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A NOTE ON QUASI-METRIZABILITY

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1. Introduction. Let X be a set. A function d from $X \times X$ into the non-negative real numbers is called a (*non-archimedean*) *quasi-metric* on X if

(i) $d(x, y) = 0$ if and only if $x = y$, and

(ii) for all $x, y, z \in X$, $d(x, z) \leqq d(x, y) + d(y, z)$
$(d(x, z) \leqq \max \{d(x, y), d(y, z)\})$.

A topological space (X, T) is said to be (*non-archimedeanly*) *quasi-metrizable* if there exists a (non-archimedean) quasi-metric on X compatible with T (i.e., the ϵ-neighborhoods form a base for the topology). Denote by N the set of positive integers, and let $g : N \times X \rightarrow T$ be a function such that for each $x \in X$, $x \in \bigcap_{n=1}^{\infty} g(n, x)$. The above notions can be simply characterized in terms of such a function g (see, e.g., Hodel [**1**]). Consider the following properties which such a function g could have:

(A) $\{g(n, x) | n \in N\}$ is a local base at x;

(B) if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$;

(C) if $y \in g(n + 1, x)$, then $g(n + 1, y) \subset g(n, x)$;

(D) for each x and each n, there exists $m \in N$ such that if $y \in g(m, x)$, then $g(m, y) \subset g(n, x)$.

Of course, first countable spaces are characterized by those spaces which admit a function g satisfying property (A). Non-archimedeanly quasi-metrizable spaces, quasi-metrizable spaces, and the so-called γ-spaces [**1**] are characterized by the existence of a function g satisfying (A) and (B), (A) and (C), and (A) and (D), respectively [**4**]. As demonstrated by Lindgren and Fletcher in [**3**], the class of γ-spaces is the same as the class of co-Nagata spaces and the class of Nagata first countable spaces. For any space, the following implications hold: n.a.-quasi-metrizable $\Rightarrow$ quasi-metrizable $\Rightarrow$ γ-space $\Rightarrow$ first countable. Kofner [**2**] has exhibited a quasi-metrizable space which is not non-archimedeanly quasi-metrizable. However, it is not known whether every γ-space is quasi-metrizable.

A base B for a space X is an *ortho-base* if whenever $B' \subset B$ and $x \in \bigcap B'$, then either $\bigcap B'$ is open or B' is a local base at x. In [**4**], Lindgren and Nyikos ask whether any of the above implications reverse in the presence of an ortho-base. To give a partial answer to this question, we consider the class of proto-metrizable spaces, i.e., the paracompact spaces with an ortho-base. We show that the first two implications do reverse in the class of proto-metrizable spaces.

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We give an example to show that the third implication does not reverse, however, even for non-archimedean spaces. Recall that X is a *non-archimedean space* if there is a base $\mathscr{B}$ for X which has rank 1 (i.e., if $B, B' \in \mathscr{B}$ and $B \cap B' \neq \emptyset$, then either $B \subset B'$ or $B' \subset B$). Non-archimedean spaces are ultraparacompact and $\mathscr{B}$ is an ortho-base [**5**]. Our example is also a linearly ordered space with a point-countable base which is not quasi-metrizable, so it also answers a question of Heath [**6**].

2. It is the purpose of this section to prove the following theorem:

THEOREM 1. *If X is a proto-metrizable space, then the following are equivalent:*
(i) *X admits a non-archimedean quasi-metric;*
(ii) *X is quasi-metrizable;*
(iii) *X is a γ-space.*

Before we embark on the proof of this theorem, we shall state another characterization of proto-metrizable spaces due to Nyikos [**5**].

Let X be a topological space, and let γ be any ordinal number. A collection $\{U_\alpha\}_{\alpha<\gamma}$ of open collections is called a *proto-uniformizing family* if
(i) $\bigcup U_\alpha = \bigcup U_{\alpha+1}$ for every $\alpha < \gamma$;
(ii) if $\beta < \alpha < \gamma$, then U_α star-refines U_β, i.e., $\{\text{st}(x, U_\alpha)|x \in X\}$ is a refinement of U_β; and
(iii) for every $x \in X$, $\{\text{st}(x, U_\alpha)|\alpha < \gamma\}$ is a base at x.
X is *proto-metrizable* if and only if there exists a proto-uniformizing family for X.

LEMMA 1. *Let X be proto-metrizable, and let O be an ortho-base for X. There exists a proto-uniformizing family $\{U_\alpha\}_{\alpha<\gamma}$, where each U_α is minimal (i.e., for each $U \in U_\alpha$, $U_\alpha - \{U\}$ is not a cover of $\bigcup U_\alpha$), and collections $V_\alpha \subset O, \alpha < \gamma$, such that for each α, $U_{\alpha+1}$ star-refines $V_{\alpha+1}$, and $V_{\alpha+1}$ star-refines U_α; also if $V \in V_\alpha$, then there exists $U \in U_\alpha$ with $U \subset V$.*

Proof. Let $V_1' = O$, and let U_1 be a minimal star-refinement of O. Let $V_1 = \{V \in V_1'|\text{there exists } U \in U_1 \text{ such that } U \subset V\}$.

Suppose U_α and V_α have been constructed for all $\alpha < \beta$. If $\beta = \beta' + 1$, we can use the hereditary paracompactness of X to find a subset $V_\beta' \subset O$ which star-refines $U_{\beta'}$. Let U_β be a minimal star-refinement of V_β' such that if $\text{st}(x, U_\beta) \subset U \in U_{\beta'}$, then either $\text{st}(x, U_\beta) \neq U$, or U is degenerate. Let $V_\beta = \{V \in V_\beta'|\text{ there exists } U \in U_\beta \text{ such that } U \subset V\}$.

If β is a limit ordinal, let

$$D_\beta = \left\{\text{Int}\left(\bigcap_{\alpha<\beta} \text{st}(x, U_\alpha)\right)\middle| x \in X\right\} - \left\{U \in \bigcup_{\alpha<\beta} U_\alpha | U = \{x\} \text{ for some } x \in X\right\}.$$

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Using the fact that for each $\alpha < \beta$ and $x \in \bigcup U_{\alpha+1}$, there exists $V \in V_{\alpha+1}$ such that $\mathrm{st}(x, U_{\alpha+1}) \subset V \subset \mathrm{st}(x, U_\alpha)$, and that O is an ortho-base, it is easy to see that if $x \notin \bigcup D_\beta$, then $\{\mathrm{st}(x, U_\alpha) | \alpha < \beta\}$ is a base at X. Let V' be a subset of O which star-refines D_β. Let U_β be a minimal star-refinement of V_β', and let $V_\beta = \{V \in V_\beta' |$ there exists $U \in U_\beta$ such that $U \subset V\}$. We continue until $D_\gamma = \{\emptyset\}$ for some ordinal γ. It is easy to check that $\{U_\alpha\}_{\alpha<\gamma}$ and $\{V_\alpha\}_{\alpha<\gamma}$ satisfy the desired properties.

Proof of Theorem 1. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). We shall prove (iii) $\Rightarrow$ (i).

Suppose X, $\{U_\alpha\}_{\alpha<\gamma}$, and $\{V_\alpha\}_{\alpha<\gamma}$ are as in Lemma 1, and suppose also that $g : N \times X \to T$ satisfies properties (A) and (D). Without loss of generality, we can assume $g(n, x) \supset g(n+1, x)$ for all $x \in X$ and $n \in N$. Call an ordered pair $(V, V') \in V_\alpha \times V_{\alpha'}$ an *(n, m)-pair of x corresponding to z* if

(i) $x \in V \cap V'$,

(ii) $V \subset g(n, z)$ and $V' \subset g(m, z)$,

(iii) $y \in g(m, z)$ implies $g(m, y) \subset g(n, z)$, and

(iv) if $(V'', V''') \in V_\beta \times V_{\beta'}$ satisfies (i)-(iii), then $\alpha \leqq \beta$ and $\alpha' \leqq \beta'$.

Fix $x \in X$, x not isolated, and integers n and m, with $n \leqq m$. Define $\alpha_0(x, n, m) = \inf \{\alpha' < \gamma |$ there is an (n, m)-pair (V, V') for x with $V' \in V_{\alpha'}\}$, providing this set is non-empty. If $\alpha(x, n, m)$ has been defined for all $\beta < \beta'$, define $\alpha_{\beta'}(x, n, m) = \inf \{\alpha' < \gamma |$ there is an (n, m)-pair (V, V') for x with $(V, V') \in V_\alpha \times V_{\alpha'}$, and $\alpha > \alpha_\beta\ (x, n, m)$ for all $\beta < \beta'\}$, providing this set is non-empty. Continue until it is in fact empty, and suppose this occurs after $\alpha_\beta(x, n, m)$ has been defined for all $\beta < \beta_0$. Now we make the following definitions:

(i) $\alpha'(x, n, m) = \sup \{\alpha_\beta(x, n, m) | \beta < \beta_0\}$;

(ii) $V'(x, n, m) = \bigcap \{V \in V_\alpha | \alpha \leqq \alpha'(x, n, m)$ and $x \in V\}$;

(iii) $\alpha(x, n, m) = \inf \{\alpha | \mathrm{st}\ (x, V_\alpha) \subset V'(x, n, m)\}$;

(iv) $V(x, n, m) = \bigcap \{V \in V_\alpha | \alpha \leqq \alpha(x, n, m)$ and $x \in V\}$.

Claim I. It is true that $\alpha'(x, n, m) < \beta_x$, where β_x is the least ordinal β such that $\{\mathrm{st}(x, V_\alpha) | \alpha < \beta\}$ is a base at x.

To see that Claim I holds, first note that $\alpha_\beta(x, n, m) < \beta_x$ for all $\beta < \beta_0$. For each $\beta < \beta_0$, there is an (n, m)-pair $(V, V') \in V_{\alpha(\beta)} \times V_{\alpha'(\beta)}$ for x corresponding to z_β with $\alpha'(\beta) = \alpha_\beta(x, n, m)$ and $\alpha(\beta) > \alpha_\delta(x, n, m)$ for all $\delta < \beta$. Now $x \in \bigcap_{\beta<\beta_0} g(m, z_\beta)$, so $g(m, x) \subset \bigcap_{\beta<\beta_0} g(n, z_\beta)$. We have assumed x is not isolated, so by property (iv) in the definition of (n, m)-pair, it must be true that $\sup \{\alpha(\beta) | \beta < \beta_0\} < \beta_x$. Since $\alpha(\beta) \leqq \alpha'(\beta) = \alpha(x, n, m) < \alpha(\beta + 1)$ for all $\beta < \beta_0$, it is now clear that $\alpha'(x, n, m) < \beta_x$.

From Claim I it follows that $V'(x, n, m)$ is open, for suppose not. Then $\{V \in V_\alpha | \alpha \leqq \alpha'(x, n, m)$ and $x \in V\}$ is a base at x. Thus there exists some $V \in V_\beta$, with $x \in V \not\subseteq \mathrm{st}\ (x, V_{\alpha'(x,n,m)})$ and with $\beta + 1 \leqq \alpha'(x, n, m)$. There exists $U \in U_\beta$ with $U \subset V$. Since U_β is minimal, there exists $p \in U -$

$\bigcup\{U' \in U_\beta | U' \neq U\}$. Now $p \in \text{st}(x, V_{\alpha'(x,n,m)}) \subset \text{st}(x, V_{\beta+1}) \subset U'$ for some $U' \in U_\beta$. Since $p \in U'$, we must have $U' = U$. The contradiction $U \subset V \subsetneqq \text{st}(x, V_{\alpha'(x,n,m)}) \subset U$ proves that $V'(x, n, m)$ is open. From this it is easy to see that $\alpha(x, n, m) < \beta_x$, and reasoning identical to the above shows that $V(x, n, m)$ is open.

Claim II. If $y \in V(x, n, m)$, then $V(y, n, m) \subset V(x, n, m)$.

To prove this claim we need only show that if $y \in V(x, n, m)$, then $\alpha(y, n, m) \geqq \alpha(x, n, m)$. Let us suppose $y \in V(x, n, m)$ and $\alpha(y, n, m) < \alpha(x, n, m)$. Note that every (n, m)-pair for x satisfies conditions (i)-(iii) in the definition of an (n, m)-pair for y. Thus $\alpha_0(y, n, m) \leqq \alpha_0(x, n, m)$. If $\alpha_0(y, n, m) < \alpha_0(x, n, m)$, then there is an (n, m)-pair (V, V') for y which is not an (n, m)-pair for x, where $V' \in V_{\alpha_0(y,n,m)}$. Thus $x \notin V \cap V'$. Since $x \in \text{st}(x, V_{\alpha(x,n,m)}) \subset \text{st}(y, V_{\alpha(x,n,m)})$ and $\text{st}(y, V_{\alpha(y,n,m)}) \subset V \cap V'$, it must be true that $\alpha(y, n, m) > \alpha(x, n, m)$, contradiction. Thus $\alpha_0(y, n, m) = \alpha_0(x, n, m)$. Now suppose $\alpha_\beta(y, n, m) = \alpha_\beta(x, n, m)$ for all $\beta < \beta'$. Then by exactly the sane reasoning as above, we can show that $\alpha_{\beta'}(y, n, m) = \alpha_{\beta'}(x, n, m)$. Thus $\alpha_\beta(y, n, m) = \alpha_\beta(x, n, m)$ for all $\beta < \beta_0$. Hence $\alpha'(y, n, m) \geqq \alpha'(x, n, m)$. From this it easily follows that $\alpha(y, n, m) \geqq \alpha(x, n, m)$, a contradiction which proves Claim II.

Let $\{(n_k, m_k) | k \in N\}$ be an enumeration of $\{(n, m) \in N \times N | n \leqq m\}$. For each non-isolated point $x \in X$, define $g'(1, x) = V(x, n_1, m_1)$. If $g'(i, x)$ has been defined for all $i < k$, let

$$g'(k, x) = \begin{cases} g'(k-1, x) & \text{if } V(x, n_k, m_k) \supset V(x, n_{k-1}, m_{k-1}) \\ V(x, n_k, m_k) & \text{otherwise.} \end{cases}$$

If x is isolated, define $g'(n, x) = \{x\}$ for all $n \in N$. Now suppose $y \in g'(k, x)$, $y \neq x$. Let k' be the least integer such that $g'(k', x) = g'(k, x)$. Then $y \in V(x, n_{k'}, m_{k'})$, so $g'(k, y) \subset g'(k', y) \subset V(y, n_{k'}, m_{k'}) \subset V(x, n_{k'}, m_{k'}) = g'(k, x)$. Thus g' satisfies property (B). That g' satisfies property (A) follows from the fact that if n and m are such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$, then $V'(x, n, m) \subset g(n, x)$. Thus X admits a non-archimedean quasi-metric, and the proof is finished.

3. It is the purpose of this section to describe an example of a first countable non-archimedean space which is not a γ-space. The space we describe is also a linearly ordered space with a point-countable base. The author is grateful to Peter Nyikos for suggesting that this space may be such an example.

Let A be an uncountable set. The points of the space X are all sequences $\{x_\alpha\}_{\alpha<\beta}$ of elements of A which are of the following type:

(i) $\beta < w_1$;

(ii) there exists an $a \in A$ which is repeated infinitely many times in the sequence; and

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(iii) if $\gamma < \beta$, then no element of A is repeated infinitely many times in the sequence $\{x_\alpha\}_{\alpha<\gamma}$.

If $x = \{x_\alpha\}_{\alpha<\beta} \in X$, and $\gamma < \beta$, we denote by $x(\gamma)$ the sequence $\{x_\alpha\}_{\alpha<\gamma}$. Let $U(x(\gamma)) = \{y \in X | y(\gamma) = x(\gamma)\}$. Let

$$U = \{U(x(\gamma)) | x = \{x_\alpha\}_{\alpha<\beta} \in X \text{ and } \gamma < \beta\}$$

be a base for a topology on X. It is easy to see that if $U(x(\gamma)) \cap U(y(\delta)) \neq \emptyset$, then either $x(\gamma) = y(\gamma)$ or $x(\delta) = y(\delta)$; hence $U(x(\gamma)) \supset U(y(\delta))$ or $U(x(\gamma)) \subset U(y(\delta))$. Thus X is a non-archimedean space. Note also that if $x = \{x_\alpha\}_{\alpha<\beta} \in X$, then the only elements of U which contain x are the sets $U(x(\gamma))$, $\gamma < \beta$. Thus X has a point-countable base. Finally, let "$<$" be any linear order on A. If $x = \{x_\alpha\}_{\alpha<\beta}$ and $x' = \{x_\alpha'\}_{\alpha<\beta'}$ are in X, define $x < x'$ and only if $x_\gamma < x_\gamma'$, where γ is the least ordinal such that $x_\gamma \neq x_\gamma'$. It is easy to check that the linear order topology induced on X is the same as the topology induced by U.

It remains to prove that X is not a γ-space. By Theorem 1, we need only show that X does not admit a non-archimedean quasi-metric. Suppose there exists a function $g' : N \times X \to T$ satisfying properties (A) and (B) given in the introduction. For each $x \in X$ and $n \in N$, there is a least ordinal γ such that $U(x(\gamma)) \subset g'(n, x)$. Define $g(n, x) = U(x(\gamma))$. It is easy to check that g also satisfies properties (A) and (B).

Let $A' = \{a^1, a^2, \ldots\}$ be any countably infinite subset of A. Since $x_0 = (a_n{}^1)_{n\in w}$, where $a_n{}^1 = a^1$ for all n, is an element of X, there exists $m(0) \in w$ such that $U(a_0{}^1, a_1{}^1, \ldots, a_{m(0)}{}^1) = g(n(0), x_0)$ for some $n(0) \in N$. Let $s_0 = (a_0{}^1, a_1{}^1, \ldots, a_{m(0)}{}^1)$. (Of course, we can take $n(0) = 1$, but this is not necessary.) Similary, there exists $m(1) \in w$ such that $U(a_0{}^1, a_1{}^1, \ldots, a_{m(0)}{}^1, a_{m(0)+1}{}^{n(0)}, \ldots, a_{m(1)}{}^{n(0)}) = g(n(1), x_1)$ for some $n(1) \in N$, where $x_1 = (a_0{}^1, \ldots, a_{m(0)}{}^1, a_{m(0)+1}{}^{n(0)}, a_{m(0)+2}{}^{n(0)}, \ldots)$. Let

$$s_1 = (a_0{}^1, \ldots, a_{m(0)}{}^1, a_{m(0)+1}{}^{n(0)}, \ldots, a_{m(1)}{}^{n(0)}).$$

Now suppose $m(\alpha)$, $n(\alpha)$, s_α and x_α have been defined for all $\alpha < \beta$. Let s denote the sequence such that $s(\gamma) = s_\alpha(\gamma)$ for every γ for which $s_\alpha(\gamma)$ is defined, and $s(\gamma)$ is not defined if $s_\alpha(\gamma)$ is not defined for any $\alpha < \beta$. Suppose that no element of A is repeated infinitely many times in s. We define $m(\beta)$, $n(\beta)$, s_β, and x_β as follows:

(i) If β is a limit ordinal, pick an element $a^\beta \in A - A'$ which does not appear in the sequence s; there exists $m(\beta) \in w$ such that

$$U(s \cap (a_0, a_1, \ldots, a_{m(\beta)})) = g(n(\beta), x_\beta)$$

for some $n(\beta) \in N$, where $x_\beta = s \cap (a_n{}^\beta)_{n\in w}$. (If s and t are sequences, $s \cap t$ denotes the sequence s followed by the sequence t.) Let
$s_\beta = s \cap (a_0, a_1, \ldots, a_{m(\beta)})$.

(ii) If $\beta = \gamma + 1$, then $s = s_\gamma$ and $U(s_\gamma) = g(n(\gamma), x_\gamma)$. There exists $m(\beta) \in w$ such that $U(s \cap (a_0{}^{n(\gamma)}, a_1{}^{n(\gamma)}, \ldots, a_{m(\beta)}{}^{n(\gamma)})) = g(n(\beta), x_\beta)$ for some

$n(\beta) \in N$, where $x_\beta = s \cap (a_k^{n(\gamma)})_{k\in w}$. Let $s_\beta = s_\gamma \cap (a_0^{n(\gamma)}, a_1^{n(\gamma)}, \ldots, a_{m(\beta)}^{n(\gamma)})$.

Continue until the sequence s as defined above contains an element of A which is repeated infinitely many times. By the construction, this element will be an element of A', say a^p. This will occur at some stage β' of the construction with $\beta' < w_1$. There exists, then, a sequence γ_n converging to β' with $U(s_{\gamma n_n} = g(p, x_{\gamma_n})$. But $s \in X$, and so $g(p, s) \subset \bigcap_{n=1}^{\infty} g(p, x_{\gamma_n}) = \{s\}$. This contradiction proves that X does not admit a non-archimedean quasi-metric, hence X is not a γ-space by Theorem 1.

In a letter to the author, P. Nyikos notes that the above example answers in the negative the following question of Hodel [**1**]: Is every space with a point-countable base a $w\theta$-space? This is due to the following theorem of Nyikos, which we include here with his permission.

THEOREM 2 (Nyikos). *Let X be a non-archimedean space. The following are equivalent:*

(i) *X is a γ-space;*
(ii) *X is a $w\gamma$-space;*
(iii) *X is a $w\theta$-space;*
(iv) *X is a θ-space.*

Proof. From [**1**] we known that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). Thus it is sufficient to show (iii) $\Rightarrow$ (i). Suppose X is a $w\theta$-space, that is, there exists a function $g : N \times X \rightarrow T$ such that $x \in \bigcap_{n=1}^{\infty} g(n, x)$, and if $\{p, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, p)$ for $n = 1, 2, \ldots$, then $\{x_n\}_{n=1}^{\infty}$ has a cluster point. We may assume the $g(n, x)$'s are elements of a rank 1 base for X.

Let $X' = \{x \in X|$ there is a neighborhood of x which is compact$\}$. Since X is hereditarily paracompact, and since compact non-archimedean spaces are metrizable, X' is an open metrizable subset of X. Thus there exists a function $g' : N \times X' \rightarrow T$ satisfying properties (A) and (D), and the $g'(n, x)$'s are elements of the rank 1 base for X, with $g'(n, x) \subset g(n, x)$.

Suppose $p \notin X'$, and fix $n \in N$. Let $\{z_n\}_{n=1}^{\infty}$ be a countable subset of $g(n, p)$ with no cluster point. Suppose that for each $m \in N$, there exists $y_m \in g(m, p)$ with $g(m, y_m) \not\subset g(n, p)$. Then $g(m, y_m) \supset g(n, p)$, and so $\{p, z_m\} \subset g(m, y_m)$ and $y_m \in g(m, p)$ for $m = 1, 2, \ldots$, yet $\{z_m\}_{m=1}^{\infty}$ has no cluster point, contradiction. Thus there exists $m \in N$ such that $y \in g(m, p)$ implies $g(m, y) \subset g(n, p)$. It is easy to verify also that $\{g(n, p)\}_{n=1}^{\infty}$ is a base at p. Thus the function $h : N \times X \rightarrow T$ defined by

$$h(n, x) = \begin{cases} g'(n, x) & \text{if } x \in X' \\ g(n, x) & \text{if } x \notin X' \end{cases}$$

satisfies properties (A) and (D), and so X is a γ-space.

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