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SOME ASPECTS OF RELATIVE INJECTIVITY

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If H and M are right R-modules, H is M-injective if every R-homonorphism $N \to H$, N a right R-submodule of M, can be extended to an R-homonorphism from M to H.

H is strongly M-injective if H is injective for inclusions whose cokernels are isomorphic to factor modules of M.

For the case of abelian groups H and M, one settles the questions "when is H M-injective" and "when is H strongly M-injective". The latter can be characterized in terms of the vanishing of Ext. Results for general module categories are also given.

Introduction

The purpose of this article is to pursue some consequences of the statement "H is M-injective", a notion studied in [1]. Sometimes we proceed by considering questions which are dual to those addressed in [3]. There are three sections; the first studies relative injectivity and Theorem 3 settles the question of when H is M-injective for

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arbitrary abelian groups H and M. The second introduces and studies relative strong injectivity which like relative strong-projectivity can be characterized in terms of the vanishing of Ext. Both the first and the second sections concentrate on abelian groups. The final section presents some results for more general categories of modules.

For the sake of completeness we will recall the following facts.

DEFINITION 1. *Let H and M be right R-modules. H is M-injective if for every submodule $N \leq M$, and every R-homomorphism $f: N \longrightarrow H$, there exists a homomorphism $g: M \longrightarrow H$ satisfying $g|N = f$.*

PROPOSITION 2. [1] *(a) $C^i(H)$ is closed under taking submodules, homomorphic images, and direct sums, where $C^i(H)$ is the class of right R-modules M such that H is M-injective.*

(b) $C_i(M)$ is closed with respect to direct sums and direct summands, where $C_i(M)$ is the class of right R-modules H such that H is M-injective.

1. Relative injectivity for abelian groups

THEOREM 3. *Let H and M be abelian groups and let $H = D \oplus K$ where D is the maximal divisible subgroup of H and K is reduced. Then;*

(i) If M is not a torsion group, H is M-injective if and only if H is divisible,

(ii) H is M-injective if and only if K is M-injective. If K is torsion-free, K is M-injective if and only if M is torsion. If K is not torsion-free, K is M-injective if and only if M is torsion and $k_pM_p = 0$ where k_p is the smallest possible integer for which K_p has a direct summand isomorphic to $Z(p^{k_p})$ if such direct summands exist. In particular K is M-injective if and only if K_t is M-injective where K_t is the torsion subgroup of K.

Proof. (i) If M is not a torsion group, Z is a subgroup of M so if H is M-injective, it is Z-injective and therefore divisible. The converse is clear.

(ii) The first statement is clear. Suppose that K is torsion-free. If M is not torsion, then by (i) K is divisible and since K

is reduced this yields the trivial case $K = (0)$.

If M is torsion then the image of every map from M to K lies in K_t. Therefore K is M-injective if and only if K_t is, and therefore K is trivially M-injective if K is torsion-free. Now suppose that K is not torsion-free. Again M is torsion lest K be divisible and hence the zero group. Since images of subgroups of M are torsion groups K is M-injective if and only if K_t is M-injective. Since both M and K_t are torsion, K is M-injective if and only if K_p is M-injective for all primes p. Suppose that for each p, K_p is M_p-injective. Then $Z(p^k)$ is M_p-injective. The assertion that $p^{k_p}M_p = 0$ is equivalent to the statement that M_p has no cyclic subgroup of order greater than p^{k_p}. Suppose, $p^{k_p}M \neq 0$, then there exists an $a \in M$ with $|a| = p^{k_p}$.
One has the diagram

$$\begin{array}{ccc} (pa) & \longrightarrow & M_p \\ \downarrow f & & \\ Z(p^{k_p}) & & \end{array}$$

where f is an isomorphism of (pa) with $Z(p^{k_p})$, which is generated say by $b \in Z(p^{k_p})$ with $f(pa) = b$. The diagram can be closed by $g : M_p \to A(p^{k_p})$ so $b = g(pa) = pg(a)$ and this contradicts the fact that the p-height of b in $Z(p^{k_p})$ is zero. Thus if K_p is M_p-injective $p^{k_p} M_p = 0$.

If $p^{k_p} M_p = 0$ then M_p is a direct sum of cyclic p-groups of order less than or equal to p^{k_p}. Since the property of being K_p-injective is closed under passing to submodules and to direct sums [1] it suffices to show that K_p is $Z(p^{k_p})$-injective for each p. We must consider a diagram of the sort:

$$\begin{array}{l} (a) = N \longrightarrow Z(p^{k_p}) \\ \downarrow f \\ K_p \end{array}$$

where $Z(p^{k_p})$ is generated by c and N is generated by $a = p^{k_p - \ell} c$ of order p^{ℓ} and a has p-height $k_p - \ell$ in $Z(p^{k_p})$. Let $f(a) = b$. Then $|b| \leq p^{\ell}$ and $h_p(b) \geq k_p - \ell$. $b = p^{k_p - \ell} b'$ for some $b' \in K_p$. Let $g : Z(p^{k_p}) \longrightarrow K_p$ be defined by $g(c) = b'$. It clearly extends f.

2. Strong relative injectivity for groups

To begin this section we need the following result:

LEMMA 4. (MacLane) *Let* $0 \to A \xrightarrow{\Theta} B \to B/A \to 0$ *be exact, let* $f : A \to H$ *be a homomorphism, and let* $P(H,B)$ *be the pushout of* Θ *and* f *yielding the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\Theta} & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & P(H,B) & \longrightarrow & B/A & \longrightarrow & 0 \end{array}$$

where both rows are exact and all squares commute: then there exists a homomorphism $g : B \longrightarrow H$ *so that* $f = g = \dot{\Theta}$ *if and only if the lower sequence splits, that is, if and only if* $Ext^1(B/A,H) = 0$.

Lemma 4 yields the following:

PROPOSITION 5. *The following are equivalent for right* R*-modules* H *and* M.

(i) *A diagram of the form*

$$\begin{array}{l} 0 \longrightarrow A \xrightarrow{f} B \\ \qquad\quad\ \downarrow \\ \qquad\quad\ H \end{array}$$

can be closed commutatively by a map from B *to* H *whenever Coker* (f) *is isomorphic to a factor module of* M *, that is,* H *is injective for* *inclusions where cokernels are isomorphic to factor modules of* M.

(ii) $\mathrm{Ext}^1(X,H) = 0$ *whenever* X *is isomorphic to a factor module of* M.

DEFINITION 6. *If the conditions of Proposition 5 hold, we say that* H *is strongly* M*-injective. It is clear that a module* H *is* M*-injective if it is strongly* M*-injective. Also if* H *is injective, then it is strongly* M*-injective for any module* M.

If M is a fixed right R-module, let $C_{si}(M)$ be the class of right R-modules H such that H is strongly M-injective. $C_{si}(M)$ is closed under taking direct products and direct summands. If R is right hereditary $C_{si}(M)$ is also closed under the passage to homomorphic images, the proof being as follows: take an exact sequence $0 \to Y \to H \to H/Y \to 0$ this induces the sequence $\mathrm{Ext}^1(X,H) \longrightarrow \mathrm{Ext}^1(X,H/Y) \longrightarrow \mathrm{Ext}^2(X,Y)$. If R is right Noetherian $\mathrm{Ext}^2(X,Y) = 0$. By assumption $\mathrm{Ext}^1(X,H) = 0$ so $\mathrm{Ext}^1(X\ H/Y) = 0$ showing that H/Y is strongly M-injective.

For direct products in $C_{si}(M)$ one has $\mathrm{Ext}^1(X,\Pi H_\alpha) \cong \Pi_\alpha \mathrm{Ext}^1(X,H_\alpha) = (0)$. The case for the passage to direct factors is straightforward.

Dually, for a fixed right R-module H, let $C^{si}(H)$ be the class of right R-modules M, for which H is strongly M-injective. $C^{si}(H)$ is closed under taking submodules by considering the diagram for injectivity. It is closed under taking homomorphic images because of Proposition 5, and it is closed under passing to direct summands because homomorphic images of direct summands of M are clearly homomorphic images of M. We now concentrate on abelian groups with these notions:

PROPOSITION 7. *Let* H *be an abelian group. If* H *is divisible* $C^{si}(H)$ *is the class of all groups. If* H *is not divisible,* $M \in C^{si}(H)$ *if and only if* M *is torsion and for all primes* p, $M_p = 0$ *if* $pH \neq H$.

Proof. If H is divisible then H is injective so H is strongly M-injective for every module M.

Suppose that H is not divisible and let $M \in C^{si}(H)$. M is torsion by Baer's criterion. Let p be a prime for which $M_p \neq 0$. Since $C^{si}(H)$ is closed under the passage to direct summands, subgroups, and homomorphic images, $Z(p) \in C^{si}(H)$. By Proposition 5 $\mathrm{Ext}(Z(p),H) = 0$. But $\mathrm{Ext}(Z(p),H) \cong H/pH$ so $pH = H$.

Conversely let M be torsion and satisfy $M_p = 0$ whenever $pH \neq H$. Let X be isomorphic to a factor group of M. X is torsion so $X \cong \oplus X_p$.

$\mathrm{Ext}(X,H) \cong \Pi_p \mathrm{Ext}(X_p,H)$. If $X_p \neq 0$, then $M_p \neq 0$ and $pH = H$. By [3, K p223] $\mathrm{Ext}(X_p,H) = 0$ because H is p-divisible. Thus $\mathrm{Ext}\ (X,H) = 0$.

COROLLARY 8. *Let M be a group. If M is not torsion, $C_{si}(M)$ is the class of all divisible groups. If M is torsion, $C_{si}(M)$ is the class of all groups H such that $pH = H$ for all primes p such that $M_p \neq 0$.*

We now consider a related question. Call a group strongly ads (absolute direct summand) if in every direct sum decomposition $G = A \oplus B$, A is strongly B-injective. Two clear examples are indecomposable, and divisible groups.

It is easy to prove the following:

PROPOSITION 9. *A group G is strongly ads if and only if*

(i) G is a torsion group such that each G_p is divisible or a single copy of $Z(p^n)$, n a positive integer, or

(ii) G is a torsion-free group which is either indecomposable or divisible, or

(iii) G is a mixed group which is divisible or of the form $D \oplus H$ where D is divisible torsion, H is torsion-free indecomposable reduced and $pH = H$ for all primes p such that $D_p \neq (0)$.

3. General modules

In this section we present some general results on relative injectivity and relative strong injectivity for modules.

PROPOSITION 10. *Let H and M be right R-modules. H is M-injective if $Ext^1(M/N,H) = 0$ for all $N \subseteq M$. If M is projective the converse holds.*

Proof. This is the dual of [2, Theorem 6].

THEOREM 11. *Let H and M be right R-modules and let A be a submodule of H. Suppose that A and H/A are M-injective and that $Ext^1(M,A) = 0$. Then H is M-injective.*

Proof. Let N be a submodule of M. We want the natural map $\mathrm{Hom}(M,H) \longrightarrow \mathrm{Hom}(N,H)$ to be onto. One has the exact sequences $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ and $0 \longrightarrow A \longrightarrow H \xrightarrow{\Theta} H/A \longrightarrow 0$ and these induce the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
(*) & \mathrm{Hom}(M,A) \arrow[r] \arrow[d] & \mathrm{Hom}(N,A) \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Hom}(M,H) \arrow[r] \arrow[d] & \mathrm{Hom}(N,H) \arrow[d] & \\
(**) & \mathrm{Hom}(M,H/A) \arrow[r] \arrow[d] & \mathrm{Hom}(N,H/A) \arrow[r] & 0 \\
 & \mathrm{Ext}^1(M,A) \arrow[d, equal] & & \\
 & 0 & &
\end{tikzcd}$$

which is commutative and exact in the obvious places. We see that $\mathrm{Hom}(M,H) \longrightarrow \mathrm{Hom}(N,H)$ is onto by a little diagram chasing. Let $f \in \mathrm{Hom}(N,H)$. $\Theta \cdot f \in \mathrm{Hom}(N,H/A)$ so from the exactness of line (**) $\exists \phi : M \longrightarrow H/A \ni \phi/N = \Theta \cdot f$. By the vertical exactness $\exists \psi : M \longrightarrow H \ni \phi = \Theta \cdot \psi$. Thus $\Theta \cdot (\psi/N) = \Theta \cdot f$. For all $x \in N$, $\Theta \cdot (\psi - f)(x) = 0$ so $\psi/N - f \in \mathrm{Hom}(N,A)$. By the exactness of row (*) $\exists \rho : M \longrightarrow A \ni \rho/N = \psi/N - f$. Let $g = \psi - \rho \in \mathrm{Hom}(M,H)$. Then $g/N = \psi/N - \rho/N = f$. Thus f has g as a pre-image.

COROLLARY 12. *If A is strongly M-injective and H/A is M-injective then H is M-injective.*

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Proof. $\mathrm{Ext}^1(M,A)$ by Proposition 5 (ii).

By the same argument as in Theorem 11 we have

COROLLARY 13. *Suppose that A is M-projective and H/A is strongly M-projective. Then H is M-projective.*

PROPOSITION 14. *Let M,A and H be right R-modules with A a submodule of H. If A and H/A are both strongly M-injective then H is strongly M-injective.*

Proof. One needs $\mathrm{Ext}^1(K,H) = 0$ for all images K of M. Now $0 \longrightarrow A \longrightarrow H \longrightarrow H/A \longrightarrow 0$ is exact, inducing $\mathrm{Ext}^1(K,A) \longrightarrow \mathrm{Ext}^1(K,H) \longrightarrow \mathrm{Ext}^1(K,H/A)$ which is exact at $\mathrm{Ext}^1(K,H)$. But $\mathrm{Ext}^1(K,A) = \mathrm{Ext}^1(K,H/A) = 0$ by hypothesis, so $\mathrm{Ext}^1(K,H) = 0$.

PROPOSITION 14. *The following are equivalent for a ring R:*

(i) submodules of projective right R-modules are projective,

(ii) for all right modules M, submodules of strongly M-projective modules are strongly M-projective,

(iii) factor modules of injective right modules are injective,

(iv) for all right modules M, factor modules of strongly M-injective modules are strongly M-injective,

(v) the global dimension of R is less than or equal to one, that is $\mathrm{Ext}^2(A,B) = 0$ for all right modules A,B,

(vi) R is right hereditary.

Proof. The equivalence of (i), (iii), (v) and (vi) is well known; see [5, p.162 4F].

(ii) $\Rightarrow$ (i). Suppose that P is a projective module and that A is a submodule of P. For every M, P is strongly M-projective so A is strongly M-projective. Clearly A (strongly) M-projective for every module M implies that A is projective.

Similarly (iv) $\Rightarrow$ (iii).

(v) $\Rightarrow$ (ii). Let H be strongly M-projective and let $A \le H$, $N \le M$. We need $\mathrm{Ext}^1(A,N) = 0$. The exact sequence $0 \longrightarrow A \longrightarrow H \longrightarrow H/A \longrightarrow 0$ induces the sequence $\mathrm{Ext}^1(H,N) \longrightarrow \mathrm{Ext}^1(A,N) \longrightarrow \mathrm{Ext}^2(H/A,N)$. Since both ends of the sequence are zero by hypothesis $\mathrm{Ext}^1(A,N) = 0$.

(v) $\Rightarrow$ (iv). Let H be strongly M-injective and let $A \leq H$. We need $\mathrm{Ext}^1(M/K,H/A) = 0$ for all submodules K of M. The sequence $0 \longrightarrow A \longrightarrow H \longrightarrow H/A \longrightarrow 0$ induces $\mathrm{Ext}^1(M/K,H) \longrightarrow \mathrm{Ext}^1(M/K,H/A) \longrightarrow \mathrm{Ext}^2(M/K,A)$. Since both ends of the sequence are zero, $\mathrm{Ext}^1(M/K,H/A) = 0$.

PROPOSITION 16. *The following are equivalent for a ring* R :

(a) all right modules are R*-projective,*

(b) all right modules are strongly R*-projective,*

(c) all right modules are R*-injective,*

(d) all right modules are strongly R*-injective,*

(e) R *is completely reducible.*

The proof is left as an exercise.

It is interesting to approach relative projectivity and relative injectivity from the point of view of the theorem which says that being quasi-Frobenius is equivalent to the coincidence of projective and injective modules.

PROPOSITION 17. *Let* R *be a ring and let* M *be a right* R*-module. Then the following are equivalent:*

(a) every right R*-module is a submodule of an* M*-projective right* R*-module,*

(b) every injective right R*-module is* M*-projective.*

Proof. (b) $\Rightarrow$ (a). Because of the existence of injective hulls. (a) $\Rightarrow$ (b). Let I be injective and suppose that $I \subset H$ where H is M-projective. I is a direct summand of H so I is M-projective too. Dually we have,

COROLLARY 18. *The following are equivalent for a ring* R *and a right module* M :

(a) every R*-module is a quotient of an* M*-injective module,*

(b) every projective module is M*-injective.*

COROLLARY 19. *If* M *is a right* R*-module that is not torsion then the following are equivalent:*

(a) every R*-module is a quotient of an* M*-injective module,*

(b) R *is quasi-Frobenius.*

Proof. (b) $\Rightarrow$ (a). Because every module is a quotient of a free module and these are injective if R is quasi-Frobenius.
(a) $\Rightarrow$ (b). By corollary 18 every projective is M-injective and, furthermore, it is injective because M contains a copy of R.

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