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INTERPOLATION UNDER A GRADIENT BOUND

GUNNAR ARONSSON

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Abstract

This paper deals with the interpolation of given real boundary values into a bounded domain in Euclidean n-space, under a prescribed gradient bound. It is well known that there exist an upper solution (an inf-convolution) and a lower solution (a sup-convolution) to this problem, provided that a certain compatibility condition is satisfied. If the upper and lower solutions coincide somewhere in the domain, then several interesting consequences follow. They are considered here. Basically, the upper and lower solutions must be regular wherever they coincide.

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1. Introduction

Consider the interpolation of given real boundary values φ into a bounded domain $\Omega \subset \mathbb{R}^n$ under a gradient bound: $|\nabla(u)| \leq g$, where g is continuous. The interpolating function must belong to $W^{1,\infty}(\Omega)$. This can be done if and only if the boundary values are compatible with the gradient bound, and the condition for this is well known. It is also known that, provided that this condition is satisfied, there exists an upper solution W in the form of an inf-convolution and a lower solution U in the form of a sup-convolution. If the upper and lower solutions coincide somewhere in the domain, then several interesting consequences follow. These will be considered here in some detail. To do so, one must look at optimal curves Γ, minimizing a line integral $\int_\Gamma g\, ds$, with prescribed endpoints. It was found convenient (though not necessary) to cast this problem into optimal control form. An existence theorem of Cesari is used, as well as the Pontryagin maximum principle. This leads to existence and some regularity for optimal curves and a uniform bound for their curvature on each compact subset of Ω. A key result is that the uniqueness set E, that is, the subset of Ω where $U = W$, either consists of optimal curves having nice properties, or is empty. As expected,

$|\nabla U| = |\nabla W| = g(x)$ on E. It is also shown that U and W belong to C^1 in the interior of E, and moreover their gradient satisfies a local Lipschitz condition there. This result is in a sense the best possible. The concepts of semiconcavity and semiconvexity play a key role in the proof. The main results are Theorems 4.2, 4.3 and 5.2.

2. On $|\nabla u|$ and two representation formulas

We start by looking at a simple interpolation lemma. When talking about a Lipschitz curve in a domain Ω with endpoints on $\partial\Omega$, we agree that the curve has only endpoints on the boundary, and stays otherwise inside Ω. The following lemma is not new. We include here a simple and concrete proof for completeness. Further discussion follows after the corollary.

LEMMA 2.1 (Interpolation under a gradient bound). *Let Ω be a bounded domain in Euclidean n-space. Let $g \geq 0$ be a given bounded and continuous function in Ω and φ be a given continuous function on $\partial\Omega$. Then there exists a solution function $u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ such that $|\nabla u| \leq g$ almost everywhere in Ω and such that $u = \varphi$ on $\partial\Omega$ if and only if*

$$\int_\Gamma g\, ds \geq |\varphi(P) - \varphi(Q)| \tag{2.1}$$

for each pair of points P, Q on $\partial\Omega$, and each Lipschitz curve $\Gamma \subset \Omega$, with endpoints P and Q.

PROOF. We sketch the easy proof in two dimensions. Assume that a function u exists, having the required properties. Let $A = (0, 0)$, $B = (1, 0)$ and the segment AB be contained in Ω. The segment $L_t = \{(x, t) \mid 0 \leq x \leq 1\}$ is contained in Ω if $|t| \leq \delta$, for some $\delta > 0$. Then u is differentiable and $|Du| \leq g$ almost everywhere on L_t for almost every t, by theorems of Rademacher and Fubini. For such t it is obvious that $|u(0, t) - u(1, t)| \leq \int_0^1 g(x, t)\, dx$. Approximation gives

$$|u(A) - u(B)| \leq \int_0^1 g(x, 0)\, dx = \int_{L_0} g\, ds,$$

by continuity of u and g. Repeated use of this, rotational invariance, addition and approximation gives $\int_\Gamma g\, ds \geq |\varphi(P) - \varphi(Q)|$, as asserted. It also follows that $u(P) \leq \varphi(Q) + \int_\Gamma g\, ds$ for $P \in \Omega$, Q on $\partial\Omega$, and Γ connecting P and Q.

Assume instead that (2.1) holds, for all P, Q on $\partial\Omega$ and all Γ. Then the desired function u can be defined by the formula $u(P^*) = \inf(\varphi(Q) + \int_\Gamma g\, ds)$, where the infimum is taken over all points $Q \in \partial\Omega$ and all Lipschitz curves $\Gamma \subset \Omega$, connecting P^* and Q. Here, P^* may be a point in the domain or on the boundary; the assumed inequality guarantees that u will have the prescribed boundary values. The bound on the gradient is trivial, and it only remains to verify that u is continuous on $\partial\Omega$.

To do so, let $Q^* \in \partial\Omega$. Let P^* approach Q^* in the Euclidean metric through a sequence $\{P_k\}$. Then $\operatorname{dist}(P_k, \partial\Omega) \to 0$. Further, $u(P_k) \leq \varphi(Q_k) + K|P_k - Q_k|$,

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where K is a bound for g, and Q_k is a point on $\partial\Omega$, visible from P_k within Ω and as close as possible to P_k. It follows that $\limsup(u(P_k)) \le \varphi(Q^*)$, showing upper semi-continuity. To show lower semi-continuity, again let $Q^* \in \partial\Omega$ and let $P^* \in \Omega$ approach Q^* in the Euclidean metric through a sequence $\{P_k\}$. Let K and Q_k be as above. Now, for each k there exist a point $R_k \in \partial\Omega$ and a Lipschitzian curve Γ_k from R_k to P_k, such that $\varphi(R_k) + \int_{\Gamma_k} g\, ds \le u(P_k) + 1/k$. Clearly,

$$\varphi(R_k) + \int_{\Gamma_k} g\, ds + K|P_k - Q_k| \ge \varphi(Q_k),$$

so that $u(P_k) \ge \varphi(Q_k) - 1/k - K|P_k - Q_k|$. Sending k to infinity gives $\liminf(u(P_k)) \ge \varphi(Q^*)$. □

REMARKS. Observe that no assumption is made about the structure of $\partial\Omega$. In particular, $\partial\Omega$ may have inaccessible points, that is, points which cannot be reached by a curve in Ω of finite length. Such points may also be inaccessible by a curve of finite weighted length $\int g\, ds$. Such points do not influence the definition of u, and can usually be ignored. Observe also that g may vanish on part of Ω; the lemma is still valid. It is possible to formulate versions of this lemma for the case of an unbounded Ω, or an unbounded g; see [L, p. 125]. The above is, however, enough for our purposes.

COROLLARY 2.2 (Minimal and maximal interpolation). *Let Ω, g and φ be as in the lemma, and let (2.1) be satisfied. For any $P^* \in \Omega$, put*

$$W(P^*) = \inf\left(\varphi(Q) + \int_{\Gamma} g\, ds\right) \quad \text{and} \quad U(P^*) = \sup\left(\varphi(Q) - \int_{\Gamma} g\, ds\right),$$

where, as above, the infimum and the supremum are both taken over all points $Q \in \partial\Omega$ and all Lipschitz curves $\Gamma \subset \Omega$ connecting P^* and Q. Let u be any function solving the interpolation problem. Then

$$U(P^*) \le u(P^*) \le W(P^*)$$

for any $P^* \in \Omega$. Further, U and W also have the prescribed boundary values, so U and W are the minimal and maximal solutions of the interpolation problem.

REMARK. It is well known that the interior distance function is useful for solving interpolation problems under a pointwise bound on the gradient; see [A1, supplement] or [J, p. 57]. Above all, one should compare [L, pp. 116–117]. A very similar situation is treated in [CDP], and the above lemma can also be seen as a consequence of Theorem 2.11 in that paper. The above formulas for W and U are clearly generalizations of the classical formulas, given by McShane and Whitney independently in 1934, for the extension of functions under Lipschitz conditions. See [A1, p. 552].

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In the rest of this paper it is assumed that (2.1) holds, unless the contrary is said. It is convenient now to introduce the two interpolation operators

$$F_1(\Omega, g, \varphi)(P) = U(P) = \sup\left(\varphi(Q) - \int_\Gamma g\, ds\right) \quad \text{and}$$
$$F_2(\Omega, g, \varphi)(P) = W(P) = \inf\left(\varphi(Q) + \int_\Gamma g\, ds\right) \quad \forall P \in \Omega.$$

LEMMA 2.3 (Self-reproducing property of these interpolation formulas). *The operators F_1 and F_2 are self-reproducing. More precisely, let ω be a subdomain of Ω, not necessarily compactly contained. Then, for all $P \in \omega$, we have $F_1(\omega, g_\omega, U_{\partial\omega})(P) = U(P)$ and $F_2(\omega, g_\omega, W_{\partial\omega})(P) = W(P)$. Here, g_ω , $U_{\partial\omega}$ and $W_{\partial\omega}$ are the restrictions of g to ω and of U and W to $\partial\omega$ respectively.*

PROOF. This is obvious from the minimal and maximal interpolation properties. □

Now, let A and B be arbitrary points in Ω; let g be as above. Define $d(A, B)$ to be $\inf(\int_\Gamma g\, ds)$, where the infimum is taken over all Lipschitz curves $\Gamma \subset \Omega$, connecting A and B. If $g > 0$ in Ω, then this defines a metric. Assuming $g \geq 0$ only, d need not separate points, so we call it a quasi-metric or a quasi-distance. Then extend the definition of d to points A and B in $\overline{\Omega}$. Does $d(A, B) \leq d(A, C) + d(C, B)$ hold? Clearly, this need not be true in general if $C \in \partial\Omega$, but if $C \in \Omega$ it clearly holds. We still call it a quasi-distance. The choice of the weight-function g will always be clear from the context.

The interpolation operators F_1 and F_2 can clearly also be written as

$$F_1(\Omega, g, \varphi)(P) = \sup\{\varphi(Q) - d(P, Q) \mid Q \in \partial\Omega\},$$
$$F_2(\Omega, g, \varphi)(P) = \inf\{\varphi(Q) + d(P, Q) \mid Q \in \partial\Omega\},$$

that is, we have a sup-convolution and an inf-convolution, though maybe not in completely orthodox form!

Conversely, one can start from a differentiable function u and define $g = |\nabla u|$. An obvious question is now: can u be reconstructed from its boundary values and g, using our interpolation formulas? A partial answer is given below.

LEMMA 2.4. *Both representations work for any $u \in C^1$ without critical points. More precisely, let Ω and φ be as in Lemma 2.1. Let $g > 0$ be bounded and continuous in Ω. Suppose that there exists a function $u \in C^1(\Omega) \cap C(\overline{\Omega})$ having boundary values φ and such that $|\nabla u| = g$ in Ω. Then both representation formulas are valid, that is,*

$$\sup\{\varphi(Q) - d(P, Q) \mid Q \in \partial\Omega\} = u(P) = \inf\{\varphi(Q) + d(P, Q) \mid Q \in \partial\Omega\}$$

for all $P \in \Omega$.

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PROOF. Let $P \in \Omega$, $Q \in \partial\Omega$ and let Γ be a Lipschitz curve in Ω, from P to Q. We know that $\varphi(Q) - d(P, Q) \leq u(P)$, that is,

$$\sup\{\varphi(Q) - d(P, Q) \mid Q \in \partial\Omega\} \leq u(P).$$

To show the converse inequality, take an arbitrary $P \in \Omega$. There exists, by Peano's theorem and since $g > 0$, one or more streamlines in the direction of increasing u, emanating from P. Uniqueness need not hold. By Zorn's lemma, a maximal streamline S exists. Being maximal, S must approach the boundary. Let R be a variable point on S. By construction, we have

$$u(R) - u(P) = \int_{S'} Du \cdot \underline{e}(s)\, ds = \int_{S'} g\, ds,$$

that is, $u(R) - \int_{S'} g\, ds = u(P)$. Here, S' is the appropriate part of S. Choose a sequence $\{R_k\}$ on S, approaching some point $Q \in \partial\Omega$. Passage to the limit gives $\varphi(Q) - \int_S g\, ds = u(P)$. Finally, approximating S by Lipschitz curves in Ω gives the desired result. The other statement is proved analogously. This proves the lemma. □

Thus, u is represented in Ω as an infimal convolution, or a supremal convolution, of its boundary values and the quasi-distance in Ω induced by $g = |\nabla u|$. This discussion continues in the last section.

3. A closer look at $\int_\Gamma g\, ds$

To get further, we must consider the question of the existence and regularity of a minimizing curve for $\int_\Gamma g\, ds$. Concerning the weight function g, it is assumed in this section and the next that $g \in C^1(\Omega) \cap C(\overline{\Omega})$, and also that there exists a positive constant α such that $\alpha \leq g$ in $\overline{\Omega}$. We assume that the basic domain $\Omega \subset \mathbb{R}^n$ is bounded. Choose β such that $g \leq \beta$ in $\overline{\Omega}$. At least two approaches are possible.

(A) *A classical calculus-of-variations approach.* The integral to be minimized can be written as $\int_0^1 G(x, \dot{x})\, dt$ for some suitable parametrization $x(t)$ and thus fits into the treatment in [BGH, Section 5.9]. In our case, $G(x, \dot{x}) = g(x)|\dot{x}|$, and condition A on p. 219 is satisfied. Given any two endpoints in $\overline{\Omega}$, accessible by some suitable arc in Ω, a minimizing curve Γ exists in a wide class C of functions according to Theorem 5.22. The further regularity analysis in [BGH] is applicable at least to any connected part of Γ, lying in the interior of Ω, which is sufficient here. Let Γ^* be such a part of Γ, possibly Γ itself. It is shown in [BGH, pp. 220–222] that Γ^* can be reparametrized with respect to arc-length s so that $x(s)$ is Lipschitzian and $|\dot{x}| = 1$ almost everywhere, indicating that nothing is lost by focusing on the next approach only.

(B) *A control-theoretic approach.* We have found it convenient to use the machinery of optimal control theory and, in particular, a variant of the so-called Pontryagin maximum principle for studying a minimizing curve. Some technical preparations must first be done. The *permissible curves* are given by arc length, that is, $x = x(s)$ in $\mathbb{R}^n$, such that $x = x(s)$ is absolutely continuous and $|\dot{x}| = 1$ almost everywhere for

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$0 \leq s \leq S < \infty$ (where S is not prescribed); further $x(s) \in \overline{\Omega}$ for all s. The endpoints x_1 and x_2 in $\overline{\Omega}$ are prescribed. It is assumed that they can be connected by some permissible curve Γ (this is not necessarily the case, if Ω has inaccessible boundary points). The problem is to minimize $I = \int_0^S g(x(s))\, ds$ over this class of curves. Now, for *any* positive function $F \in C^1(\Omega) \cap C(\overline{\Omega})$ bounded away from 0, each permissible curve can be rewritten as a solution to a control system $\dot{x}(t) = F(x)u(t)$ for some measurable and bounded control $u(t)$ on some interval $0 \leq t \leq T < \infty$, where T is not prescribed. Then $s = \int_0^t ds/d\tau\, d\tau = \int_0^t |dx/dt|\, dt$ is an absolutely continuous function of t. Therefore, in control formulation, the functional to be minimized is

$$\int_0^S g(x(s))\, ds = \int_0^T g(x(t)) \frac{ds}{dt}\, dt = \int_0^T g(x(t)) \left| \frac{dx}{dt} \right| dt$$
$$= \int_0^T g(x(t)) F(x(t)) |u(t)|\, dt.$$

From now on, we choose $F(x) = 1/g(x)$ for all x, so that the functional is then simply $\int_0^T |u(t)|\, dt$ (minimum effort control). A 'canonical' control representation (that is, satisfying $|u| = 1$) is obtained by simply putting

$$t = \int_0^s \frac{d\sigma}{F(x(\sigma))}.$$

Then $t = t(s)$ is bi-Lipschitz and $dx/dt = (dx/ds)(ds/dt) = (dx/ds)F = F \cdot u(t)$ where $|u(t)| = 1$ almost everywhere. Further, there is only one such representation, except for translations in time.

Next, we go from arbitrary control form to arc length. Take an arbitrary (AC) solution $(x(t), u(t))$ such that $\dot{x}(t) = F(x(t))u(t)$ for almost all $t \in [0, T]$, for some $u \in L^\infty$, and with prescribed endpoints. For each $t \in [0, T]$, define $\sigma(t) = \int_0^t |dx/d\tau|\, d\tau$; this function is increasing, though not necessarily strictly increasing. Put $S = \sigma(T)$. For each $s \in [0, S]$, define $Z(s) = \min\{t \mid \sigma(t) = s\}$. This function is strictly increasing, but not necessarily continuous. Then define $X(s) = x(Z(s))$. If $s < s'$, then clearly

$$|X(s) - X(s')| = \left| \int_{Z(s)}^{Z(s')} \frac{dx}{d\tau}\, d\tau \right| \leq \int_{Z(s)}^{Z(s')} \left| \frac{dx}{d\tau} \right| d\tau = s' - s,$$

so X is Lipschitzian with constant 1. Thus $|dx/ds| \leq 1$, and we will verify that equality holds almost everywhere.

Let E_1 consist of all t^* such that $dx(t^*)/dt$ does not exist or such that t^* is not a Lebesgue point for dx/dt. Thus, E_1 and $\sigma(E_1)$ have Lebesgue measure zero. Let E_2 consist of all t^*, not in E_1, such that $dx(t^*)/dt = 0$. We claim that $\sigma(E_2)$ also has measure zero. Clearly, E_2 can be covered by arbitrarily short intervals I, over which the slope of $x(t)$ is less than some δ (that is, $\int_I |dx/d\tau|\, d\tau \leq \delta |I|$) and δ is at our disposal. According to Vitali's covering theorem, a finite number of these intervals,

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pairwise disjoint, can cover E_2 except for a set of arbitrarily small measure. It follows that $\sigma(E_2)$ has measure zero.

Take $s \in [0, S]$, not belonging to $\sigma(E_1)$ or $\sigma(E_2)$. Then $dx(Z(s))/d\tau \neq 0$. For small $h > 0$, we have

$$|X(s+h) - X(s)| = \left| \int_{Z(s)}^{Z(s+h)} \frac{dx}{d\tau}\, d\tau \right| \quad \text{and} \quad h = \int_{Z(s)}^{Z(s+h)} \left| \frac{dx}{d\tau} \right| d\tau.$$

Since $Z(s)$ is a Lebesgue point for dx/dt, the ratio of these expressions tends to 1 as $h \to 0$. So the control trajectory is transferred to arc-length form, and $|dx/ds| = 1$ for almost all s. Finally, $\int_0^T |u(t)|\, dt = \int_0^S g(x(s))\, ds$, as above, so if two control solutions are transferred to the same curve, then the functional values agree.

Concerning the minimization problem, it is obvious that we need only consider curves Γ such that

$$|x_1 - x_2| \le L(\Gamma) \le \frac{I(\Gamma_0)}{\alpha},$$

where $L(\Gamma)$ denotes the length of Γ. In the control formulation, we may impose the general restriction $|u(t)| \le 1$ without loss of generality. For any admissible pair $(x(t), u(t))$ on an interval $[0, T]$, we may therefore assume that $T \ge |x_1 - x_2|/\alpha > 0$. An upper bound is also needed. If the same admissible pair is transformed to arc-length form, we may assume that $S \le I(\Gamma_0)/\alpha$. A simple scaling argument now shows that we can assume $T \le \beta I(\Gamma_0)/\alpha$ without any loss of generality. Consequently, as far as a minimum is concerned, consideration can be restricted to admissible pairs $(x(t), u(t))$ on an interval $[0, T]$, where T is trapped between two positive bounds. This means that for the optimal control problem, there is a fixed starting point $(x_1, 0)$ and a compact target set consisting of all (x_2, t), where $|x_1 - x_2|/\alpha \le t \le \beta I(\Gamma_0)/\alpha$. Now the setting up is complete for invoking a general existence theorem by L. Cesari, namely Theorem 1 in [C, p. 478]. It asserts the *existence* of an optimal control pair $(x^*(t), u^*(t))$, defined for $0 \le t \le t^*$. Assume, without loss of generality, that this solution is already parametrized so that $|u^*(t)| = 1$ almost everywhere. If I^* is the optimal value of the functional I, then $I^* = t^*$. Let $(x_1(\cdot), u_1(\cdot))$, on an interval $(0, t_1)$, be another admissible pair. Then $t^* = I^* \le I(x_1(\cdot), u_1(\cdot)) \le t_1$. Therefore, this particular pair $(x^*(\cdot), u^*(\cdot))$ is *time optimal*, a fact which will be extremely useful.

Clearly, the optimal curve Γ^* need not be contained in Ω, even if the endpoints are. The following analysis is based on a variational technique and is applicable only to each connected part $\overline{\Gamma}$ of Γ^* that is contained in Ω. Now $\overline{\Gamma}$ obviously minimizes the line integral $\int_\Gamma g\, ds$ among all curves in Ω with the same endpoints, so the preceding analysis is applicable to $\overline{\Gamma}$. Thus, $\overline{\Gamma}$ corresponds to a time-optimal solution of the *control system* $\dot{x} = F(x)u(t)$. As above, the restriction on the control variable is just $|u| \le 1$. We therefore have a time-optimal trajectory $\overline{x}(t)$, contained in Ω. The corresponding control is written $\overline{u}(t)$ and the time interval is $[0, T]$.

In this fortunate situation we can invoke the maximal principle, also called *the Pontryagin maximum principle*. Our main reference is [LM, pp. 314–315 (Cor. 1)];

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other helpful references are [PBGM, p. 81]; [CLSW, p. 252]; or [P, p. 80]. According to that principle, there exists a nontrivial absolutely continuous 'adjoint response' (also called a co-state) $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_n(t))$ over the interval $[0, T]$ satisfying the system

$$\frac{d\eta_i}{dt} = -\sum_{k=1}^{n} \eta_k \cdot \overline{u_k}(t) \cdot \frac{\partial F}{\partial x_i}(\overline{x}(t)), \quad i = 1, 2, \ldots, n,$$

almost everywhere on $[0, T]$ such that *the maximality relation* also holds almost everywhere on $[0, T]$. The latter means that the quantity $F(\overline{x}(t)) \cdot \sum_{k=1}^{n} \eta_k(t) \cdot u_k$, where the variable u is subject to $|u| \leq 1$ and otherwise free, is maximized by $u = \overline{u}(t)$ for almost all t. (Actually, the maximum principle gives more information, but this is sufficient for the moment.) The maximality relation immediately gives

$$\overline{u}(t) = \frac{\eta(t)}{|\eta(t)|} \tag{3.1}$$

for almost all t, a very useful relation. It follows from the adjoint system that $\eta(t)$ is continuously differentiable on $[0, T]$ and nonzero. Thus, the same holds for $\overline{u}(t)$. It immediately follows that $\overline{x}(t) \in C^2[0, T]$. (For obtaining higher smoothness, we would need to require more from $F(x)$.)

Next, the adjoint system takes the interesting form

$$\frac{d\eta_i}{dt} = -|\eta(t)| \cdot \frac{\partial F}{\partial x_i}(\overline{x}(t)), \quad i = 1, 2, \ldots, n.$$

This can be written

$$\frac{1}{|\eta(t)|} \cdot \frac{d\eta}{dt} = -\nabla F(\overline{x}(t)) \tag{3.2}$$

for future use. The above relations for $\overline{u}(t)$, $d\eta/dt$, and so on are not new; they can for instance be found in [BH, pp. 97–98]. Further, $ds/dt = F(\overline{x}(t))$, and clearly $s(t) \in C^2[0, T]$. It also follows that $\overline{x}(s) \in C^2$ for $0 \leq s \leq S$. Now the curvature of Γ is found from the relations $d\overline{u}/dt = F(\overline{x}(t))\overline{u}(t)$ and $F(x) = 1/g(x)$, as well as (3.1) and (3.2), by easy computations. The result is

$$\frac{d\overline{u}}{ds} = \frac{\nabla g}{g} - \left(\frac{\nabla g}{g} \cdot \overline{u}\right)\overline{u}. \tag{3.3}$$

Since $\overline{u}(t)$ is the unit tangent vector of Γ, this means that *the curvature $d\overline{u}/ds$ is the orthogonal part of the logarithmic gradient of the weight function g*. This is of course not unexpected, and somewhat similar considerations for minimizing a weighted line integral occur in image processing; one can for instance compare [AK, Section 4.3.2, pp. 176–179].

It seems convenient to summarize all this as follows.

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THEOREM 3.1. *Let Ω be bounded, and impose conditions on g as at the start of this section. Define permissible curves as above, and assume that at least one such curve exists.*

Then there exists at least one optimal curve, that is, one minimizing $\int_\Gamma g\,ds$. Further, if Γ^* is an arbitrary optimal curve, it has an essentially unique representation $(x^*(t), u^*(t))$ such that $|u^*(t)| = 1$ for almost all t. This pair (x^*, u^*) is a time-optimal pair for the control system $\dot{x}(t) = F(x)u(t)$ under the condition $|u(t)| \le 1$. This holds also if Γ^* is not contained in Ω. (It is certainly contained in $\overline{\Omega}$.)

Finally, let $\overline{\Gamma}$ be a 'relatively' optimal curve, contained in Ω, not necessarily having the initially prescribed endpoints. Then $\overline{x} \in C^2$ and the curvature $d\overline{u}/ds$ is the part of the logarithmic gradient of the weight function g orthogonal to $\overline{\Gamma}$. This holds along all of $\overline{\Gamma}$.

REMARK. It is natural to ask which of these results hold for an optimal curve Γ_1 that is situated in Ω, except for one or two endpoints? One can verify that the above relations can be extended up to endpoints on $\partial\Omega$, by assuming that $g \in C^1(\overline{\Omega})$. These results will, however, not be used in this work.

4. Structure of the uniqueness set

Recall the interpolation problem from Section 2. The same assumptions as in the beginning of Section 3 are still valid, that is, $g \in C(\overline{\Omega}) \cap C^1(\Omega)$, and there exist positive constants α and β such that $\alpha \le g \le \beta$ in $\overline{\Omega}$. The basic domain $\Omega \subset \mathbb{R}^n$ is still bounded; φ is defined and continuous on $\partial\Omega$. The two interpolation formulas from Section 2 must now be modified a little, since we will need the existence of minimizing (or maximizing) couples Q, Γ. The following assumption is now made:

$$\int_\Gamma g\,ds \ge |\varphi(P) - \varphi(Q)| \tag{4.1}$$

for each pair of points P, Q on $\partial\Omega$, and each Lipschitz curve $\Gamma \subset \overline{\Omega}$, with endpoints P and Q. The two interpolation formulas will be modified accordingly: in the formula $W(P) = \inf(\varphi(Q) + \int_\Gamma g\,ds)$, Q varies over $\partial\Omega$ as before, and Γ now varies over all permissible curves in $\overline{\Omega}$ which connect P and Q. The formula for $U(P)$ is changed similarly.

We claim that the functions U and W are unchanged. To see that, consider W. Assume that the new version of $W(P)$ is less than the old version. Then there exist Q and $\Gamma \subset \overline{\Omega}$ such that $\varphi(Q) + \int_\Gamma g\,ds < \inf(\varphi(R) + \int_{\Gamma_1} g\,ds)$, where Γ_1 is restricted to Ω, except for endpoints. But Γ must have a 'last' point S on $\partial\Omega$ and a corresponding last section Γ_2 in Ω. Now choose $R = S$ and $\Gamma_1 = \Gamma_2$. Invoke (4.1) and a contradiction follows, proving the result.

LEMMA 4.1. *The infimum defining $W(P)$ and the supremum defining $U(P)$ are both attained for each P in Ω.*

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PROOF. Consider the first formula only. Take a sequence $\{Q_k, \Gamma_k\}$ of points on $\partial\Omega$ and curves in $\overline{\Omega}$ such that $(\varphi(Q_k) + \int_{\Gamma_k} g\, ds)$ approaches its lower bound. One can assume that $Q_k \to Q^* \in \partial\Omega$. A standard selection argument gives a sequence of curves converging uniformly (as functions of arc length) to a limit curve Γ^*. The couple (Q, Γ^*) delivers the wanted infimum. We refer to theorems by Hilbert and Tonelli; see [AK, p.150], or [BGH, p. 108]. $\square$

Recall from Section 2 that $W(P) \geq U(P)$ for all $P \in \Omega$.

THEOREM 4.2. *Assume that $W(P) = U(P)$ for some $P \in \Omega$. Then there is an optimal curve Γ passing through P, having only its endpoints on $\partial\Omega$, and such that $W = U$ on this curve. Further, U and W are both differentiable on Γ, have the same gradient, and $|\nabla U| = |\nabla W| = g$ there. Finally, $\nabla U = \nabla W$ is tangential to Γ. Thus, Γ is a common streamline for ∇U and ∇W.*

PROOF. According to the preceding lemma, there exist points Q_1, Q_2 on $\partial\Omega$, and associated curves Γ_1, Γ_2 in $\overline{\Omega}$ connecting these points to P, such that

$$\varphi(Q_1) - \int_{\Gamma_1} g\, ds = U(P) = W(P) = \varphi(Q_2) + \int_{\Gamma_2} g\, ds.$$

One can assume that Γ_1 has only Q_1 common with $\partial\Omega$, and that Γ_2 has only Q_2 common with $\partial\Omega$. This follows from an argument similar to the one above showing that U and W are unchanged. The combined curve $\Gamma_1 + \Gamma_2$ is therefore contained in Ω, except for endpoints. Now the relations above imply that $\varphi(Q_1) - \varphi(Q_2) = \int_{\Gamma_1+\Gamma_2} g\, ds$. On the other hand, $\int_\Gamma g\, ds \geq \varphi(Q_1) - \varphi(Q_2)$ for any curve Γ connecting Q_1 and Q_2, according to condition (4.1). Thus the combined curve $\Gamma_1 + \Gamma_2$ is optimal in the sense of Section 3, and enjoys the regularity properties stated in Theorem 1. Therefore the combined curve is in C^2 as a function of arc length. Further, we see that $W = U$ at each point of $\Gamma_1 + \Gamma_2$. Differentiation of U and W along the curve is no problem; the tangential derivative is clearly equal to g. Instead, ΔU and ΔW, caused by a small 'sideways' shift of the curve must be estimated. Consider again P as above. Let P' be a neighbouring point, such that $(P' - P)$ is orthogonal to Γ at P. From the simple estimate below, $W(P') \leq W(P) + \psi(|P' - P|)$ and analogously $U(P') \geq U(P) - \phi(|P' - P|)$, where ϕ and ψ are nonnegative functions such that $\psi(t) = o(t)$ for small t, and similarly for ϕ. Since $U(P') \leq W(P')$, it follows that

$$-\phi(|P' - P|) \leq U(P') - U(P) \leq W(P') - W(P) \leq \psi(|P' - P|).$$

Therefore any normal derivative of U or W vanishes along Γ. Note also that all estimates are locally uniform.

A simple estimate. Let π be the hyperplane orthogonal to Γ through P. Thus P' belongs to π. We will estimate $W(P')$ from above by adding a linear perturbation to Γ near P. A similar estimate for $U(P')$ is then obtained analogously. Write Γ as $x^*(s)$ for $0 \leq s \leq S$, that is, $x^*(0) = Q_2$ and $x^*(S) = P$. Let B be a unit vector with the

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direction of $(P' - P)$. We can assume that $|P' - P| < S^2$. Put $a = |P' - P|^{1/2}$; thus $0 < a < S$. Now define Γ' to be $x^*(s)$ for $0 \le s \le S - a$ and $x^*(s) + Ba(s - S + a)$ for $S - a \le s \le S$. Thus $x(S) = P'$. Let s' denote arc length along Γ', and let

$$I = \int_{S-a}^{S} g(x^*(s))\, ds \quad \text{and} \quad I' = \int_{S-a}^{S} g(x^*(s) + Ba(s - S + a))\, \frac{ds'}{ds}\, ds.$$

A comparison of I and I' is needed. Now $ds'/ds = |dx^*/ds + Ba|$, where $|dx^*/ds|$ is 1 for all s. Thus, $|ds'/ds|^2 = 1 + a^2 + 2a\,(dx^*/ds) \cdot B$ for $S - a \le s \le S$. The scalar product $(dx^*/ds) \cdot B$ vanishes for $s = S$, because the plane π is orthogonal to Γ at P. Therefore, we can write $|ds'/ds|^2 \le 1 + a^2 + a\varphi$ for some continuous $\varphi \ge 0$, easily estimated, such that $\varphi(s) \to 0$ if $s \to S$. Consequently,

$$0 < \frac{ds'}{ds} \le 1 + \frac{1}{2}(a^2 + a\varphi(s))$$

for $S - a \le s \le S$. Put $\psi(s) = \frac{1}{2}(a^2 + a\varphi(s))$. For convenience, also put

$$\Delta g(s) = g(x^*(s) + Ba(s - S + a)) - g(x^*(s)).$$

This easily gives (recall that $g \le \beta$)

$$\begin{aligned} I' - I &\le \int_{S-a}^{S} \Delta g(s)\, ds + \int_{S-a}^{S} g(x^*(s) + Ba(s - S + a))\psi(s)\, ds \\ &\le \int_{S-a}^{S} |\Delta g(s)|\, ds + \int_{S-a}^{S} \beta\psi(s)\, ds. \end{aligned}$$

The integral involving Δg is clearly bounded by $a^3 M = |P' - P|^{3/2} M$, where M is a local bound for $|\nabla g|$. Further, the last integral is dominated by

$$\frac{\beta}{2} \int_{S-a}^{S} (a^2 + a\varphi(s))\, ds = \frac{\beta}{2}|P' - P|^{3/2} + \frac{\beta}{2}|P' - P| \cdot o(1),$$

where the last factor goes to 0 as $P' \to P$, uniformly on compact subsets of Ω. It follows that $W(P')$ is no more that the sum of $W(P)$ and the preceding expressions, which is exactly an estimate of the kind we wanted. Thus, Theorem 4.2 is completely proved. □

THEOREM 4.3. *The set* $\{P \in \Omega \mid U(P) = W(P)\}$, *that is,* the uniqueness set *for the interpolation problem, is either empty or consists of a family of complete optimal curves as in Theorem 4.2. If the uniqueness set contains a subdomain* $\omega \subset \Omega$, *then* $U = W$ *there and both belong to* $C^1(\omega)$. *Moreover,* $|\nabla U| = g$ *in* ω. *Finally, if* $U \in C^1(\Omega)$, *then* $U = W$ *in* Ω, *so the interpolation is unique; and similarly if* $W \in C^1(\Omega)$.

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PROOF. The first statement is clear from the preceding. Let the solution be unique in $\omega \subset \Omega$. It is already proved that $|\nabla U| = g$ there, and g is continuous. It remains to show that the direction of ∇U is continuous. Argue indirectly: let P be arbitrary in ω and assume there is a sequence $\{P_k\}$ in ω approaching P such that $\nabla U(P_k)$ does not converge to $\nabla U(P)$, that is, the direction vectors $e(P_k)$ do not approach $e(P)$. Let $|e(P_k) - e(P)| > \delta > 0$ for all k. For each P_k there is an optimal curve Γ_k passing through P_k with the direction $e(P_k)$. But now, as in Lemma 4.1, it is possible to select a subsequence of points P_k and curves Γ_k approaching an optimal curve Γ^* through P. This curve must have a direction $e^*(P)$ at P. It now follows from Theorem 3.1 that all the curves Γ_k have uniformly bounded curvature near P. But then the convergence of the curves implies that $e(P_k) \to e^*(P)$, that is, $|e^*(P) - e(P)| \geq \delta$. This leads to two different optimal directions through P, which is clearly impossible by the previous theorem. Finally, if $U \in C^1(\Omega)$, then one simply looks at complete streamlines for ∇U. It follows that $U \equiv W$. □

EXAMPLES. (1) Choose $n = 2$, take Ω to be the unit circle and $g \equiv 1$. Put $A = (0, -1)$ and $B = (0, 1)$. Choose $\varphi(A) = -1$ and $\varphi(B) = 1$. Finally, let φ be a linear function of arc length on each of the two semicircles connecting A and B. Then, as is easily verified, the uniqueness set consists of the diameter AB and nothing else.

(2) An example where the uniqueness set is empty is obtained by choosing Ω and g as in the previous example, and $\varphi(x, y) = \arctan y$.

5. More regularity

The statement in the previous theorem that U and W belong to $C^1(\omega)$ will be strengthened here. By requiring one more derivative for g, we can show that U and W actually belong to $C^{1,1}_{\text{loc}}(\omega)$, that is, $\nabla U = \nabla W$ satisfies a Lipschitz condition on each compact part of ω. (As above, ω is a subdomain of the uniqueness set.) *It is assumed in this section that* $g \in C(\overline{\Omega}) \cap C^2(\Omega)$, *besides the earlier assumptions*. The plan of the proof is simple: verify that W is semiconcave and U semiconvex and then use known results from the theory for such functions; see [CS]. The definition of semiconcave to be used here is found in [CS, p. 2]. Further, a function h is semiconvex by definition if $-h$ is semiconcave.

LEMMA 5.1. *W is semiconcave and U is semiconvex on each compact part of Ω.*

PROOF. We will verify that W satisfies a slightly modified version of Definition 1.1.1 in [CS, p. 2], which requires a kind of upper bound for second differences of W. Accordingly, it must be shown that, for some positive constant C, one has

$$W(x + h) + W(x - h) - 2W(x) \leq C|h|^2 \tag{5.1}$$

for all x and h in question. Since W is bounded, this inequality is crucial only for small h. We must specify which x and h will be considered. Take $\delta > 0$ and consider the open set $E = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > 8\delta\}$; this is assumed nonempty. Take an

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arbitrary $x^* \in E$, and $h \in \mathbb{R}^n$ such that $|h| < \delta$. An inequality like (5.1) will be proved when $x = x^*$, and the existence of C will become clear. There exist Q and Γ such that $W(x^*) = \varphi(Q) + \int_\Gamma g\, ds$. There also exists a uniform upper bound M for the length S of Γ, where M is independent of x^*. Further, Γ starts on $\partial\Omega$ and ends in E. Thus, one can define y^* to be the last point $z \in \Gamma$ for which $\operatorname{dist}(z, \partial\Omega) \le 4\delta$. Write Γ as $x(s)$, where $x(0) = Q$, $x(A) = y^*$, and $x(S) = x^*$. Now A is uniquely defined, and clearly $S - A > 4\delta$. Put $H = |h|^{-1}h$. Define the linear function y by $y(s) = ((s - A)/(S - A)) \cdot H$, for $A \le s \le S$ and let $t \in [-|h|, |h|]$ be a parameter at our disposal. Consider the perturbed curve Γ', defined by $x'(s) = x(s)$ for $0 \le s \le A$, and $x'(s) = x(s) + ty(s)$ for $A \le s \le S$. Thus, for instance $x'(S) = x^* + tH$. Also observe that

$$\left| t \cdot \frac{dy(s)}{ds} \right| \le \frac{|h|}{S - A} < \frac{1}{4}$$

for $A \le s \le S$. Further, $(x(s) + t \cdot y(s)) \in \Omega$ and $\operatorname{dist}(x(s) + t \cdot y(s), \partial\Omega) > 3\delta$ for $A \le s \le S$. For ease of notation, put $\varphi(Q) = 0$. Consider

$$F(t) := \int_{\Gamma'} g\, ds = \int_0^A g(x(s))\, ds + \int_A^S g(x(s) + ty(s)) \frac{ds'}{ds}\, ds$$

where $ds'/ds = |dx/ds + t\, dy/ds|$. Thus, $F(0) = W(x^*)$, $F(|h|) \ge W(x^* + h)$ and $F(-|h|) \ge W(x^* - h)$, and so

$$W(x^* + h) + W(x^* - h) - 2W(x^*) \le F(|h|) + F(-|h|) - 2F(0).$$

The right-hand side here can simply be estimated using Taylor's formula. Therefore, the first and second derivatives of $F(t)$ must be considered. Now dx/ds is a unit vector in $\mathbb{R}^n$, and thus

$$\left| \frac{ds'}{ds} \right|^2 = 1 + \frac{t^2}{(S - A)^2} + 2t\, \frac{dx}{ds} \cdot \frac{H}{S - A} \ge 1 - 2|t|\, \frac{1}{S - A} \ge 1 - 2|h|\, \frac{1}{4\delta} \ge \frac{1}{2}.$$

Consequently,

$$\frac{ds'}{ds} = \sqrt{1 + \frac{t^2}{(S - A)^2} + 2t\, \frac{dx}{ds} \cdot \frac{H}{S - A}}$$

can be differentiated with respect to t without any problems. Therefore,

$$F'(t) = \int_A^S \nabla g(x(s) + ty(s)) \cdot y(s)\, \frac{ds'}{ds}\, ds + \int_A^S g(x(s) + ty(s)) \frac{\delta}{\delta t} \left(\sqrt{1 + \frac{t^2}{(S - A)^2} + 2t\, \frac{dx}{ds} \cdot \frac{H}{S - A}} \right) ds.$$

Here $x(s) + ty(s)$ is contained in the set $G = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 3\delta\}$. Clearly, $F'(t)$ depends continuously on t. Now $g \in C^2$ and all ingredients in the above

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expression can be differentiated with respect to t. All derivatives appearing will be uniformly continuous and bounded, and there is a uniform bound M for the length $S - A$ of the interval. Therefore, $F''(t)$ is continuous and bounded: $|F''(t)| \leq K$ for $|t| \leq |h|$, where K can be chosen independently of $x^* \in E$, though it may depend on δ. By Taylor's formula,

$$F(|h|) + F(-|h|) - 2F(0) = \frac{|h|^2}{2}(F''(\theta_1) + F''(\theta_2))$$

for suitable θ_1 and θ_2. Consequently, $F(|h|) + F(-|h|) - 2F(0) \leq K|h|^2$, that is, (5.1) holds. Since $\delta > 0$ is arbitrary, W is semiconcave.

An analogous argument shows that U is semiconvex. □

THEOREM 5.2. *Let ω be a subdomain of the uniqueness set. Then U and W belong to $C^{1,1}_{\text{loc}}(\omega)$, that is, $\nabla U = \nabla W$ satisfies a Lipschitz condition on each compact part of ω.*

PROOF. Since $U = W$ is both semiconcave and semiconvex in ω, this follows from [CS, Corollary 3.3.8, p. 61]. □

REMARK. This result is best possible in the sense that $U$and W need not have continuous second derivatives on ω ; not even if g is constant and $\omega = \Omega$. See Example 3 in [A1, p. 555], where u_{yy} is discontinuous on the negative x-axis.

6. Interpretation in terms of an L^∞ extremum problem

The interpolation problem above in $W^{1,\infty}(\Omega)$ can be considered as an extremum problem: Ω is a bounded domain in $\mathbb{R}^n$, the function g in $C^1(\Omega) \cap C(\overline{\Omega})$ satisfies $\alpha \leq g \leq \beta$ for some positive α and β, and a continuous function φ is defined on $\partial\Omega$.

PROBLEM. Find the smallest $M \geq 0$, such that the interpolation problem (as in Section 2) has a solution $u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with boundary values φ, such that $|\nabla u| \leq Mg(x)$ almost everywhere.

This is a variant of the problem of interpolation under a Lipschitz condition or under a bound for the gradient; see for instance [A1] or [ACJ].

It follows from Lemma 2.1 that the optimal value is the smallest number M such that $M \int_\Gamma g\, ds \geq |\varphi(P) - \varphi(Q)|$ for all P, Q and Γ as in Lemma 2.1, provided that this number exists. We assume now that it does exist, and so M exists and is finite.

This characterization of M is not new; it is found in [CDP, Theorem 2.11]. By our Corollary 2.2 there exists a pointwise biggest and a pointwise smallest solution to the extremum problem. This also holds in a more general situation, as shown in [CDP, Theorem 2.11]. The results in Theorems 4.2 and 4.3 are applicable to this extremum problem. *In particular, if the extremum problem has a unique solution u, then $u \in C^1(\Omega)$ and $|\nabla u| = Mg$ in Ω.* This should be compared to Theorem 3 in [A1, p. 554]. Further, our result here that the uniqueness set (if nonempty) consists of complete optimal curves can be compared to Theorem 2 in [A1, p. 553]. One can also compare [A1, supplement].

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Another observation. Defining $H(p, u, x) \equiv |p|^2 (g(x))^{-2}$, the problem here is of the form: minimize the functional $F(u) = \operatorname{ess\,sup} H(Du, u, x)$ over the appropriate function class. For such a problem an associated variational equation can be found; see for instance [Y, pp. 153–154], or [A1, p. 557]. It has the form

$$H_p(Du(x), u(x), x) \cdot D_x(H(Du(x), u(x), x)) = 0. \tag{6.1}$$

Here, H_p is the vector of partial derivatives of H with respect to all p_i, and the second factor D_x is the gradient of the function $x \mapsto H(Du(x), u(x), x)$. The best known case of (6.1) is obtained for $H(p, u, x) \equiv p^2$ and $n = 2$, in which case (6.1) takes the nice form

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0.$$

A good deal is known here, see [A1, A2] or [ACJ]; this is the *infinity Laplace equation* in the plane.

There is also a rich theory treating the relation between (6.1) and the L^∞ extremum problem above. In particular, under various conditions on H, it has been proved that viscosity solutions of (6.1) are minimizers, even local minimizers, for the extremum problem. Such functions are called *absolute minimizers*. It has also been proved that absolute minimizers are viscosity solutions of (6.1). We refer to [Y, Cr] and related work for more details on that relationship. Note that, in the present case, H does not depend on u. The following results are consequences of the above discussion.

If the extremum problem has a unique solution u, then u is a classical solution of (6.1), since $u \in C^1(\Omega)$ and $H(\nabla u(x), x)$ is constant. Further, Ω is covered by complete optimal curves, that is, streamlines, for ∇u. More generally, let E be the uniqueness set for the extremum problem. Then E consists of complete optimal curves, or is empty. Finally, $u \in C^1$ on the interior of E and $H(\nabla u(x), x) \equiv M^2$ there.

7. Some further observations

The connection between *eikonal equations* (including eikonal inequalities) and the extreme functions U and W has been known for a long time. See [L] for a detailed treatment, based on the use of viscosity solutions, or [CDP]. We will make some further easy observations.

Recall the 'plus formula' $W(P) = \inf(\varphi(Q) + \int_\Gamma g\, ds)$, and the 'minus formula' $U(P) = \sup(\varphi(Q) - \int_\Gamma g\, ds)$. Clearly, $|\nabla W| = g$ at each point where W is differentiable, and the same holds for U. The question of pointwise differentiability will not be discussed here, nor will we use viscosity solutions. Consider instead a different problem: is a function $u \in C^1(\Omega)$, having 'decent' boundary values φ, uniquely determined by $g = |\nabla u|$ and φ? And can it be computed? These nontrivial questions appear as the simplest case of a problem in image analysis, called *shape from shading*, see [DFS]. It is trivial to give examples where u is not uniquely determined, so the problem is not generally well posed, but still quite interesting. The problem is easy, if there are no critical points (that is, $g > 0$), as seen from Lemma 2.4. The presence of critical points makes the problem more difficult, whatever solution concept is used. Things are easier in some cases.

LEMMA 7.1. *Consider a bounded domain $\omega \subset \mathbb{R}^2$. Let $u \in C^1(\omega) \cap C(\overline{\omega})$ and let u be constant (m) on $\partial\omega$. Assume that the set $E = \{x \in \omega \mid \nabla u(x) = 0\}$ is connected and $\overline{E} \cap \partial\omega = \emptyset$.*

Then $u(x) = M \pm \inf(\int_\Gamma g\, ds)$ in ω, where $u = M$ on E and $g = |\nabla u|$. The infimum is taken over all Lipschitz curves in ω, connecting x and E. The plus sign holds if $M < m$; otherwise the minus sign holds.

PROOF. Obviously $M \neq m$, since otherwise there must be some critical point in $\omega \setminus E$. Assume that $M > m$; the other case is analogous. Then $m < u(x) < M$ for each $x \in \omega \setminus E$, since otherwise there would be a critical point in $\omega \setminus E$. Now take an arbitrary $x \in \omega \setminus E$, and follow a maximal streamline through x in the increasing direction (compare the proof of Lemma 2.4). This streamline must terminate on E, which proves the minus formula. If $M < m$, the plus formula follows. □

REMARK. Note that it is not assumed that $|\nabla u|$ is bounded. Thus, u is simply M plus or minus the *weighted distance* to the critical set. Clearly, $u(x) = m \pm \inf(\int_\Gamma g\, ds)$, based on curves Γ connecting x and $\partial\omega$.

The following is a slight improvement.

THEOREM 7.2. *Let Ω be a bounded domain in the plane and $u \in C^1(\Omega) \cap C(\overline{\Omega})$. Assume that the set $E = \{x \in \Omega \mid \nabla u(x) = 0\}$ is nonempty, connected, and $E \subset\subset \Omega$. Put $u = M$ on E without loss of generality. Assume that u has a local maximum on E, that is, there is an open set ω such that $E \subset \omega \subset \Omega$ and such that $u < M$ on $\omega \setminus E$. Then $u(x) = \inf(u(y) + \int_\Gamma g\, ds)$ for all $x \in \Omega$, and here the infimum is taken over all $y \in \partial\Omega$ and all Lipschitz curves $\Gamma \subset \Omega$, connecting x and y. Hence, u is uniquely determined in the case of a local maximum (or minimum) on E. Further, there is an open set ω' such that $E \subset \omega' \subset \omega$ and $u(x) = M - \inf(\int_\Gamma g\, ds)$ in ω'. This infimum is taken over all Lipschitz curves in ω' connecting x and E.*

The case of a local minimum on E is analogous.

PROOF. Take an arbitrary x in $\omega \setminus E$ and consider a maximal streamline from x in the direction of decreasing u (Zorn's lemma!). Clearly, E is repelling for such a curve, so it cannot terminate in Ω. Thus, it must approach $\partial\Omega$, which proves the representation formula on $\omega \setminus E$. It holds on E by continuity, and so M is determined. Then, for any $\theta > 0$, put $E_\theta = \{x \in \Omega \mid \operatorname{dist}(x, E) \leq \theta\}$ and choose θ so small that $E_\theta \subset\subset \omega$. Put $\mu = \max\{u(x) \mid x \in \partial E_\theta\}$. Thus $\mu < M$. Consider then $\omega' = \{x \in E_\theta \mid u(x) > \mu\}$. For any $x \in \omega' \setminus E$, follow a maximal streamline from x for increasing u. It is trapped in ω' and must approach E. This proves the theorem. Note also that $u(x) = \mu$ on $\partial\omega'$. □

REMARKS. One can easily verify that Proposition 5.3 in [L, pp. 142–143], follows from the above theorem. For numerical computation of the above solution, the author would suggest the 'fast marching' algorithm, explained in [S, Ch. 9].

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GUNNAR ARONSSON, Department of Mathematics, Linköping University,
S-58183 Linköping, Sweden
e-mail: guaro@mai.liu.se