study some conditions and some ring extensions to extend the class of strongly one-sided AB rings. Let R be a ring; $R[x]$ (resp. $R[[x]]$) denotes the polynomial ring (resp. power series ring) over R with an indeterminate x over R .

The class of strongly one-sided AB rings is not closed under subrings by Example 3.2. In the following we see a kind of subring that inherits the strong ABness.

PROPOSITION 4.1. *Let R be a ring and $0 \neq e = e^2 \in R$ such that $eIe \neq 0$ for any non-zero ideal I of R . If R is strongly right AB, then so is eRe .*

Proof. Let R be a strongly right AB ring and $S = eRe$. Suppose that $r_S(X) \neq 0$ for $X \subseteq S$. Then since R is strongly right AB there is a non-zero ideal I of R such that $I \subseteq r_R(X)$. Moreover $r_S(X) = r_R(X) \cap S \supseteq I \cap S = eIe$. By the condition eIe is a non-zero ideal of S , concluding that S is strongly right AB. □

We next consider polynomial rings over a strongly right AB ring. By [17], there exists a strongly right bounded ring R whose polynomial ring $R[x]$ is not strongly right bounded. By [20, Example 2], there exists an IFP ring R whose polynomial ring $R[x]$ is not IFP. So it is natural to ask whether $R[x]$ is strongly right AB if a ring R also is. We do not know the answer, but we find some conditions under which it is affirmative.

PROPOSITION 4.2. *Let R be a ring and J be an ideal of R such that every element in $R \setminus J$ is regular and $J^2 = 0$. Then R , $R[x]$ and $R[[x]]$ are strongly AB.*

Proof. Set $K = R \setminus J$. Let $A \subset R$ with $r_R(A) \neq 0$. Then $A \subseteq J$, and so $AJ = 0$; hence R is strongly right AB. Similarly R is strongly left AB.

Let $B \subset R[x]$ with $r_{R[x]}(B) \neq 0$, and suppose that $f(x)g(x) = 0$ for $0 \neq f(x) = \sum_{i=0}^m a_i x^i \in B$ and $0 \neq g(x) = \sum_{j=0}^n b_j x^j \in r_{R[x]}(B)$. We can let a_0 and b_0 be both non-zero and $m = n$. Since R/J is a domain, [4, Lemma 1] implies that $a_i b_j \in J$ for all i and j ; $a_0 b_0 = 0$ gives $a_0, b_0 \in J$. Assume $f(x) \notin J[x]$. Then $g(x) \in J[x]$ because each $a_i b_j$ is in J . Say that k is smallest such that $a_k \in K$. Then $k \geq 1$. From $J^2 = 0$ we get $0 = a_0 b_k + a_1 b_{k-1} + \dots + a_{k-1} b_1 + a_k b_0 = a_k b_0$, entailing $b_0 = 0$, a contradiction. Thus $f(x) \in J[x]$, and we get $B \subset J[x]$. Consider the non-zero ideal $J[x]$ of $R[x]$. From $J[x]^2 = 0$ we have $BJ[x] = 0$, concluding that $R[x]$ is strongly right AB. Similarly $R[x]$ is strongly left AB. The proof for $R[[x]]$ is similar. □

Given a ring R and a bimodule ${}_R M_R$, the *trivial extension* of R by M , denoted by $T(R, M)$, is the ring $R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Let R be a commutative ring and h be a ring endomorphism of R . For an R -module M , the multiplication $(a, m)(b, n) = (ab, h(a)n + bm)$ gives a ring structure to $R \oplus M$, denoted by $R(+)_h M$.

COROLLARY 4.3. (1) *Let R be a commutative domain and M be an R -module. If M is torsion-free, then $T(R, M)$ and $T(R, M)[x]$ are both strongly AB.*

(2) *Let K be a field, h be a non-zero ring endomorphism of K and V be a K -vector space. Then $K(+)_h V$ and $(K(+)_h V)[x]$ are both strongly AB.*

(3) *Let R be a local ring with $J(R)^2 = 0$. Then R and $R[x]$ are both strongly AB.*

Proof. (1) Let $J = \{(a, m) \in T(R, M) \mid a = 0\}$; then $J^2 = 0$ and $R \setminus J$ is the set of all regular elements in $T(R, M)$. (2) Let $J = \{(k, v) \in K(+)_h V \mid k = 0\}$; then $J^2 = 0$ and $R \setminus J$ is the set of all regular elements in $K(+)_h V$. (3) Let R be a local ring. Then $R \setminus J(R)$ is clearly the set of all regular elements in R . Thus we get the results from Proposition 4.2. □

The following is similar to Proposition 4.2.

PROPOSITION 4.4. *If R is a local ring with nilpotent $J(R)$, then R and $R[[x]]$ are strongly AB.*

Proof. Let R be a local ring such that $J(R)^k = 0$ and $J(R)^{k-1} \neq 0$. Then R is strongly AB by Lemma 3.4(6), and $\frac{R[[x]]}{J(R)[[x]]} \cong \frac{R}{J(R)}[[x]]$ is a domain such that $(J(R)[[x]])^{k-1}$ is a non-zero nilpotent ideal of $R[[x]]$. Let $A \subset R[[x]]$ with $r_{R[[x]]}(A) \neq 0$, say $f(x)g(x) = 0$ for $0 \neq f(x) \in B$ and $0 \neq g(x) \in r_{R[[x]]}(B)$. Here we can suppose that a_0 and b_0 are both non-zero. Put $T = \frac{R[[x]]}{J(R)[[x]]}$, and use $h(x)$ in place of $h(x) + J(R)[[x]]$ for simplicity. Assume $f(x) \notin J(R)[[x]]$. Then $f(x) = x^n f_0(x) = f_0(x)x^n$ in T for some non-negative integer n and $f_0(x) \in T$ with invertible constant term. So $f_0(x)$ is invertible, say $f_0(x)h(x) = h(x)f_0(x) = 1$; hence $h(x)f_0(x)x^n = x^n$ and $(1 - h(x)f_0(x))x^n \in J(R)[[x]]$. But $J(R)[[x]]^k = 0$, and so we get $(1 - h_0(x)f_0(x))x^{nk} = 0$ (hence $1 - h_0(x)f_0(x) = 0$) for some $h_0(x) \in R[[x]]$. Now we have $0 = h_0(x)f(x)g(x) = h_0(x)f_0(x)g(x)x^n = g(x)x^n$, and so $g(x) = 0$, a contradiction. Thus $f(x) \in J(R)[[x]]$ and $A(J(R)[[x]])^{k-1} = 0$, entailing that $R[[x]]$ is strongly right AB. Similarly $R[[x]]$ is strongly left AB. \square

Consider R_n as in Section 2 over a local ring with nilpotent Jacobson radical. Then R_n and $R_n[[x]]$ are both strongly AB by Theorem 2.2, Lemma 3.4(6) and Proposition 4.4.

PROPOSITION 4.5. *Let R be a ring and n any positive integer. Then R is strongly right AB if and only if so is $R[x]/(x^n)$, where (x^n) is the ideal of $R[x]$ generated by x^n .*

Proof. Put $S = R[x]/(x^n)$, and use $h(x)$ in place of $h(x) + (x^n)$ for simplicity. Suppose that R is strongly right AB. Let $A \subset S$ with $r_S(A) \neq 0$, and take non-zero $g(x) = \sum_{j=0}^n b_j x^j$ in $r_S(A)$. Here we can assume $b_0 \neq 0$. Consider the subset $B = \{a \in R \mid a = a_m \text{ for } 0 \neq f(x) = \sum_{i=m}^n a_i x^i \in A\}$ of R . Then $r_R(B) \neq 0$, since $b_0 \in r_R(B)$. But R is strongly right AB, and so there is a non-zero ideal of R such that $I \subseteq r_R(B)$. Now we have $A(I[x])^{n-1} = 0$, concluding that S is strongly right AB.

Conversely suppose that S is strongly right AB. Let $A \subseteq R$ with $r_R(A) \neq 0$. Note that $r_R(A) = r_S(A) \cap R$. Since $r_R(A) \neq 0$, we get $r_S(A) \neq 0$. But S is strongly right AB, so there is a non-zero ideal L of S such that $r_S(A) \supseteq L$. Let L_0 be the set of all coefficients of all polynomials in L . Then clearly L_0 is a non-zero ideal of R . Now $AL_0 = 0$, and so R is strongly right AB. \square

Due to Rege and Chhawchharia [31], a ring R is called *Armendariz* if $a_i b_j = 0$ for all i and j whenever polynomials $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$. Reduced rings are Armendariz by [4, Lemma 1], and Armendariz rings are abelian by the proof of [1, Theorem 6]. Commutative (hence strongly AB) rings need not be Armendariz by [31, Example 3.2].

PROPOSITION 4.6. *Let R be an Armendariz ring. Then R is strongly right AB if and only if so is $R[x]$.*

Proof. Suppose that R is strongly right AB. Let $A \subseteq R[x]$ with $r_{R[x]}(A) \neq 0$ and B be the set of all coefficients of polynomials in A . Take non-zero $f(x) = a_0 + a_1 x + \dots + a_n x^n$ in $r_{R[x]}(A)$. Then for any $g(x) = b_0 + b_1 x + \dots + b_m x^m \in A$, $g(x)f(x) = 0$. Since R is Armendariz, $b_i a_j = 0$ for all i, j . Thus $a_j \in r_R(B)$ for any $j = 1, 2, \dots, n$, entailing $r_R(Y) \neq 0$. Since R is strongly right AB, there exists a non-zero ideal I of R such that

$r_R(B) \supseteq I$. Then $I[x]$ is a non-zero ideal of $R[x]$ such that $r_{R[x]}(A) \supseteq I[x]$. Thus $R[x]$ is strongly right AB. The proof of the converse is similar to the second part of the proof of Proposition 4.5. \square

PROPOSITION 4.7. *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Then R is strongly right AB if and only if so is $\Delta^{-1}R$.*

Proof. Suppose that R is strongly right AB, and let $A \subseteq \Delta^{-1}R$ such that $r_{\Delta^{-1}R}(A) \neq 0$. Let $B = \{a \in R \mid u^{-1}a \in A\} \subseteq R$.

Take $0 \neq v^{-1}b \in r_{\Delta^{-1}R}(A)$; then $Bb = 0$, and so $r_R(B) \neq 0$. Since R is strongly right AB, there exists a non-zero ideal I of R such that $r_R(B) \supseteq I$. Then for any $u^{-1}a \in A$, $u^{-1}aI = 0$, since $a \in B$ and $AI = 0$. Note that $\Delta^{-1}I$ is a non-zero ideal of $\Delta^{-1}R$. Since $I \subseteq r_{\Delta^{-1}R}(A)$, we have $\Delta^{-1}I \subseteq r_{\Delta^{-1}R}(X)$. Thus $\Delta^{-1}R$ is strongly right AB.

Conversely, suppose that $\Delta^{-1}R$ is strongly right AB, and let $A \subseteq R$ such that $r_R(A) \neq 0$. Then $r_{\Delta^{-1}R}(A) \neq 0$. Since $\Delta^{-1}R$ is strongly right AB, there exists a non-zero ideal L of $\Delta^{-1}R$ such that $r_{\Delta^{-1}R}(A) \supseteq L$. Take $0 \neq v^{-1}b \in L$. Since v^{-1} is central, we have $v^{-1}RbR = Rv^{-1}bR \subseteq L$. Thus $Av^{-1}RbR = v^{-1}ARbR = 0$, and so $ARbR = 0$. Thus R is strongly right AB. \square

The ring of *Laurent* polynomials in x , coefficients in a ring R , consists of all formal sums $\sum_{i=k}^n m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

COROLLARY 4.8. *Let R be an Armendariz ring. Then the following conditions are equivalent:*

- (1) R is strongly right AB;
- (2) $R[x]$ is strongly right AB;
- (3) $R[x; x^{-1}]$ is strongly right AB.

Proof. (1) \Leftrightarrow (2) is Proposition 4.6. (2) \Leftrightarrow (3): Let $\Delta = \{1, x, x^2, \dots\}$. Then Δ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Note that $R[x; x^{-1}] = \Delta^{-1}R[x]$. So the equivalence is obtained from Proposition 4.7. \square

Let R, S be rings and $f : R \rightarrow S$ be an isomorphism. $R \oplus S$, denoted by $R(+)_f S$, is a ring with the usual addition and the multiplication $(r_1, s_1)(r_2, s_2) = (r_1 r_2, f(r_1) s_2 + s_1 f(r_2))$.

PROPOSITION 4.9. *Let R, S be rings and $f : R \rightarrow S$ be an isomorphism. Then R is strongly right AB if and only if so is $R(+)_f S$.*

Proof. Apply the proof of Theorem 2.2. \square

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